

Plan

- Problem of sequence alignment
 - Algorithm
 - Global Alignment
 - Local Alignment
- Substitution matrices
- Fast database search: BLAST

Sequence Alignment

The Problem: Given:

a = MVPAGIW

b = MVAGLRW

find *the best* alignment:

a* = MVPAGI-W

b* = MV-AGLRW

6 matches

5 identities

1 substitution (I ↔ L)

2 gaps (P ↔ - and R ↔ -)

Scoring

$$S = \#identities + \mu \#substitutions - \delta \#gaps$$

Number of possible alignments

$$\Omega = \binom{M+N}{N} = \frac{(M+N)!}{M!N!}$$

$$\Omega(M = N = 100) \approx 10^{59}$$

BUT: Dynamic programming can find the optimal solution!

(Due to local additivity of the scoring function and the lack of “loops”).

Global Alignment Needleman-Wunsch

Given:

Sequences: $\mathbf{a} = a_1 a_2 \cdots a_n$; $\mathbf{b} = b_1 b_2 \cdots b_m$.

Matrix: $s(x, y)$ and gap penalty $s(x, -) = s(-, x) = g(x)$

Find:

$$S(\mathbf{a}, \mathbf{b}) = \max \sum_{i=1}^L s(a_i^*, b_i^*)$$

Solution:

1. Define $S_{ij} = S(a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_m)$

and set $S_{00} = 0$,

$$S_{0j} = \sum_{k=1}^j s(-, b_k), \text{ and } S_{i0} = \sum_{k=1}^i s(a_k, -)$$

2.

$$S_{ij} = \max \begin{cases} S_{i-1, j-1} + s(a_i, b_j), \\ S_{i-1, j} + s(a_i, -), \\ S_{i, j-1} + s(-, b_j) \end{cases}$$

Proof:

$$\begin{array}{ccc} \cdots a_i & \cdots a_i & \cdots - \\ \cdots b_j & \cdots - & \cdots b_j \end{array}$$

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Gap
penalty

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Boundary
conditions

2.

$$S_{ij} = \max \begin{cases} S_{i-1, j-1} + s(a_i, b_j), \\ S_{i-1, j} + s(a_i, -), \\ S_{i, j-1} + s(-, b_j) \end{cases}$$

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Global Alignment Needleman-Wunsch

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Forward
propagation

Proof:

$$\begin{array}{ccc} \cdots a_i & \cdots a_i & \cdots - \\ \cdots b_j & \cdots - & \cdots b_j \end{array}$$

$s(x, x) = 2$ MATCH
 $s(x, y) = -1$ for $x \neq y$ MISMATCH
 $s(x, -) = s(-, x) = -2$ GAP PENALTY

$$S_{ij} = \max \begin{cases} S_{i-1, j-1} + s(a_i, b_j), \\ S_{i-1, j} + s(a_i, -), \\ S_{i, j-1} + s(-, b_j) \end{cases}$$

Score of the optimal alignment that ends at (i,j)

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		$j-1$	j		
		G	A	A	T
	G				
$i-1$	G		1	-1	
i	C		4	?	
	T				

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Match/mismatch

		$j-1$	j		
		G	A	A	T
	G				
$i-1$	G		1	-1	
i	C		4		
	T				

$\dots a_{i-1}^* \text{ C}$
 $\dots b_{i-1}^* \text{ A}$

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Match/mismatch
 Gap in sequence #2

		$j-1$		j	
		G	A	A	T
	G				
$i-1$	G		1	-1	
i	C		4		
	T				

An orange arrow points from the cell (i, j-1) containing '4' to the cell (i, j) containing '-1'.
 A green arrow points from the cell (i, j) containing '-1' to the cell (i, j+1) which is empty.

$\dots a_{i-1}^* C$
 $\dots b_{i-1}^* A$

$\dots a_{i-1}^* C$
 $\dots b_{i-1}^* -$

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Match/mismatch
 Gap in sequence #2
 Gap in sequence #1

		$j-1$	j		
		G	A	A	T
	G				
$i-1$	G		1	-1	
i	C		4		
	T				

(Note: In the original image, an orange arrow points from 1 to 4, a green arrow points from -1 to 4, and a blue arrow points from 4 to the empty cell at row i , column j .)

..... a^*_{i-1} C
 b^*_{i-1} A

..... a^*_{i-1} C
 b^*_{i-1} --

..... a^*_{i-1} --
 b^*_{i-1} A

$s(x, x) = 2$ MATCH
 $s(x, y) = -1$ for $x \neq y$ MISMATCH
 $s(x, -) = s(-, x) = -2$ GAP PENALTY

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Match/mismatch
 Gap in sequence #2
 Gap in sequence #1

		$j-1$	j		
		G	A	A	T
	G				
$i-1$	G		1	-1	
i	C		4	2	
	T				

$\dots a^*_{i-1} \text{ C}$
 $\dots b^*_{i-1} \text{ A}$

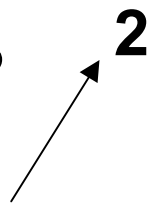
$1-1=0$

$\dots a^*_{i-1} \text{ C}$
 $\dots b^*_{i-1} \text{ --}$

$-1-2=-3$

$\dots a^*_{i-1} \text{ --}$
 $\dots b^*_{i-1} \text{ A}$

$4-2=2$



Example

Algorithm

1. Build S_{ij} matrix

2. Trace it back

Memory: $O(NM)$

Time: $O(NM)$

- There are EXACT algorithms that take less memory $O(N)$.
- There are APPROXIMATE algorithms that take less time $O(kN)$ or $aN+O(pN)$.

Global Alignment (cont)

Arbitrary form of gap penalty

$g(k)$, where k is the length of the gap

Solution:

$S_{00} = 0$, $S_{0j} = g(j)$, and $S_{i0} = g(i)$

$$S_{ij} = \max \begin{cases} S_{i-1,j-1} + s(a_i, b_j), \\ \max_{1 \leq k \leq j} \{S_{i-k,j} + g(k)\}, \\ \max_{1 \leq l \leq i} \{S_{i,j-l} + g(l)\} \end{cases}$$

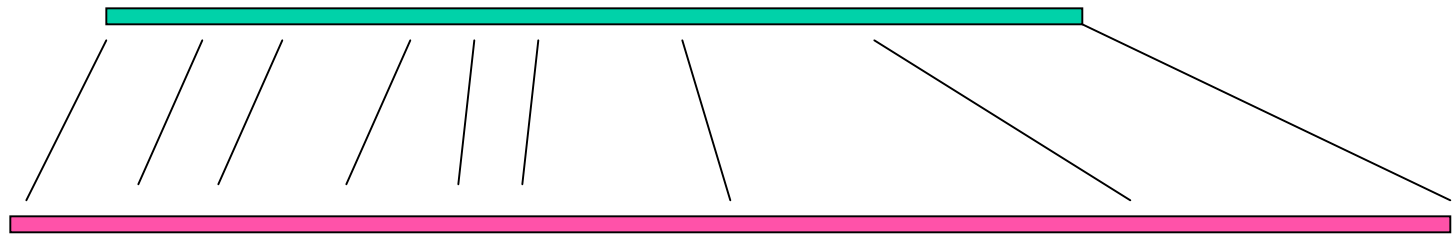
Computation time $O(n^3)$

	G	A	A	C	T
A					
G					
G		1	-1		
C		4	2		
T					

Boundary conditions

- Global
- Local
- Global-local

Global alignment



and set $S_{00} = 0$,

$$S_{0j} = \sum_{k=1}^j s(-, b_k), \text{ and } S_{i0} = \sum_{k=1}^i s(a_k, -)$$

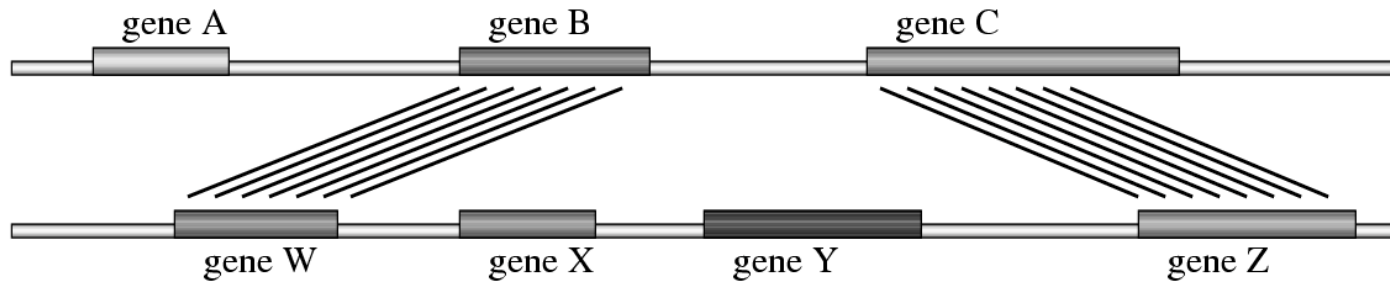
GGATCC..

---AAT..

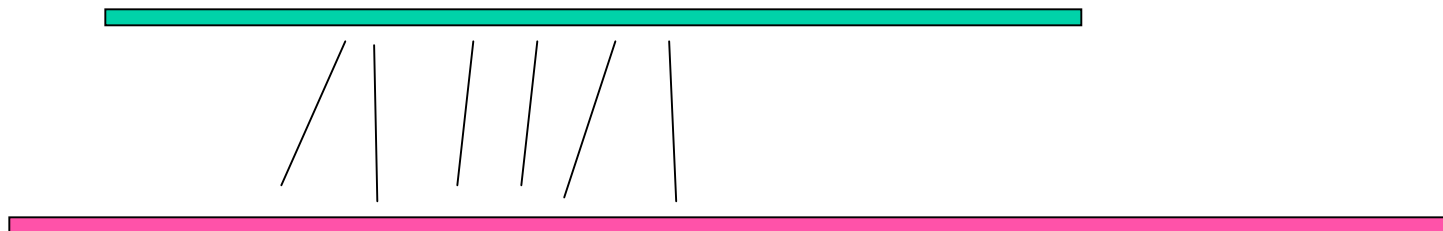


Gap penalty at the head

Local Alignment



IDEA: No penalty for gaps at the ends



Local Alignment

Local Alignment Smith-Waterman
No penalties for head/tail gaps!

Solution:

$$S_{00} = S_{0j} = S_{i0} = 0$$

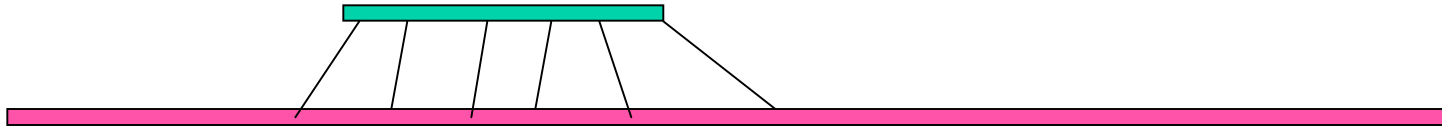
$$S_{ij} = \max \begin{cases} 0 \\ S_{i-1,j-1} + s(a_i, b_j), \\ \max_{1 \leq k \leq j} \{S_{i-k,j} + g(k)\}, \\ \max_{1 \leq l \leq i} \{S_{i,j-l} + g(l)\} \end{cases}$$

Start trace-back from $\max\{S_{ij}\}$

(GGA) ----TCCAGT----- (ATTC)
 ----TCC-GT----- (GGCC)

Global-local Alignment

IDEA: No penalty for gaps at the ends of ONE SEQUENCE



Fitting one sequence into another (Global-Local Alignment)

No penalties for head/tail gaps for one sequence

Solution:

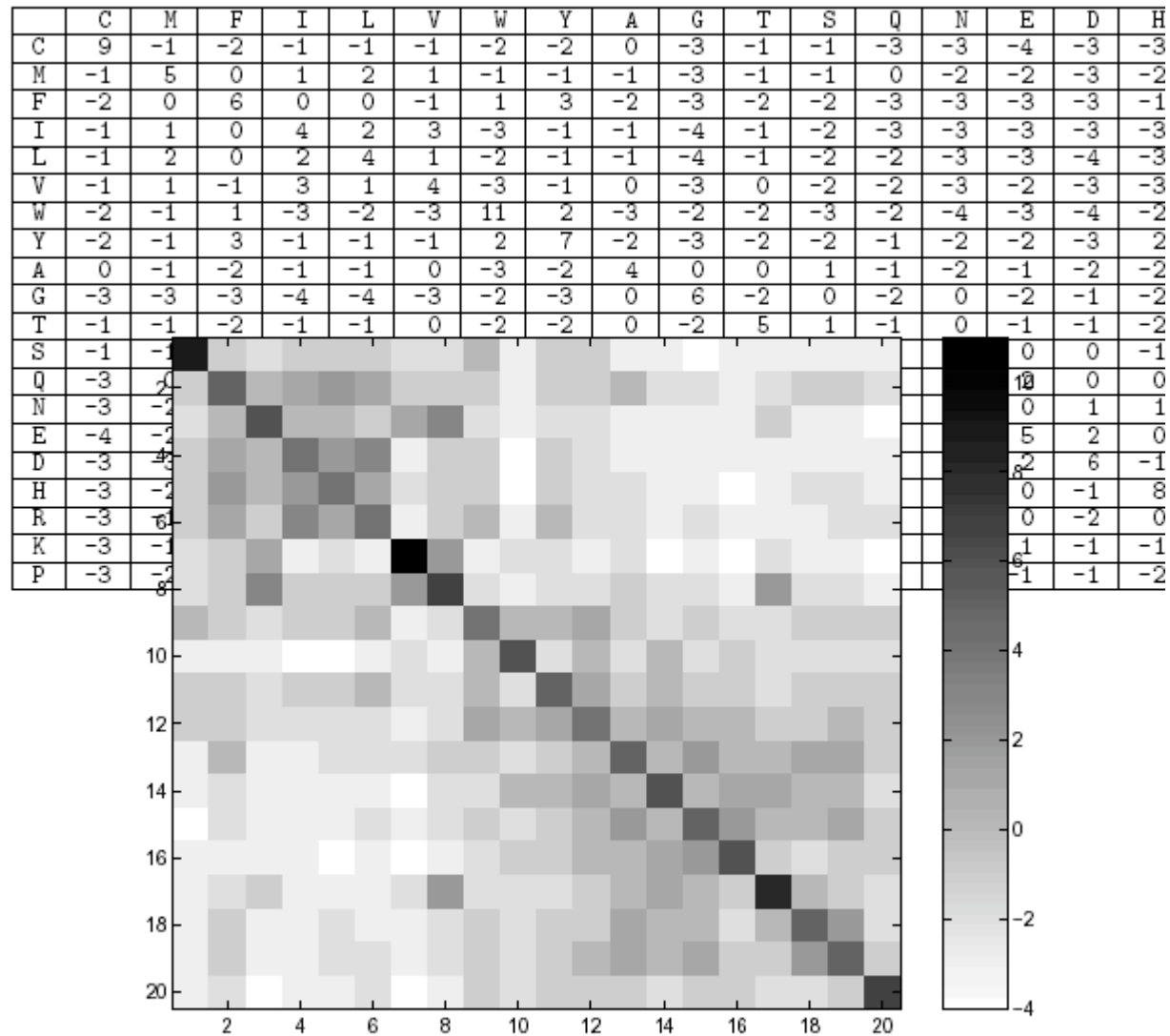
$$S_{00} = S_{0j} = 0, \text{ but } S_{i0} = g(i)$$

$$S_{ij} = \max \begin{cases} S_{i-1,j-1} + s(a_i, b_j), \\ \max_{1 \leq k \leq j} \{S_{i-k,j} + g(k)\}, \\ \max_{1 \leq l \leq i} \{S_{i,j-l} + g(l)\} \end{cases}$$

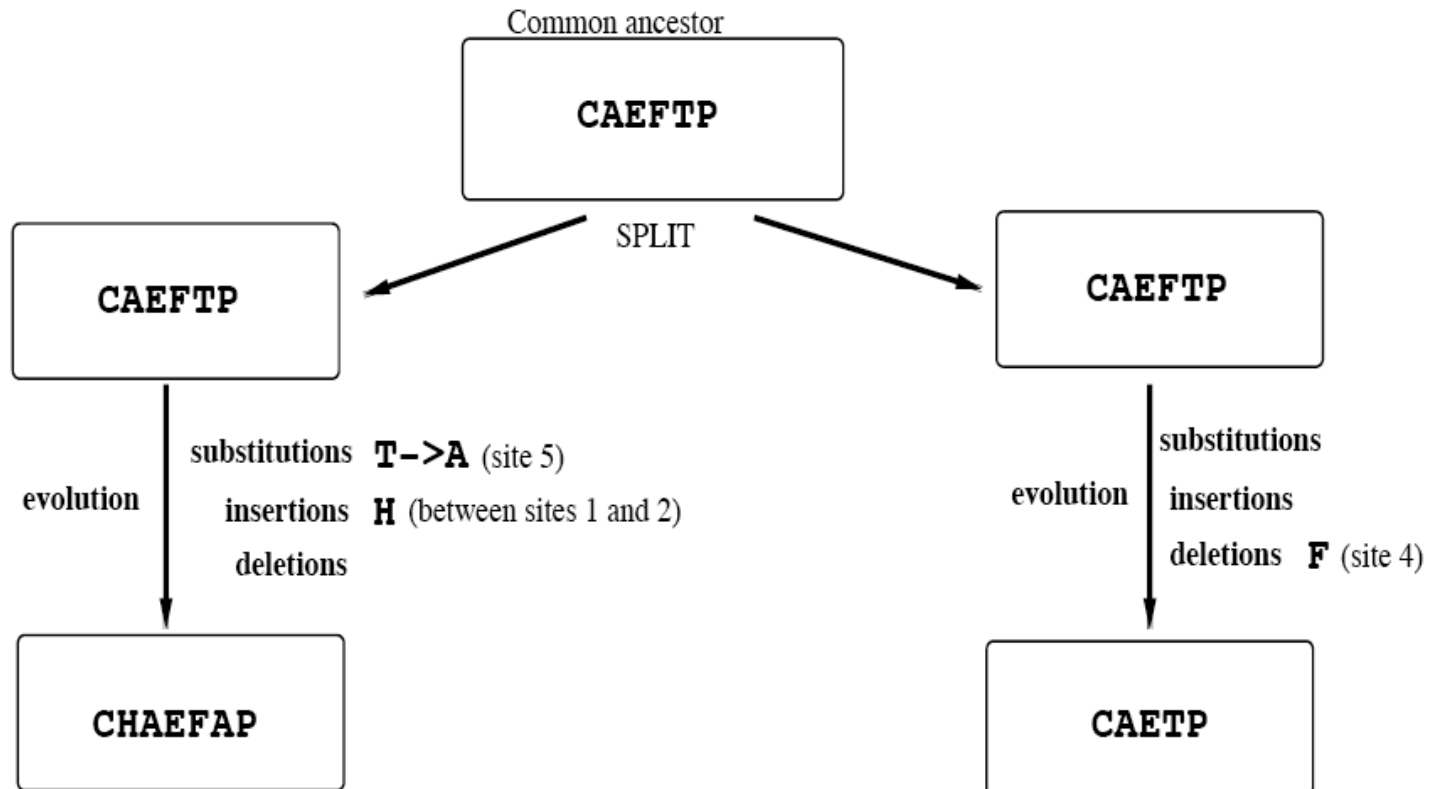
Start trace-back from $\max\{S_{nj}\}$

Scoring/substitution matrices

Realistic Scoring



EVOLUTIONARY MODEL OF SEQUENCE ALIGNMENT

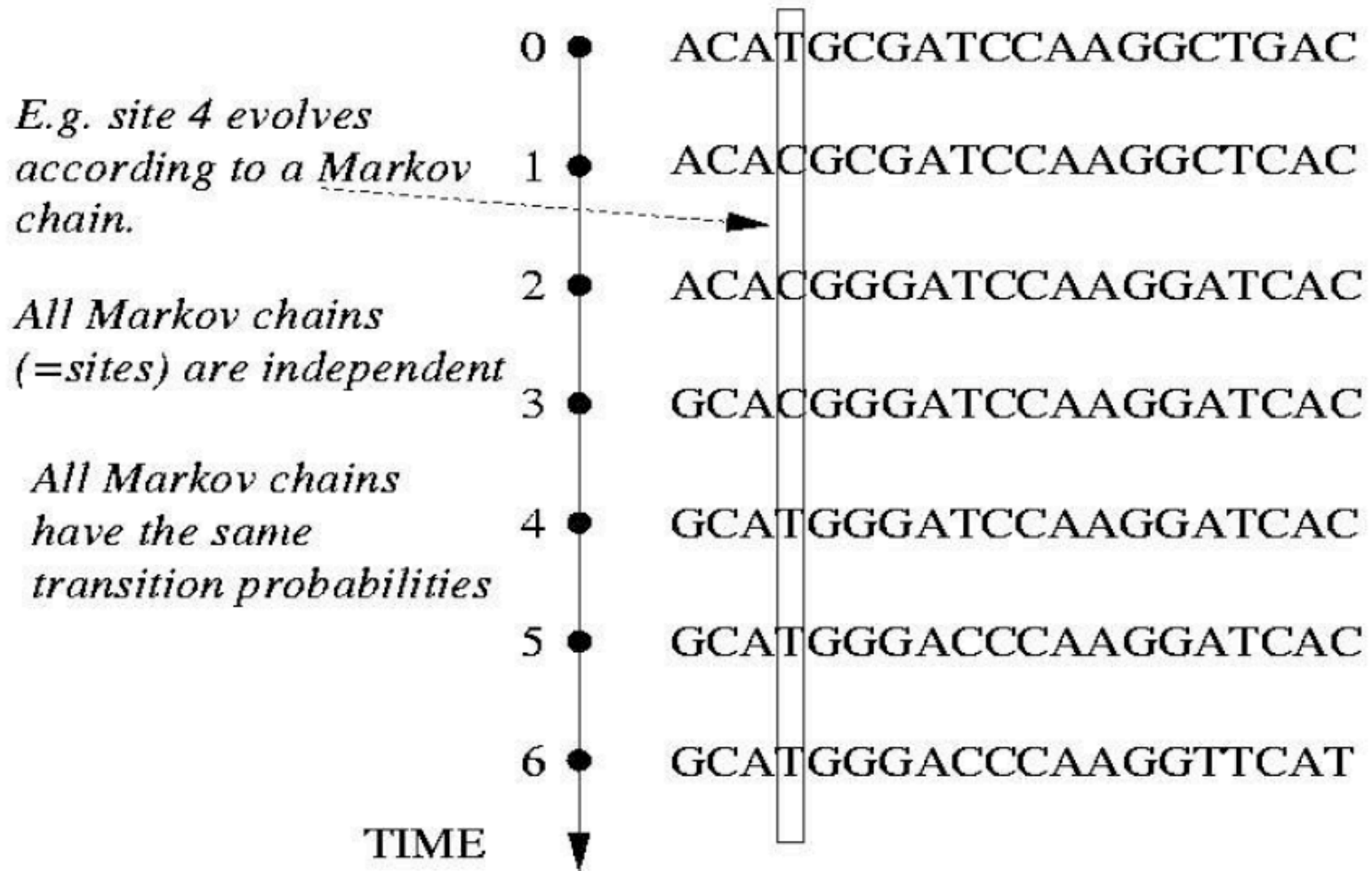


TIME



Evolutionarily correct alignment: **CHAEFAP**
C-AE-TP

Models for sequence evolution (DNA): Each site of the DNA sequence evolves according to a Markov Chain with state space $\{A,C,G,T\}$.



MARKOV CHAIN

Let $X_0, X_1, X_2, X_3, \dots$ be a Markov chain with **state space** S , for example $S = \{a, c, g, t\}$.

TRANSITION MATRIX

$$P = \begin{pmatrix} p_{a,a} & p_{a,c} & p_{a,g} & p_{a,t} \\ p_{c,a} & p_{c,c} & p_{c,g} & p_{c,t} \\ p_{g,a} & p_{g,c} & p_{g,g} & p_{g,t} \\ p_{t,a} & p_{t,c} & p_{t,g} & p_{t,t} \end{pmatrix}.$$

Here

$$p_{i,j} = \mathbf{P}(X_{n+1} = j | X_n = i)$$

for $n \geq 0$, where $i, j \in \{a, c, g, t\}$.

Simplest model for sequence evolution: Jukes-Cantor

$$\begin{pmatrix} p_{a,a} & p_{a,c} & p_{a,g} & p_{a,t} \\ p_{c,a} & p_{c,c} & p_{c,g} & p_{c,t} \\ p_{g,a} & p_{g,c} & p_{g,g} & p_{g,t} \\ p_{t,a} & p_{t,c} & p_{t,g} & p_{t,t} \end{pmatrix} = \begin{pmatrix} 1 - 3\alpha & \alpha & \alpha & \alpha \\ \alpha & 1 - 3\alpha & \alpha & \alpha \\ \alpha & \alpha & 1 - 3\alpha & \alpha \\ \alpha & \alpha & \alpha & 1 - 3\alpha \end{pmatrix}$$

The stationary distribution is $\vec{\pi} = (0.25, 0.25, 0.25, 0.25)$.

The parameter α depends on the time scale

(if the unit time is 100.000 generations, α would take a smaller value than if the unit time were chosen as 200.000 generations).

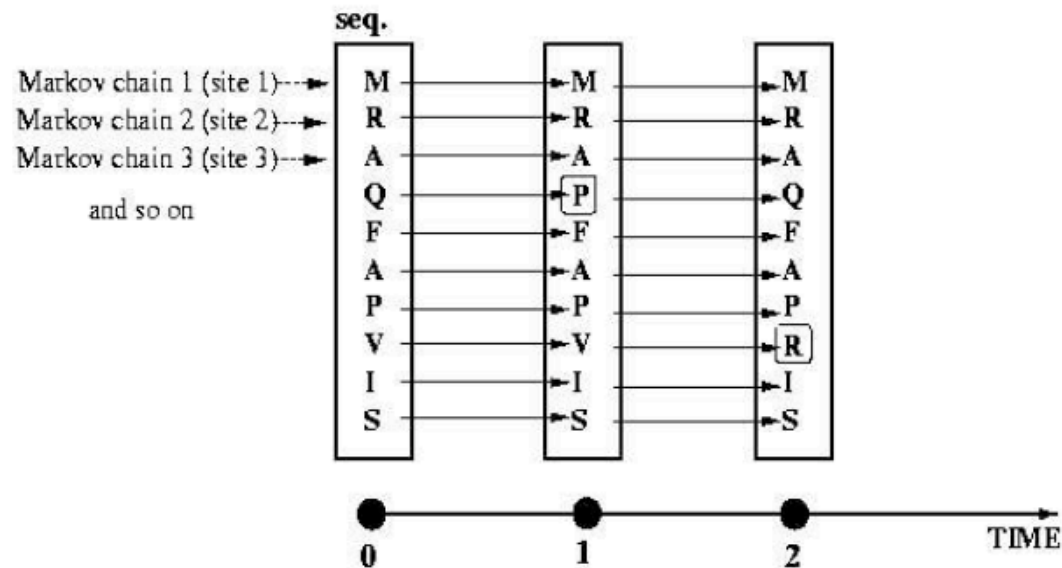
Necessary: $\alpha < 1/3$.

The n -step transition probabilities can be computed:

$$\mathbf{P}(X_n = i | X_0 = i) = 0.25 + 0.75 \cdot (1 - 4\alpha)^n, \text{ for } i \in \{a, c, g, t\}.$$

$$\mathbf{P}(X_n = j | X_0 = i) = 0.25 - 0.25 \cdot (1 - 4\alpha)^n, \text{ for } i, j \in \{a, c, g, t\}, i \neq j.$$

Underlying model: Each site in the sequence evolves *according to a Markov chain*, and *independently* of the other sites.



All the Markov chains have the *same* transition matrix P (matrix with dimension 20×20).

FROM TRANSITION MATRIX TO ALIGNMENT SCORES

Two hypothesis:

1. Sequences S1 and S2 are unrelated (=random matching)
2. Sequences S1 and S2 have a common ancestor.

$$\text{Score} = \text{Log} (P1/P2)$$

P1 - probability of observed alignment given model 1

P2 - probability of observed alignment given model 2

Dayhoff et al. (1978) used ungapped multiple alignments of certain well-conserved regions from closely related proteins.
(71 groups of proteins, all in all 1572 changes.)

AAEE	AATG...G	C	E
CAPP	AATH...G	T	E
PPAV	ASTH...G	C	G
VVIG	AAAH...G	A	I

>85%



Dr. Margaret Oakley Dayhoff (1925-1983)

The most parsimonious tree

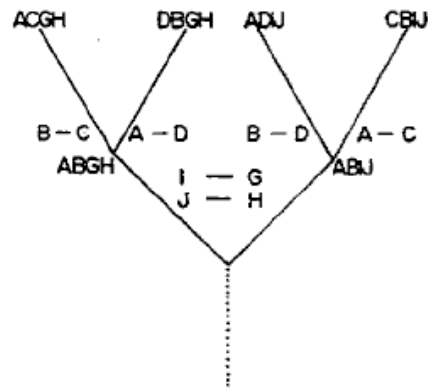
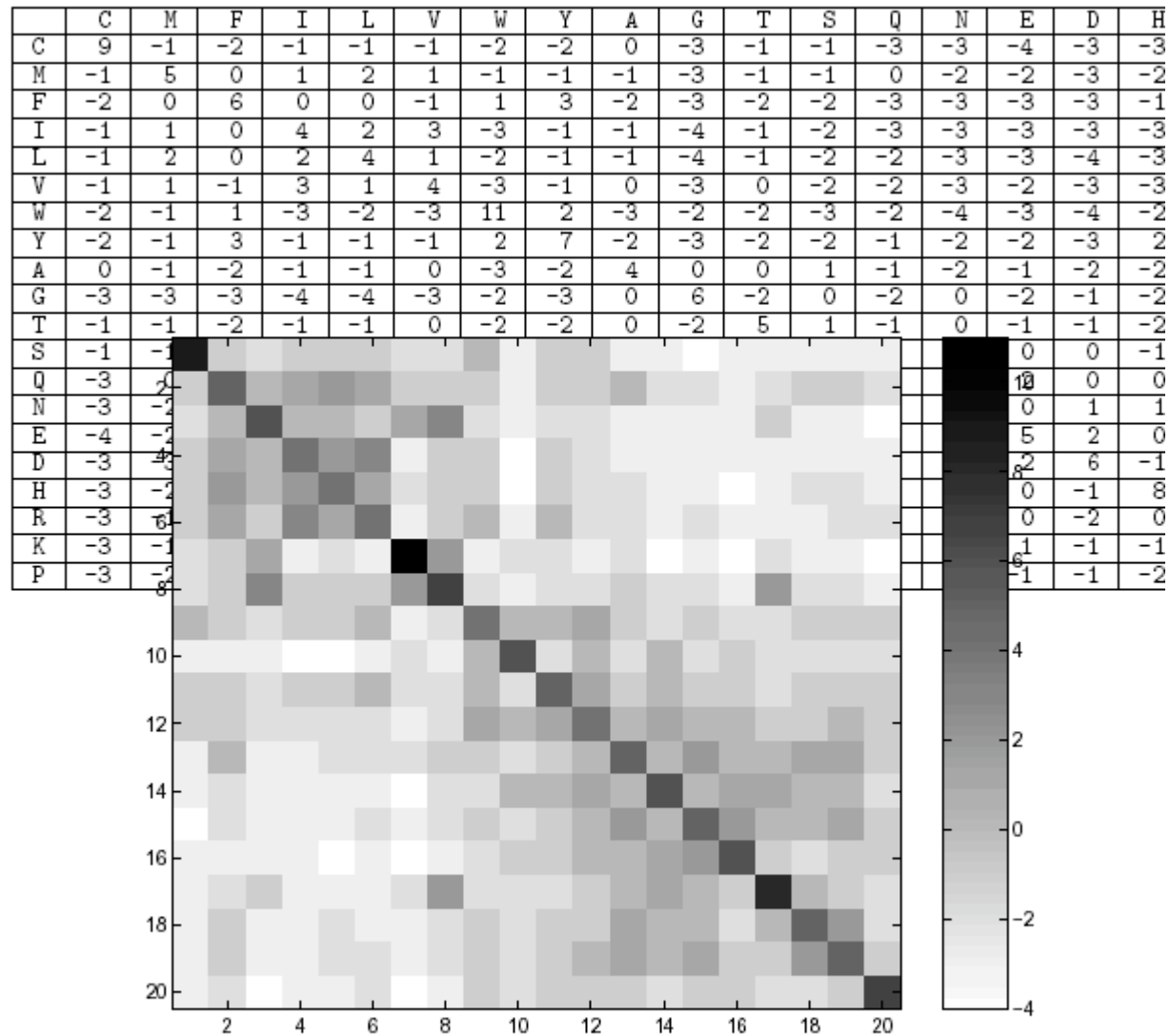


Figure 78. Simplified phylogenetic tree. Four "observed" proteins are shown at the top. Inferred ancestors are shown at the nodes. Amino acid exchanges are indicated along the branches.

	A	B	C	D	G	H	I	J
A			1	1				
B			1	1				
C	1	1						
D	1	1						
G							1	
H								1
I					1			
J						1		

Figure 79. Matrix of accepted point mutations derived from the tree of Figure 78.

Realistic Scoring



Database searches

Problem:

Alignment of a gene 1000bp
against the Human genome $3 \cdot 10^9$ bp
... 10^{12} operations...

- protein against a database of 10^5 proteins
-> 10^9 operations
- genome against genome
-> 10^{14} operations

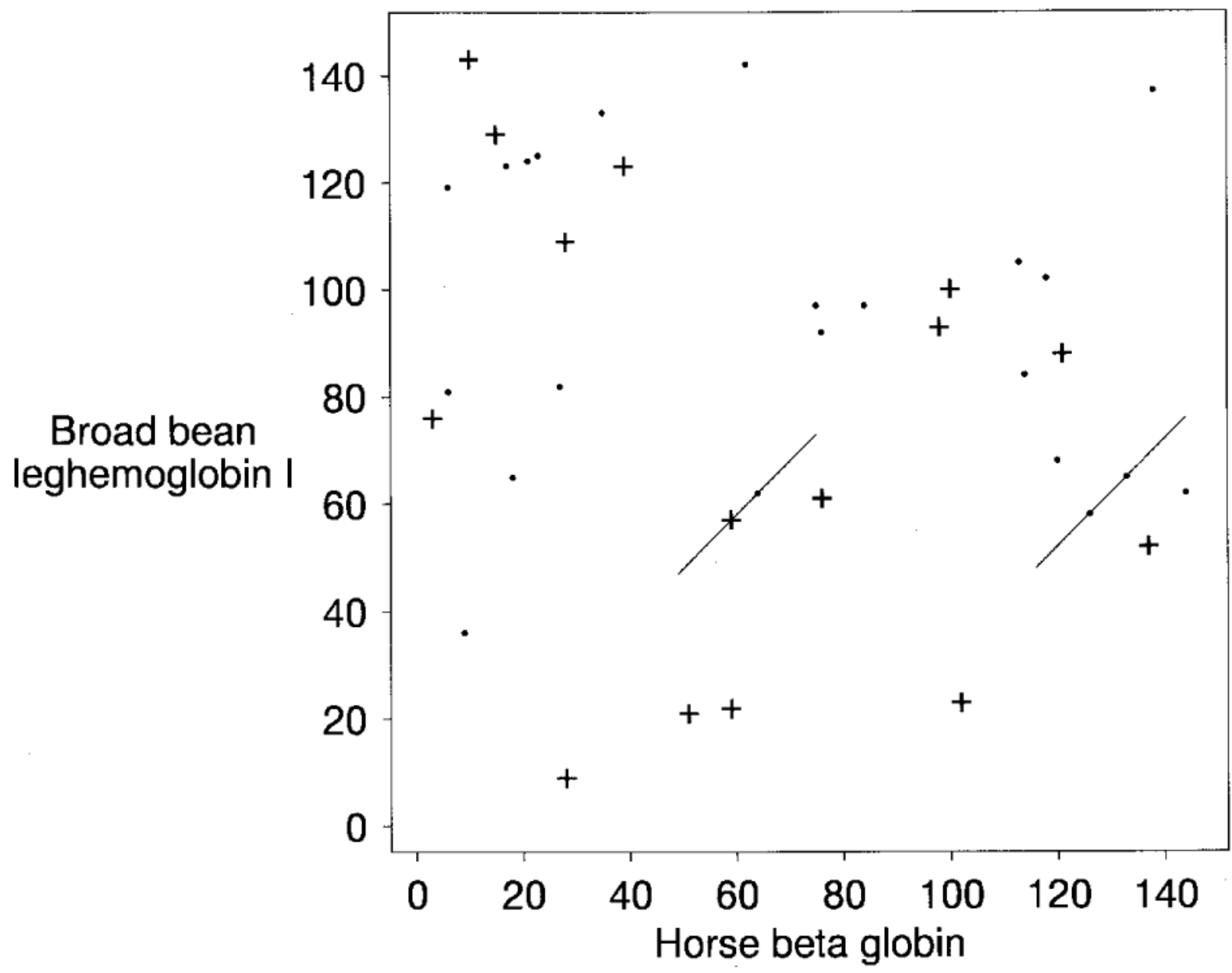
NEED **FASTER** ALGORITHMS

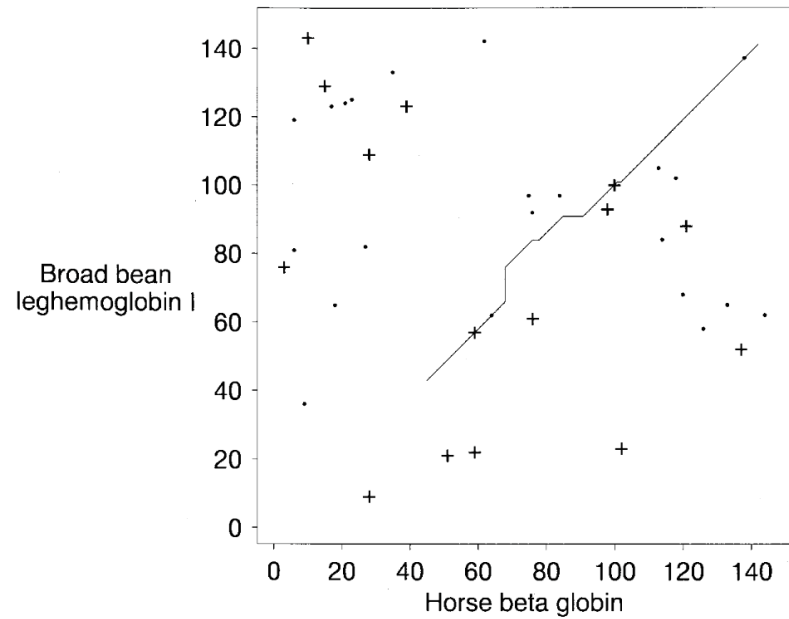
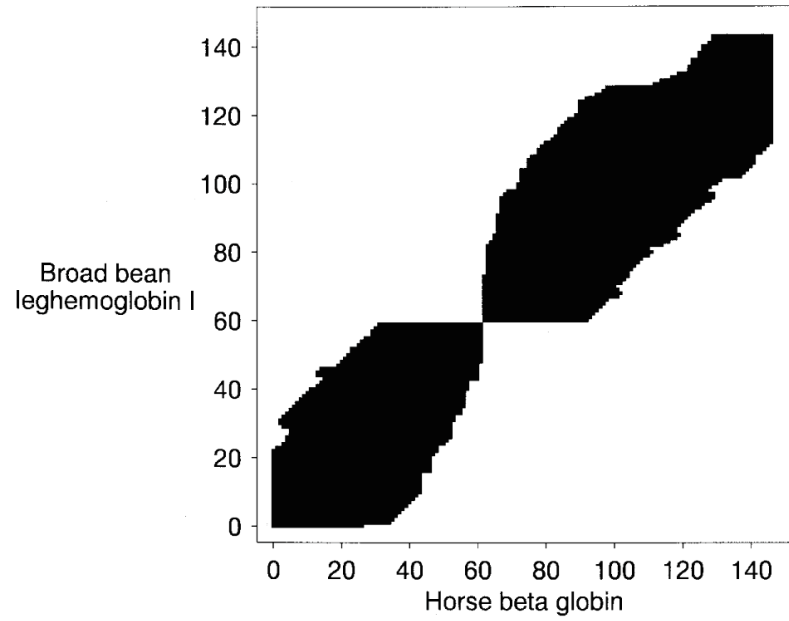
Database searches

BLAST

- Pre-processing: Low Complexity Regions (LCRs)
- Scanning for common words (hits)
- Two-hit heuristic
- HSP (high-scoring segment pair) $> S$
- Constrained gaped extension
- E-value

WARNING: The final alignment is not very good!





```

Leghemoglobin 43 FSFLKDSAGVVDSPKLGHAHAEKVFGMVRDSAVQLRATGEVV--LDGKDGS----- 90
                F L + V+ +PK+ AH +KV                L + GE V LD G+
Beta globin   45 FGDLSNPGAVMGNPKVKAHGKKV-----LHSPGEGVHHLNLKGTFAALSE 90

```

```

Leghemoglobin 91 IHIQKGLDP-HFVVVKEALLKTIKEASGDKWSEELSAAWEVAYDGLATAI 140
                +H K +DP +F ++ L+ + G ++ EL A+++ G+A A+
Beta globin   91 LHCDKLHVDPENFRLLGNVLVVVLLARHFGKDFTPELQASYQKVVAGVANAL 141

```