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Fluctuation Induced Forces (FIF) in Equilibrium

Fluctuation-induced forces (FIFs) are prevalent in nature, covering a plethora of phenomena spanning the realms of biophysics to cosmology.The ingredients common in these phenomena are: **(i)** A fluctuating medium that can be described by a probability distribution; and **(ii)** External objects whose presence constrains (or in some way modifies) these fluctuations. The overall strength of the interaction is proportional to the magnitude of undistorted fluctuations (set by k_BT and $\hbar$ for thermal and quantum fluctuations in equilibrium); while its range is set by the extent of correlations of the fluctuations. The most interesting cases are when the interactions are long–ranged, corresponding to scale free fluctuations.

1.1 Constraining a probability distribution functional

Consider a scalar field $\phi(\mathbf{x})$ in d-dimensions ($\mathbf{x} \in \mathcal{R}^d$) whose fluctuations are governed by a probability distribution functional $\mathcal{P}[\phi]$. Now impose constraints on the value of the field at two points as $\phi(\mathbf{x}_1) = a_1$ and $\phi(\mathbf{x}_2) = a_2$. The probability for the constrained condition is

$$W(\mathbf{x}_1, \mathbf{x}_2) \equiv \mathcal{P}[\phi(\mathbf{x}_1) = a_1, \phi(\mathbf{x}_2) = a_2] = \langle \delta[\phi(\mathbf{x}_1) - a_1][\delta[\phi(\mathbf{x}_2) - a_2]\rangle \tag{1.1}$$

$$= \int \frac{dQ_1}{(2\pi)}\frac{dQ_2}{(2\pi)}\langle \exp[iQ_1(\phi(\mathbf{x}_1) - a_1) + iQ_2(\phi(\mathbf{x}_2) - a_2)]\rangle. \tag{1.2}$$

To simplify the algebra, and since this will be the case to the majority of examples discussed, let us assume that the field ϕ is Gaussian distributed with zero mean, in which case (denoting by $\langle - \rangle_c$ the covariance)

$$W(\mathbf{x}_1, \mathbf{x}_2) = \int \frac{dQ_1}{(2\pi)}\frac{dQ_2}{(2\pi)} \exp\left[-iQ_1a_1 - iQ_2a_2 - \frac{1}{2}\langle (Q_1\phi(\mathbf{x}_1) + Q_2\phi(\mathbf{x}_2))^2\rangle_c\right]. \tag{1.3}$$

For the special case of $a_1 = a_2 = 0)$[1], we have

$$W(\mathbf{x}_1, \mathbf{x}_2) = \int \frac{dQ_1}{(2\pi)}\frac{dQ_2}{(2\pi)} \exp\left[-\frac{Q_1^2}{2}\langle \phi(\mathbf{x}_1)^2\rangle_c - \frac{Q_2^2}{2}\langle \phi(\mathbf{x}_2)^2\rangle_c - Q_1Q_2\langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle_c\right]. \tag{1.4}$$

The above equation can be interpreted as follows: $W(\mathbf{x}_1, \mathbf{x}_2)$ can be interpreted as a partition function for fluctuating "charges" Q_1 and Q_2 at positions x_1 and x_2. Creating

[1]**Exercise:** Compute the result for nonzero a_1 and a_2, and/or non-zero means for the Gaussian field ϕ.

a charge of magnitude Q carries a cost (reduction in log-probability) of $\langle\phi(\mathbf{x})^2\rangle_c Q^2/2$ (analogous to a capacitative energy), while the charges interact through a "potential" proportional to $\langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle_c$. The dual perspectives of a constrained fluctuating field, and interacting fluctuating charges enforcing the constraints are equivalent.

The integrals for Q_1 and Q_2 are also Gaussian, leading to

$$W(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{(2\pi)}\exp\left(-\frac{1}{2}\ln\det\begin{vmatrix}\langle\phi(\mathbf{x}_1)^2\rangle_c & \langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle_c\\ \langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle_c & \langle\phi(\mathbf{x}_2)^2\rangle_c\end{vmatrix}\right) \tag{1.5}$$

$$= \frac{1}{(2\pi)}\exp\left[-\frac{1}{2}\ln\left(\langle\phi(\mathbf{x}_1)^2\rangle_c\langle\phi(\mathbf{x}_2)^2\rangle_c - \langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle_c^2\right)\right] \tag{1.6}$$

$$= \frac{1}{(2\pi)}\exp\left[-\frac{\ln\langle\phi(\mathbf{x}_1)^2\rangle_c}{2} - \frac{\ln\langle\phi(\mathbf{x}_2)^2\rangle_c}{2} - \frac{1}{2}\ln\left(1-\frac{\langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle_c^2}{\langle\phi(\mathbf{x}_1)^2\rangle_c\langle\phi(\mathbf{x}_2)^2\rangle_c}\right)\right]. \tag{1.7}$$

If we now move the position of the constraints with respect to each other, the constrained probability changes. Regarding variations of this probability as originating from the Boltzmann weight of a potential $U(\mathbf{x}_1,\mathbf{x}_2)$ between the two points (as in $W \propto \exp\left[-\frac{U}{k_BT}\right]$), then the first two terms correspond to the cost of individual constraints (one-body potentials), while the last term can be interpreted as a fluctuation-induced potential

$$V(\mathbf{x}_1,\mathbf{x}_2) = \frac{k_BT}{2}\ln\left(1-\frac{\langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle_c^2}{\langle\phi(\mathbf{x}_1)^2\rangle_c\langle\phi(\mathbf{x}_2)^2\rangle_c}\right). \tag{1.8}$$

In most physically interesting situations, due to translational symmetry $\langle\phi(\mathbf{x}_1)^2\rangle_c = \langle\phi(\mathbf{x}_2)^2\rangle_c \equiv \langle\phi^2\rangle$, with correlations $\langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle_c = \langle\phi(\mathbf{x}_1-\mathbf{x}_2))\phi(\mathbf{0})\rangle$ that typically fall off rapidly with the relative separation, resulting in

$$V(\mathbf{r}) \simeq -\frac{k_BT}{2}\frac{\langle\phi(\mathbf{r})\phi(\mathbf{0})\rangle_c^2}{\langle\phi^2\rangle_c^2}. \tag{1.9}$$

1.2 Gaussian (free) field theory

Let us know specialize to a Gaussian field theory distributed as

$$\mathcal{P}[\phi(\mathbf{x})] \propto \exp\left\{-\frac{K}{2}\int d^d\mathbf{x}\left[\xi^{-2}+(\nabla\phi)^2\right]\right\}. \tag{1.10}$$

Taking advantage of translational invariance, we can change variables to Fourier modes (in a system of volume V)

$$\tilde{\phi}_\mathbf{q} = \int d^d\mathbf{x}\, e^{i\mathbf{q}.\mathbf{x}}\,\phi(\mathbf{x}), \quad \text{with} \quad \phi(\mathbf{x}) = \frac{1}{V}\sum_\mathbf{q} e^{-i\mathbf{q}.\mathbf{x}}\tilde{\phi}_\mathbf{q} = \int\frac{d^dq}{(2\pi)^d}e^{-i\mathbf{q}.\mathbf{x}}\tilde{\phi}(\mathbf{q}), \tag{1.11}$$

in terms of which the probability distribution is becomes a product of independent random variables as

$$\mathcal{P}[\tilde{\phi}_\mathbf{q}] \propto \prod_\mathbf{q}\exp\left\{-\frac{K}{2V}\left[\xi^{-2}+q^2\right]|\tilde{\phi}_\mathbf{q}|^2\right\}, \quad \text{with} \quad \langle|\tilde{\phi}_\mathbf{q}|^2\rangle = \frac{V}{K(\xi^{-2}+q^2)}. \tag{1.12}$$

Correlations in real space are now obtained as

$$\langle \phi(\mathbf{x_1})\phi(\mathbf{x_2})\rangle = \int \frac{d^dq}{(2\pi)^d}\frac{e^{-i\mathbf{q}.(\mathbf{x_1}-\mathbf{x_2})}}{K(\xi^{-2}+q^2)} \equiv -\frac{1}{K}I_d(\mathbf{x_1}-\mathbf{x_2},\xi), \tag{1.13}$$

where we have defined

$$I_d(\mathbf{x},\xi) = -\int \frac{d^dq}{(2\pi)^d}\frac{e^{i\mathbf{q}.\mathbf{x}}}{q^2+\xi^{-2}}. \tag{1.14}$$

The solution to the above equation is spherically symmetric, satisfying

$$\frac{d^2I_d}{dx^2} + \frac{d-1}{x}\frac{dI_d}{dx} = \frac{I_d}{\xi^2} + \delta^d(\mathbf{x}). \tag{1.15}$$

We can try out a solution that decays exponentially at large distances as

$$I_d(x) \propto \frac{\exp(-x/\xi)}{x^p}. \tag{1.16}$$

(We have anticipated the presence of a subleading power law.) The derivatives of I_d are given by

$$\frac{dI_d}{dx} = -\left(\tfrac{p}{x}+\tfrac{1}{\xi}\right)I_d, \tag{1.17}$$

$$\frac{d^2I_d}{dx^2} = \left(\tfrac{p(p+1)}{x^2}+\tfrac{2p}{x\xi}+\tfrac{1}{\xi^2}\right)I_d. \tag{1.18}$$

For $x \neq 0$, we find

$$\frac{p(p+1)}{x^2}+\frac{2p}{x\xi}+\frac{1}{\xi^2}-\frac{p(d-1)}{x^2}-\frac{(d-1)}{x\xi} = \frac{1}{\xi^2}. \tag{1.19}$$

The choice of ξ as the decay length ensures that the constant terms in the above equation cancel. The exponent p is determined by requiring the next largest terms to cancel. For $x \ll \xi$, the $1/x^2$ terms are the next most important; we must set $p(p+1) = p(d-1)$, and $p = d-2$. This is the familiar exponent for Coulomb interactions, and indeed at this length scale the correlations don't feel the presence of ξ, and decay as

$$I_d(x) \simeq C_d(x) = \frac{x^{2-d}}{(2-d)S_d} \qquad (x \ll \xi). \tag{1.20}$$

(Note that a constant term can always be added to the solution to satisfy the limits appropriate to the correlation function under study.) At large distances $x \gg \xi$, the $1/(x\xi)$ term dominates and its vanishing implies $p = (d-1)/2$. Matching to $x \approx \xi$ yields

$$I_d(x) \simeq \frac{\xi^{(3-d)/2}}{(2-d)S_d x^{(d-1)/2}}\exp(-x/\xi) \qquad (x \gg \xi). \tag{1.21}$$

We now focus on the limit $x \ll \xi$, and eliminate dependence on K as follows: To deal with the divergence as $x \to 0$ in $d > 2$, we introduce a short-distance (lattice) cutoff a, defining the value at $x = 0$ such that

$$\langle \phi(\mathbf{x})\phi(\mathbf{0})\rangle = \langle \phi(\mathbf{0})^2\rangle \left(\frac{a}{x}\right)^{d-2} \quad (d>2). \tag{1.22}$$

In $d = 2$, we need to take care of both short and long distance divergences, and

$$\langle \phi(\mathbf{x})\phi(\mathbf{0})\rangle = \langle \phi(\mathbf{0})^2\rangle \left(\frac{\ln(L/x)}{\ln(L/a)}\right) \quad (d=2). \tag{1.23}$$

Finally, in $d = 1$, it is best to keep a finite correlation length, and set

$$\langle \phi(\mathbf{x})\phi(\mathbf{0})\rangle = \langle \phi(\mathbf{0})^2 \; e^{-x/\xi} \quad (d=1). \tag{1.24}$$

1.2.1 Dipole-dipole FIF in three dimensions

With the imposed cutoff, in $d = 3$ with the imposed constraints, we obtain a fluctuation induced potential

$$V_m(r) = \frac{k_BT}{2}\ln\left[1-\left(\frac{a}{r}\right)^2\right] \approx -\frac{k_BT}{2}\frac{a^2}{r^2}. \tag{1.25}$$

The subscript m is to emphasize that the (Dirichlet) constraints $\phi = 0$ can be interpreted as leading to monopole fluctuations. What happens if we instead impose (Neumann) constraints $\nabla\phi = 0$? The previous calculations should now be modified with the replacements

$$\delta(\phi) \to \delta^3(\nabla\phi) \qquad , \qquad e^{iQ\phi} \to e^{i\mathbf{P}.\nabla\phi}. \tag{1.26}$$

Integrating the Gaussian field ϕ now leads to

$$W_d(\mathbf{x}_1,\mathbf{x}_2) = \int \frac{d^3\mathbf{P}_1}{(2\pi)^3}\frac{d^3\mathbf{P}_2}{(2\pi)^3} \exp\left[-\frac{P_1^2}{2}\langle\nabla\phi^2\rangle - \frac{P_2^2}{2}\langle\nabla\phi^2\rangle - P_1^\alpha P_2^\beta \langle \partial_\alpha\phi(\mathbf{x}_1)\partial_\beta\phi(\mathbf{x}_2)\rangle\right]. \tag{1.27}$$

Note that rather than monopole fluctuations, the constraint on $\nabla\phi$ leads to dipole fluctuations, modifying the Coulomb interaction to the dipolar form proportional to $P_1^\alpha P_2^\beta(r^2\delta_{\alpha\beta} - d\; x_\alpha x_\beta)/r^{d+2}$ in d-dimensions. The dipolar fluctuations can be decomposed into a longitudinal component $P_\parallel$, and $(d-1)$ transverse components $\mathbf{P}_\perp$. The d-components are independent with interactions of strength $P_{1,\parallel}P_{1,\parallel}(d-1)/r^d$ between the longitudinal components, and $P_{1,\parallel}P_{1,\parallel}(-1)/r^d$ between the transverse components.

In three dimensions, once more introducing a short-distance cutoff a', the corresponding fluctuation-induced interaction is

$$V_d(r) = \frac{k_BT}{2}\ln\left[1-\left(\frac{2a'^3}{r^3}\right)^2\right] + 2\ln\left[1-\left(\frac{-a'^3}{r^3}\right)^2\right] \approx -3k_BT\frac{a'^6}{r^6}. \tag{1.28}$$

That the above answer has the same power law form as van der Waals interactions is not accidental, and will be discussed in more detail later. For the time being, I note that for two objects with polarizabilities χ_1 and χ_2, dipole fluctuations lead to an interaction that falls off with separation as $V_d(r) = -3k_BT(\chi_1\chi_2/r^6)$. [2]

1.2.2 Different approaches to FIF in one dimension

(a) The method described above in the case of one dimension, and for $x \ll \xi$ leads to

$$V_m(x) = \frac{k_BT}{2}\ln\left[1 - e^{-2x/\xi}\right] \approx \frac{k_BT}{2}\ln\left(\frac{2x}{\xi}\right), \tag{1.29}$$

and a corresponding force of

$$F(x) = -\frac{\partial V}{\partial x} = -\frac{1}{2}\frac{k_BT}{x}. \tag{1.30}$$

On dimensional grounds, the force must indeed be proportional to (k_BT/x). The coefficient $(-1/2)$ is an important proportionality factor which we now obtain from two other approaches.
(b) In one dimension, the Dirichlet constraints are equivalent to confining the field to an interval of length $x = H$, with partition function ((henceforth with $\xi \to \infty$)

$$Z(H) = \int \mathcal{D}[\phi(x)] \exp\left\{-\frac{K}{2}\int dx(\partial_x\phi)^2\right\}. \tag{1.31}$$

Discretizing the interval $0 \le x \le H$ into $N = H/\delta$ segments, the partition function can be interpreted as that of a N harmonic springs, with one constraints that the two end-points coincide. Without the constraint, we expect $Z \propto z^N$, the constraint reduces the result to $Z \propto z^N/N^{1/2}$. (This reduction is similar to that of a random walk constrained to return to its origin, and will be discussed in detail later.) Since $N \propto H$, we have

$$\ln Z(H) = N \ln z' - \frac{1}{2}\ln H. \tag{1.32}$$

The leading part of the free energy is a extensive contribution. Fortunately, this non-universal contribution does not contribute to the force when the constraint is moved along an interval with the same fluctuations on both sides. The FIF arises from the

[2] **Exercise:** The interaction between two dipole moments $\vec{D}_1$ and $\vec{D}_2$, at a separation $\vec{r} = r\,\hat{r}$ is given by

$$V(\vec{r}) = \frac{3\left(\vec{D}_1 \cdot \hat{r}\right)\left(\vec{D}_2 \cdot \hat{r}\right) - \left(\vec{D}_1 \cdot \vec{D}_2\right)}{r^3}.$$

(a) Consider *permanent dipoles* of fixed magnitude $|\vec{D}_1| = D_1$ and $|\vec{D}_2| = D_2$ which can point in any direction in three dimensions. Find the expression for the partition function $Z(r)$, obtained by integrating over all possible dipole orientations, at the lowest non-trivial order in βV. (**Hint:** Angular averages of vector components satisfy $\langle D_\alpha D_\beta\rangle_0 = D^2\delta_{\alpha\beta}/3$.) **(b)** Interpreting the partition function $Z(r)$ as resulting from an effective fluctuation-induced potential $U(r)$, find $U(r)$ at the lowest non-trivial order, and comment on its temperature dependence. **(c)** Most atoms an molecules do not have a permanent dipole moment, but are *polarizable*, i.e. there is an energy cost of $D^2/(2\chi)$ to create a dipole moment of magnitude D. Now consider the dipolar interaction $V(r)$ emerging from two polarizable particles with polarizabilities χ_1 and χ_2. Repeat the calculation of $Z(r)$, including the energy costs of creating the dipoles. **(d)** Find effective fluctuation-induced potential between polarizable particles at the lowest non-trivial order, and comment on its temperature dependence.

sub-dominant contribution growing as $\ln H$, with a universal amplitude which agrees with the previous calculation:

$$F(H) = k_B T \frac{\partial \ln Z}{\partial H} = -\frac{1}{2}\frac{k_B T}{H}. \tag{1.33}$$

(c) The partition function can also be evaluated following the earlier decomposition into independent Fourier modes, as

$$Z(H) = \int \mathcal{D}[\phi(x)] \exp\left\{-\frac{K}{2}\int dx (\partial_x \phi)^2\right\} \tag{1.34}$$

$$\propto \prod_n \int d\tilde{\phi}_q \exp\left\{-\frac{K q_n^2}{2V}|\tilde{\phi}_q|^2\right\} \tag{1.35}$$

$$\propto \prod_n \sqrt{\frac{2\pi}{K q_n^2}} \propto \exp\left[-\frac{1}{2}\sum_n \ln(K q_n^2)\right]. \tag{1.36}$$

The appropriate Fourier modes satisfying the boundary condition are proportional to $\sim (n\pi x/H)$, with $q_n = n\pi/H$ for $n = 1, 2, \cdots$. This leads to

$$\ln Z(H) = -\frac{1}{2}\sum_{n=1} \ln[K(n\pi/H)^2] = -\frac{1}{2}\sum_{n=1}\ln(K) - \sum_{n=1}\ln(n\pi/H). \tag{1.37}$$

The force is then formally computable from

$$F(H) = k_B T \frac{\partial \ln Z}{\partial H} = -\frac{k_B T}{H}\sum_{n=1} 1 \,! \tag{1.38}$$

The sum is of course divergent as it includes the extensive part of the free energy which does not contribute to force.

There is, however, a mathematical procedure for subtracting this infinity, such that $\sum_{n=1} 1 = \zeta(0) = -1/2$. This *zeta-function regularization* can be achieved by introducing a convergence factor $e^{-\epsilon n}$ such that

$$\sum_{n=1} 1 = \lim_{\epsilon\to 0} e^{-\epsilon n} = \lim_{\epsilon\to 0}\frac{e^{-\epsilon}}{1-e^{-\epsilon}} = \lim_{\epsilon\to 0}\frac{1-\epsilon+O(\epsilon^2)}{1-1+\epsilon+\epsilon^2/2+O(\epsilon^3)} \tag{1.39}$$

$$= = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(1-\epsilon+\epsilon/2+O(\epsilon^2)\right) = \frac{1}{\epsilon} - \frac{1}{2} + O(\epsilon). \tag{1.40}$$

Focusing only on the finite term in the above expansion, we again obtain the FIF

$$F(H) = k_B T \frac{\partial \ln Z}{\partial H} = -\frac{1}{2}\frac{k_B T}{H}. \tag{1.41}$$

1.2.3 Quantum Casimir force in one dimension

In quantum mechanics, after including kinetic energies, a Fourier mode of wavenumber $\mathbf{q_n}$ morphs into a harmonic oscillator of frequency $\omega_n = c|\mathbf{q_n}|$, where c is an appropriate speed (e.g. speed of light for electromagnetic waves). Each harmonic oscillator can accommodate quanta of energy $\hbar\omega_n$ above the ground state energy of $\hbar\omega_n/2$.

Thus even at zero temperature, the collection of normal modes is assigned a ground state energy of

$$E_0 = \sum_{n=1} \frac{\hbar\omega_n}{2} = \frac{\hbar c}{2}\frac{\pi}{H}\sum_{n=1} n \ . \tag{1.42}$$

Once more, we are dealing with a sum $\sum_{n=1} n$ that is infinite, but hides a finite value of $\zeta(-1) = 1/12$, which can be extracted by regularization as

$$\sum_{n=1} n = \lim_{\epsilon\to 0} ne^{-\epsilon n} = -\lim_{\epsilon\to 0}\frac{d}{d\epsilon}\sum_{n=1} e^{-\epsilon n} \tag{1.43}$$

$$= -\lim_{\epsilon\to 0}\frac{d}{d\epsilon}\frac{1}{e^{\epsilon}-1} = \lim_{\epsilon\to 0}\frac{e^{\epsilon}}{(e^{\epsilon}-1)^2} \tag{1.44}$$

$$= \lim_{\epsilon\to 0}\frac{1+\epsilon+\epsilon^2/2+O(\epsilon^3)}{[\epsilon(1+\epsilon/2+\epsilon^2/6+O(\epsilon^3)]^2} \tag{1.45}$$

$$= = \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\left(1+\epsilon-\epsilon+\epsilon^2(1-7/12-1+1/2)+O(\epsilon^3)\right) \tag{1.46}$$

$$= \frac{1}{\epsilon^2} - \frac{1}{12} + O(\epsilon). \tag{1.47}$$

The finite part of the ground state energy then leads to a quantum FIF of

$$F(H) = -\frac{dE_0}{dH} = -\frac{\hbar c}{H^2}\cdot\frac{\pi}{24} \quad . \tag{1.48}$$

The Riemann zeta-function is defined through the series

$$\zeta(s) = \sum_{n=1}\frac{1}{n^s} \, , \tag{1.49}$$

convergent for for $s > 1$. For $s \leq 0$, a finite component can be extract through the regularization procedure described above, leading to[3]

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12}, \quad \zeta(-2) = 0, \quad \zeta(-3) = \frac{1}{120}, \zeta(-4) = 0, \quad \cdots . \tag{1.50}$$

1.2.4 Thermal Casimir force in two dimensions

As discussed later, Casimir first studied the FIF per unite area (i.e. pressure) due to quantum fluctuations of the electromagnetic field confined between two mirrors. We shall hence forth refer to forces arising due to confinement of a fluctuating field in d-dimensions, between two plates of $(d-1)$ dimensions, as Casimir FIF.

Now consider a thermally fluctuating free field theory in two dimensions, confined between two one-dimensional plates at separation H. Following decomposition into

[3] **Exercise:** Using the procedure described compute $\zeta(-2)$ and $\zeta(-3)$.

Fourier modes of wave-vector $\mathbf{q} = (q, n\pi/H)$, the classical (log) partition function is obtained as

$$\ln Z(H) = -\frac{1}{2} \sum_{q,n} \ln \left[K \left(q^2 + (\frac{n\pi}{H})^2 \right) \right] . \tag{1.51}$$

Assuming long plates of dimension $L \gg H$, the sum over the modes q can be replaced with an integral, using density of states $L/(2\pi)$, such that

$$\ln Z(H) = -\frac{L}{2} \int \frac{dq}{2\pi} \sum_{n=1} \ln \left[K \left(q^2 + (\frac{n\pi}{H})^2 \right) \right] . \tag{1.52}$$

The (two dimensional) pressure in now obtained as

$$\beta P = \frac{\partial \ln Z}{\partial (HL)} = \frac{1}{L} \frac{\partial \ln Z(H)}{\partial H} = \frac{1}{H} \sum_{n=1} \left(\frac{n\pi}{H} \right)^2 \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{q^2 + (n\pi/H)^2} \tag{1.53}$$

$$= \frac{1}{H} \sum_{n=1} \left(\frac{n\pi}{H} \right)^2 \frac{2\pi i}{2\pi \cdot 2 \cdot in\pi/H} = \frac{1}{H^2} \frac{\pi}{2} \sum_{n=1} n \tag{1.54}$$

$$= -\frac{1}{H^2} \frac{\pi}{24} , \tag{1.55}$$

where the last expression follows from zeta-function regularization.

Note the similarity to the previous quantum result in the appearance of $\sum_{n=1} n$. Quite generally the "partition function" of a quantum field theory in d-dimensions is related to that of a classical field theory in $(d+1)$-dimensions, with $k_B T \to \hbar c$, and noting the extra dimension.

1.2.5 Thermal Casimir force in three dimensions

An interesting example of the thermal Casimir force in $d = 3$ occurs for wetting films of superfluid helium. The free energy of superfluid phonons (Goldstone modes) can still be treated classically below (but not too far below) the transition temperature.

The previous calculation from two dimensions is now modified to

$$\ln Z(H) = -\frac{A}{2} \int \frac{d^2 q}{(2\pi)^2} \sum_{n=1} \ln \left[K \left(q^2 + (\frac{n\pi}{H})^2 \right) \right] , \tag{1.56}$$

where A is the area of the film. The corresponding pressure is

$$\beta P = \frac{\partial \ln Z}{\partial (HA)} = \frac{1}{A} \frac{\partial \ln Z(H)}{\partial H} = \frac{1}{H} \sum_{n=1} \left(\frac{n\pi}{H} \right)^2 \int_0^{\Lambda} \frac{2\pi q \; dq}{4\pi^2} \frac{1}{q^2 + (n\pi/H)^2} \tag{1.57}$$

$$= \frac{1}{4\pi H} \sum_{n=1} \left(\frac{n\pi}{H} \right)^2 \ln \left(q^2 + (n\pi/H)^2 \right) \Big|_0^{\Lambda} \tag{1.58}$$

$$= \frac{1}{4\pi H} \sum_{n=1} \left(\frac{n\pi}{H} \right)^2 \ln \left[\frac{\Lambda^2 + (n\pi/H)^2}{(n\pi/H)^2} \right] \tag{1.59}$$

$$= -\frac{2}{4\pi H} \cdot \frac{\pi^2}{H^2} \sum_{n=1} n^2 \left(\ln n + \ln(\pi/H)\right) + \Lambda\text{-dependent terms.} \tag{1.60}$$

We don't need to bother about the term proportional to $\ln(\pi/H)$ as $\sum_{n=1} n^2 = \zeta(-2) = 0$. The other term is related to the derivative of a zeta-function, since

$$\zeta'(s) = \sum_{n=1}^{\infty} \frac{-\ln n}{n^s}, \qquad \sum_{n=1}^{\infty} n^s \ln n = -\zeta'(-s). \tag{1.61}$$

Finally noting that $\zeta'(-2) = -\zeta(3)/4\pi^2$, the Casimir pressure becomes

$$\beta P = -\frac{1}{H^3} \cdot \frac{2\pi^2}{4\pi} \cdot \frac{\zeta(3)}{4\pi^2} = -\frac{1}{H^3} \cdot \frac{\zeta(3)}{8\pi}. \tag{1.62}$$

1.2.6 Quantum Casimir force in three dimensions

Let us finally compute the celebrated quantum Casimir pressure in $d = 3$. The frequencies of the Harmonic oscillators are given by $\omega_n(\mathbf{q}) = c\sqrt{q^2 + (n\pi/H)^2}$. The zero point energy assigned to the ground state is now

$$E_0(H) = \frac{\hbar c}{2} \sum_{n=1} A \int \frac{d^2 q}{(2\pi)^2} \sqrt{q^2 + (n\pi/H)^2}. \tag{1.63}$$

Therefore,

$$\frac{E_0(H)}{A} = \frac{\hbar c}{2} \sum_{n=1} \int_0^{\Lambda} \frac{2\pi q \; dq}{4\pi^2} \sqrt{q^2 + (n\pi/H)^2} \tag{1.64}$$

$$= \frac{\hbar c}{2} \frac{1}{4\pi} \sum_{n=1} \frac{2}{3} \left(q^2 + (\frac{n\pi}{H})^2\right)^{3/2} \Bigg|_0^{\Lambda} \tag{1.65}$$

$$= -\frac{\hbar c}{12\pi} \cdot \frac{\pi^3}{H^3} \sum_{n=1}^{\infty} n^3 \tag{1.66}$$

$$= -\frac{\hbar c}{H^3} \cdot \frac{\pi^2}{1440}. \tag{1.67}$$

The corresponding pressure is

$$\beta P = -\frac{\partial E_0(H)}{A \partial H} = -\frac{\hbar c}{H^4} \cdot \frac{\pi^2}{480}. \tag{1.68}$$

The actual Casimir force of QED is twice the above result, as the electromagnetic field has two sets of normal modes (TE and TM) corresponding to Dirichlet and Neumann boundary conditions. [4]

[4] **Exercise:** Complete the set of examples by computing the quantum Casimir force in $d = 2$ dimensions.