

Networks

1. Percolation on the Cayley tree: In simple models of percolation, elements of a lattice (sites or bonds) are independently occupied with a probability p . A cluster is defined as a connected (by neighboring bonds) set of these occupied elements. At small p , only small clusters exist, and the probability that two sites, separated by a distance r , are connected to each other decays as $\exp(-r/\xi)$. The correlation length $\xi(p)$ grows with increasing p , diverging at the *percolation threshold* p_c as $\xi(p) \sim |p_c - p|^{-\nu}$. A so-called *infinite cluster*, spanning the entire system first appears at the percolation threshold p_c , covering more and more sites for $p > p_c$. We can define a probability $P(p)$ that a site belongs to this infinite cluster which, on approaching p_c from above, vanishes as $P(p) \sim |p_c - p|^\beta$.

(a) The *Cayley tree* is a hierarchical lattice in which each site at one level is connected to z sites at the level below. Thus the n -th level of the tree has z^n sites. For $z = 2$, obtain a recursion relation for the probability $P_n(p)$ that the top site of a tree of n levels is connected to some site at the bottom level.

(b) Find the limiting behavior of $P_\infty \equiv P(p)$ for infinitely many levels. Give the exponent β characterizing the vanishing of $P(p)$ at p_c .

(c) Show that for starting values close to $P(p)$, the recursion relations admit solutions of the form $P_n = P_\infty e^{-n/\xi}$. Find expressions for ξ for both $p < p_c$ and $p > p_c$, and hence obtain the exponent ν for the divergence of ξ at the percolation threshold (for $z = 2$).

(d) Find the value of p_c for any branching number z . Do the exponents β and ν depend on z ?

2. Preferential network growth with node removal: Consider the following extension of network growth by preferential attachment, as explored in *C. Moore, G. Ghoshal, and M.E.J. Newman, Phys. Rev. E 74, 036121 (2006)*: At each time step a new node is created, its m links attached preferentially proportional to the number of links already present for a pre-existing node. However, before the next node is added, a randomly selected node is removed with probability r .

(a) Show that after many steps t , the average number of nodes and links grow as $N(t) = t(1 - r)$ and $L(t) = tm(1 - r)/(1 + r)$, respectively.

(b) Write down the recursion relation governing the probability $p(k, t)$ of nodes with k links at time t .

(c) Show that in steady state $p^*(k) \propto k^{-\gamma}$, and find the exponent γ .

3. 'Feed-down' network: Consider a set of non-negative variables $\{x_n(t)\}$ (e.g. chemical concentrations), evolving in time according to first order differential equations

$$\frac{dx_n}{dt} = f_n(x_{n+1}, x_{n+2}, x_{n+3}, \dots) - g_n(x_n).$$

While the equations are quite general, we make the following assumptions:

(i) Each component decays at a rate $g_n \geq 0$ which only depends on its value x_n ; $g_n(x_n)$ is a monotonically increasing function of x_n , with $g_n(0) = 0$.

(ii) Each component is generated at a rate $0 \leq f_n \leq \max(g_n)$, which can depend only on variables numbered higher than n ; i.e. f_n does not depend on x_1, x_2, \dots, x_n .

(a) By considering eigenvalues of the stability matrix show that these equations admit a *stable* fixed point.

(b) If the variables $\{x_n(t)\}$ are made space dependent and allowed to diffuse, such that a term $D_n \nabla^2 x_n$ is added to $\partial x_n / \partial t$, can these equations admit Turing patterns?

(c) If all f_n are monotonically increasing functions of their arguments, show that starting from $\{x_n(t=0) = 0\}$, the variables proceed monotonically to the fixed point values.

4. Hopfield network with correlated states: In a recent work (arXiv:1211.3133) a type of Hopfield Lyapunov function is used to characterize the epigenetic landscape of cells. The expression profiles of transcription factors (simplified to a binary code of off or on for roughly hundred TFs) are specific to each cell type (e.g. liver, skin, heart, \dots), and are modeled as ‘associative memories’ in the parlance of neural networks. An important subtlety is that unlike typical ‘memories’ stored in a neural net, the expression profiles are highly correlated. A corresponding variant of the Hopfield model is examined in this problem.

The desired states are characterized by set of binary vectors $\{\xi^\mu\}$ for $\mu = 1, 2, \dots, M$; each vector has components $\xi_i^\mu = \pm 1$ with $i = 1, 2, \dots, N$. We would like to encode these states into the couplings $\{J_{ij}\}$ of a Hopfield network, composed of variables $\{-1 \leq x_i \leq +1\}$, evolving as

$$\frac{dx_i}{dt} = -\frac{x_i}{\tau} + \tanh \left(h_i + \sum_j J_{ij} x_j \right).$$

(a) Compute a Lyapunov function that is minimized by the above dynamics.

(b) For one state vector $\vec{\xi}$, show that the couplings $J_{ij} = \xi_i \xi_j / N$ (with $h_i = 0$) enable recovery of the pattern, provided $\tau > \tau_c$.

(c) Consider a set of M uncorrelated states, corresponding to orthogonal binary vectors such that $\sum_i \xi_i^\mu \xi_i^\nu = N \delta_{\mu\nu}$. Show that multiple states can be encoded via $J_{ij} = \sum_\mu \xi_i^\mu \xi_j^\mu / N$ (again with $h_i = 0$).

(d) We now relax the condition of orthogonal memories. Show that in this case states can be encoded through the couplings $J_{ij} = \sum_{\mu\nu} [\xi_i^\mu (C^{-1})_{\mu\nu} \xi_j^\nu] / N$, where C^{-1} is the inverse of the correlation matrix, whose elements are $C_{\mu\nu} = \sum_i \xi_i^\mu \xi_i^\nu / N$.
