An Underlying Link for Scaling of Fluctuations in

Growth Fronts, Fracture Lines, Strong Localization, ...

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⋮

What do the following topics have in common?

Growing Interfaces

Shock Waves

Traffic Flow

Optimized Paths

Vortex Lines

Insulator Resistivity

Fracture Lines

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Answer # 1. Nothing at all.

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Answer # 2. They all exhibit Fluctuations

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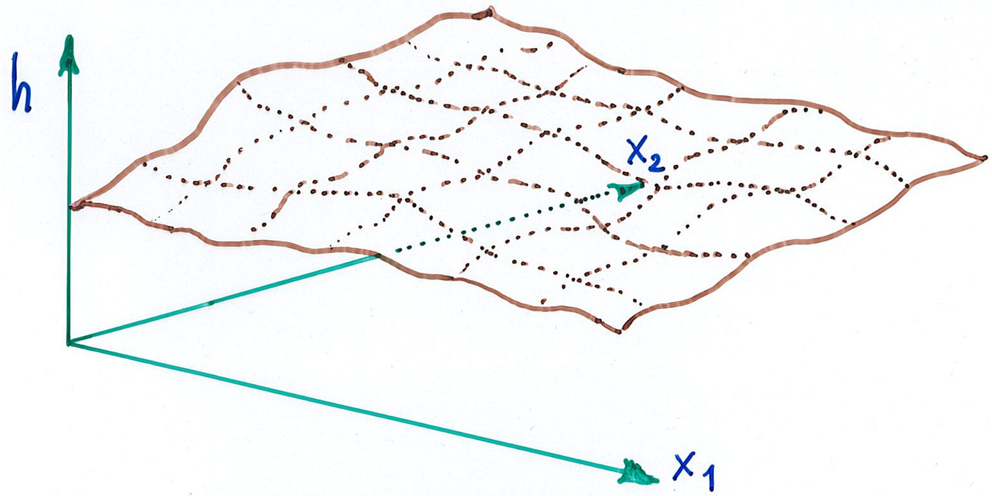
Answer # 1. Nothing at all .

Answer # 2. They all exhibit Fluctuations

governed by the same Generalized Diffusion Equation.

(Equilibrium) Fluctuations of a Surface



- Energy: $H = \int d^{d'}x \, \sigma \left\{ \sqrt{1+(\nabla h)^2} - 1 \right\} \approx \frac{\sigma}{2} \int d^{d'}x \, (\nabla h)^2$

- Equation of Motion:

$$\underset{\text{(Inertia)}}{\cancel{\rho \frac{\partial^2 h}{\partial t^2}}} + \underset{}{\overset{\text{Friction}}{\mu \frac{\partial h}{\partial t}}} = \overset{\text{Force}}{- \frac{\delta H}{\delta h}} + \overset{\text{Thermal Noise}}{f(x,t)}$$

$$\Rightarrow \quad \boxed{\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta(x,t)} \qquad \langle \eta(x,t)\eta(x',t')\rangle = D\, \delta(x-x')\, \delta(t-t')$$

- Fluctuations:

$$\langle | h(\underset{\sim}{x},t) - h(x',t') | \rangle \sim |x-x'|^{\chi} f\left(\frac{|t-t'|}{|x-x'|^{z}} \right)$$

$$\boxed{\begin{aligned} \chi_0 &= \frac{3-d}{2} \\ z_0 &= 2 \end{aligned}}$$

Roughness Exponent

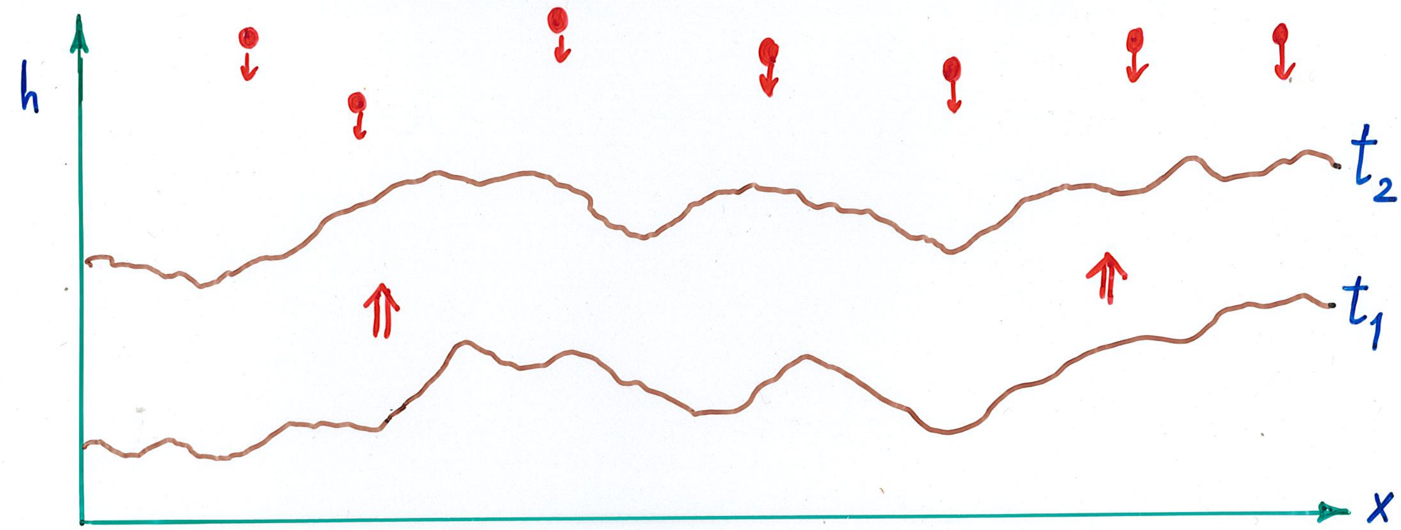
Dynamic Exponent

- $d=3$, $\quad \delta h \sim (\ln L)^{1/2}$ $\qquad d=2 \quad \delta h \sim L^{1/2}$

(Non-Equilibrium) Fluctuations of a Growing Surface

- Growth by deposition of particles (M.B.E.)
- Eden model for growing tumor, ...

$\Rightarrow$ Compact cluster with a rough surface.



- Equation of Motion : (M.K., G. Parisi, Y.-C. Zhang, 1986)

$$\boxed{\frac{\partial h}{\partial t} = v + \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x,t)}$$

v: Average Growth velocity

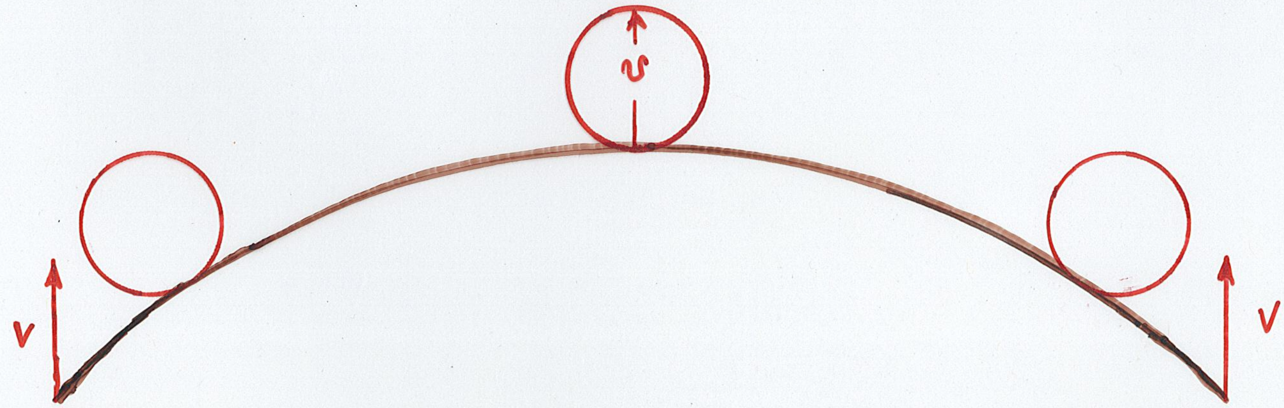
ν: Smoothening by Evaporation

$\lambda \propto v$: Growth normal to the surface

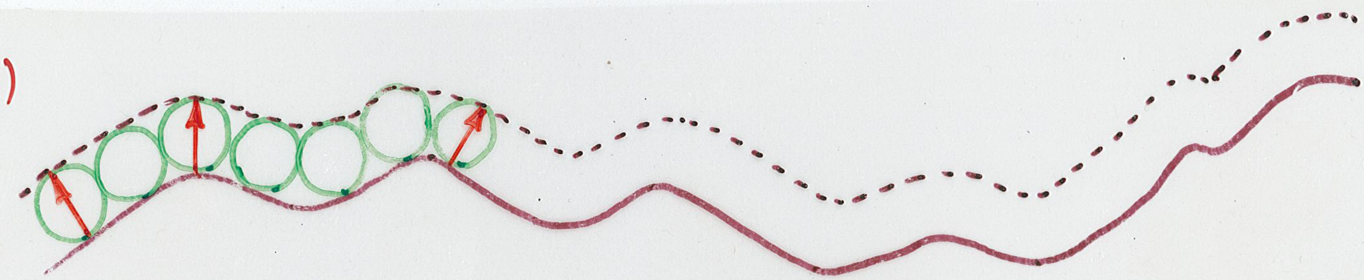
$\langle \eta(x,t)\, \eta(x',t') \rangle = D\,\delta(x-x')\,\delta(t-t')$: Noise of growth events (beam

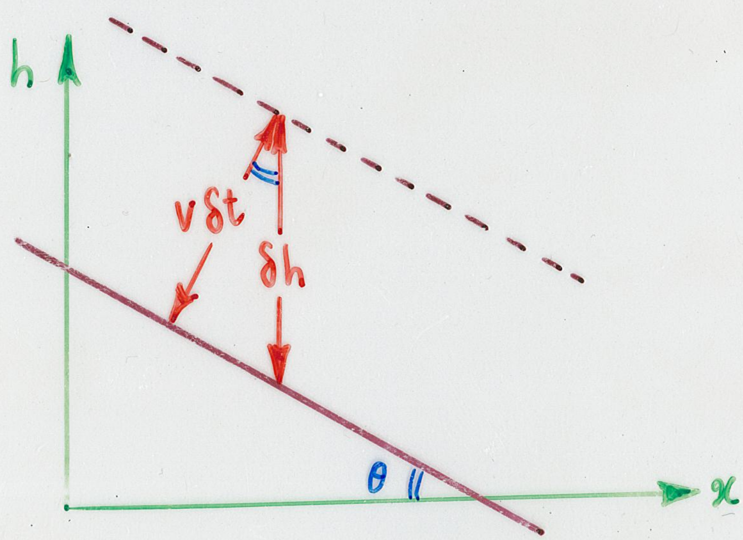
- Geometrical origin of the nonlinear term $\frac{\lambda}{2}(\nabla h)^2$

(a)



(b)





$$\delta h = \frac{v\,\delta t}{\cos\theta} = v\,\delta t\sqrt{1+(\nabla h)^2} \approx v\,\delta t\left(1+\frac{(\nabla h)^2}{2}+\cdots\right)$$

$$\dot{h} \sim \frac{\delta h}{\delta t} \sim v + \frac{v}{2}(\nabla h)^2 + \cdots$$

Related Equations

1. Growing Interface:

$$\boxed{\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t)}$$

2. Burgers' Equation for Incompressible Fluid Flow:

let $\vec{v} = -\nabla h$ such that $\nabla \times \vec{v} = 0$

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \lambda\, \vec{v}\cdot\nabla\vec{v} = \nu \nabla^2 \vec{v} - \nabla \eta(x,t)}$$

3. Diffusion with random sources and sinks:

Cole-Hopf transformation $W(x,t) = \exp\left[\frac{\lambda}{2\nu} h(x,t)\right]$

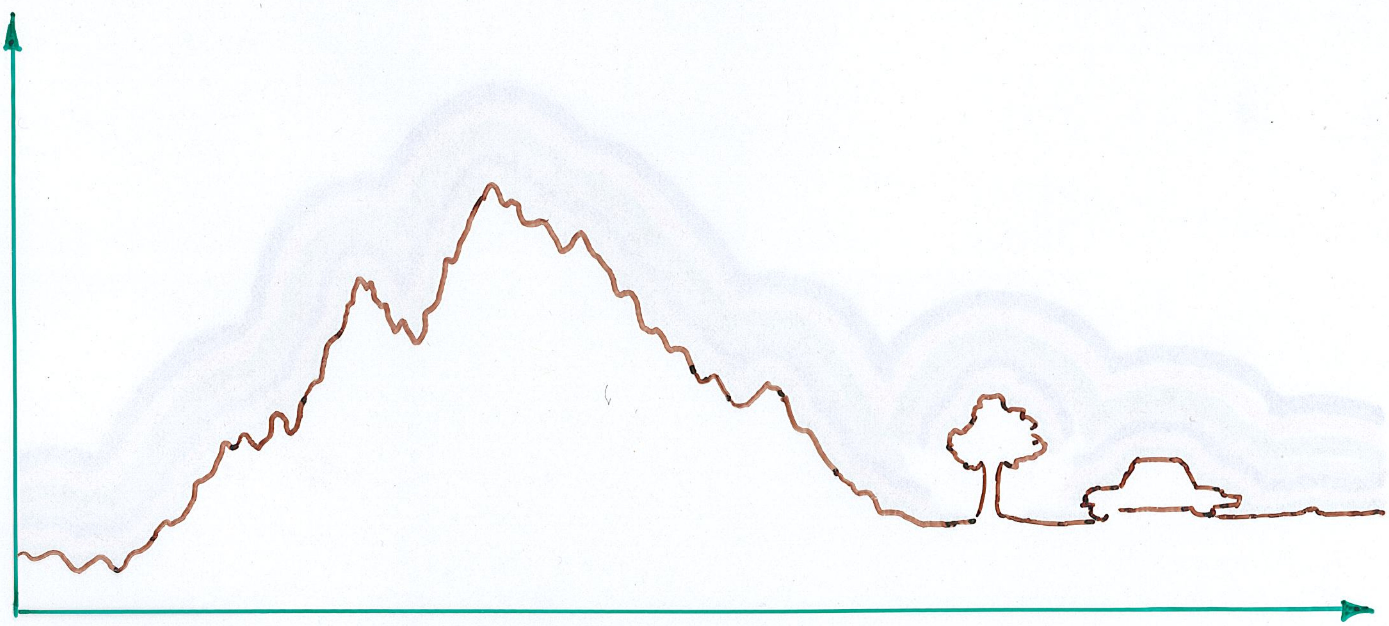
$$\boxed{\frac{\partial W}{\partial t} = \nu \nabla^2 W + \left(\frac{\lambda}{2\nu}\eta(x,t)\right) W}$$

4. Sums over paths in random media:

$$\boxed{W(x,t) = \int_{(0,0)}^{(x,t)} \mathcal{D}x(\tau)\, \exp\left\{-\int_0^t d\tau \left[\frac{1}{4\nu}\left(\frac{dx}{d\tau}\right)^2 - \frac{\lambda}{2\nu}\eta(x(\tau),\tau)\right]\right\}}$$

Deterministic Growth Patterns

- Consider a "uniform Snowfall" on a Random Landscape.



- For $\eta(x,t) = 0$, use the mapping $\quad W = e^{\frac{\lambda}{2\nu} h}$

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 \quad \Leftrightarrow \quad \frac{\partial W}{\partial t} = \nu \nabla^2 W$$

- Initially: $\quad h(x, t=0) = h_0(x) \quad \Leftrightarrow \quad W(x,0) = e^{\frac{\lambda}{2\nu} h_0(x)}$

$\therefore$
$$\boxed{h(x,t) = \frac{2\nu}{\lambda} \ln \left\{ \int d^d x' \exp\left[- \frac{(x-x')^2}{2\nu t} + \frac{\lambda}{2\nu} h_0(x') \right] \right\}}$$

- $\nu \to 0$ limit is easily obtained by maximum of integrand as

$$\lim_{\nu \to 0} h(x,t) = \max \left[h_0(x') - \frac{(x-x')^2}{2\lambda t} \right]_{x'}$$

i.e. $\quad h(x,t) \longrightarrow$ Collection of Paraboloids

$\quad v = -\nabla h \longrightarrow$ Shock Waves

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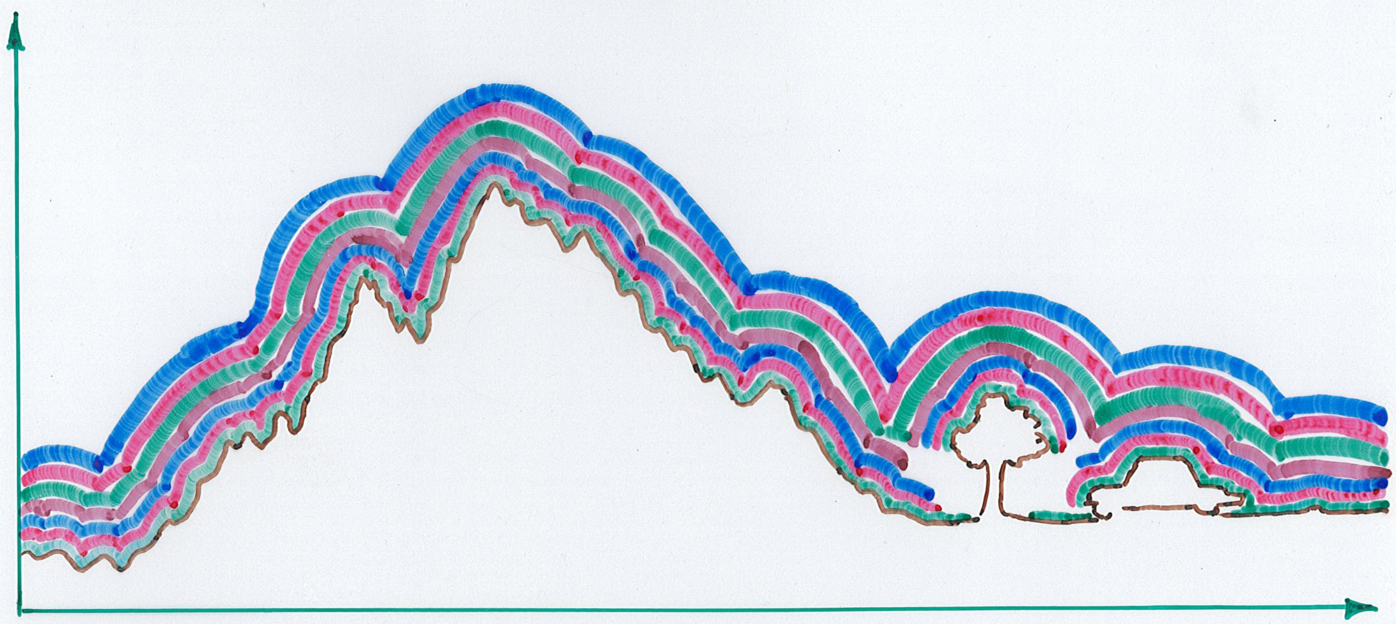

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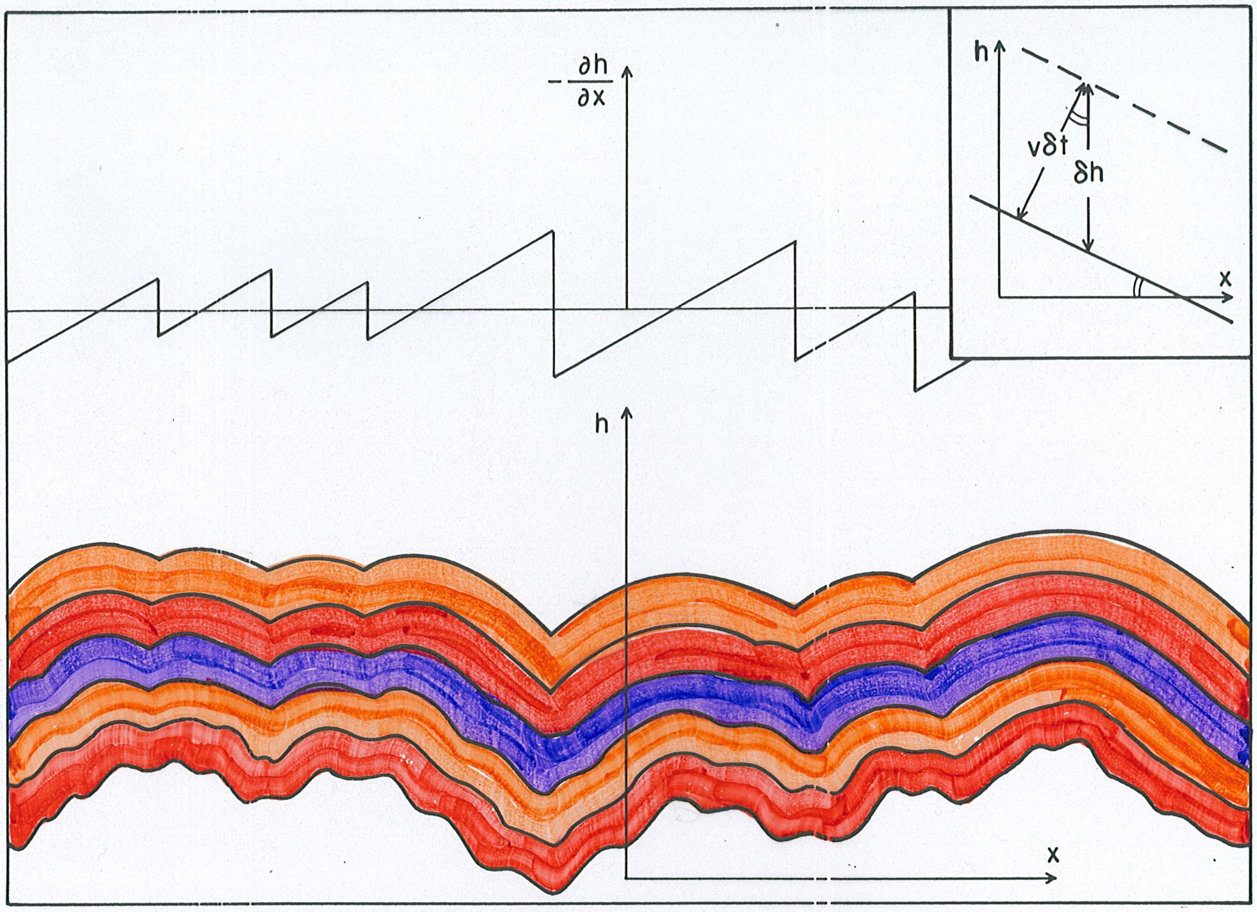
Scaling of Paraboloids



- If the initial profile is a random walk: $\overline{|h_0(x) - h_0(x')|} \sim |x - x'|^{1/2}$

$$h(x,t) = \max\left[h_0(x') - \frac{(x-x')^2}{2\lambda t} \right]_{x'}$$

$$\delta x^{1/2} \sim \frac{(\delta x)^2}{t} \quad \Rightarrow \quad \delta x \sim t^{2/3}$$

- In general if: $\quad |h_0(x) - h_0(x')| \sim |x - x'|^{\chi}$

$$\delta x \sim t^{\frac{1}{z}} \sim t^{\frac{1}{2-\chi}}$$

$$\therefore \quad |h(x,t) - h(x',t')| \sim |x - x'|^{\chi} f\left(\frac{|t-t'|}{|x-x'|^{z}} \right)$$

with $\boxed{\chi + z = 2}$

Stochastic Growth Fluctuations

- Noisy equation : $$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x,t)$$

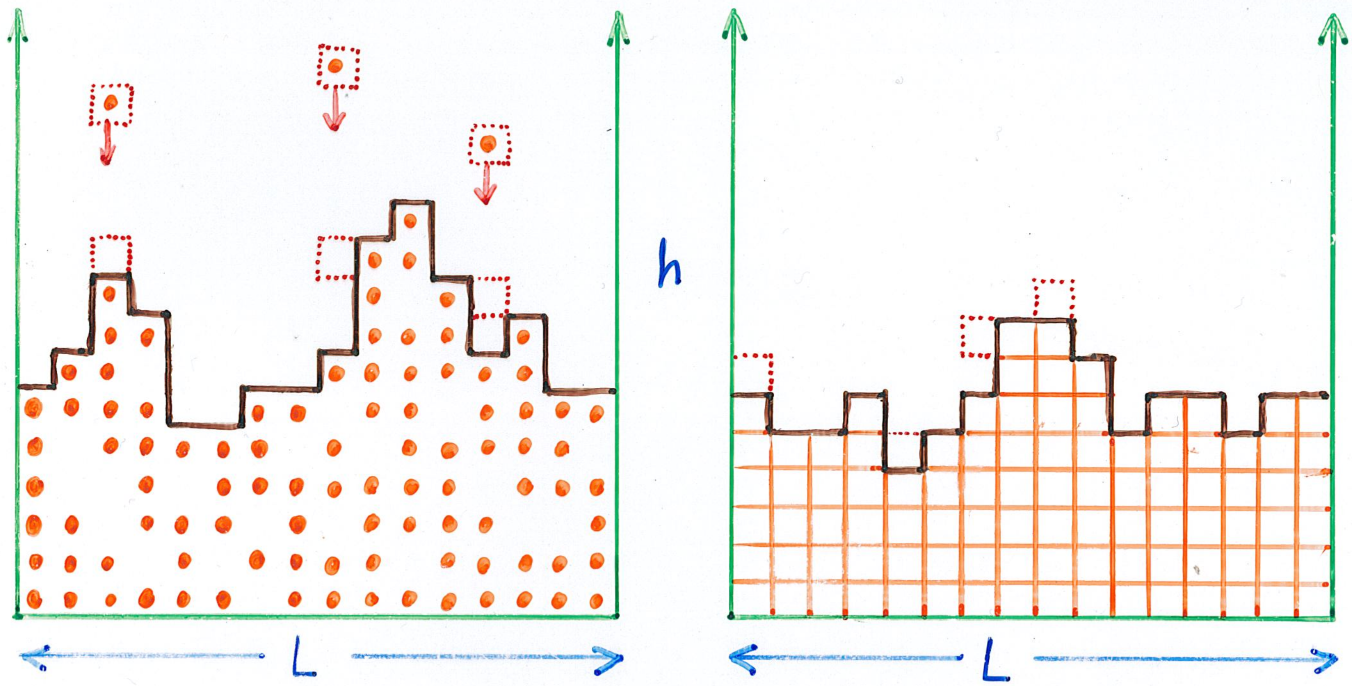
- Fluctuations : $$\langle |h(x,t) - h(x',t')| \rangle \sim |x - x'|^{\chi} f\left(\frac{|t-t'|}{|x-x'|^{z}}\right) \qquad \chi, z = ?$$

	$\lambda = 0$		$\lambda \neq 0$
$d = 2$	$\begin{cases} \chi_0 = 1/2 \\ z_0 = 2 \end{cases}$		$\begin{cases} \chi = 0.5 \\ z = 1.5 \end{cases}$
$d = 3$	$\begin{cases} \chi_0 = 0 \text{ "ln"} \\ z_0 = 2 \end{cases}$		$\begin{cases} \chi \simeq 0.4 \\ z \simeq 1.6 \end{cases}$
$d = 4$	$\begin{cases} \chi_0 = -1/2 \\ z_0 = 2 \end{cases}$	λ^*	$\begin{cases} \chi \simeq 0.3 \\ z \simeq 1.7 \end{cases}$

- Results from a Dynamic Renormalization Group Approach
  (Forster, Nelson, Stephen, 1977)

- For $d \geqslant 3$ only an exponent identity $\chi + z = 2$

Numerical Simulations of Growing Surfaces



BALLISTIC DEPOSITION EDEN MODEL

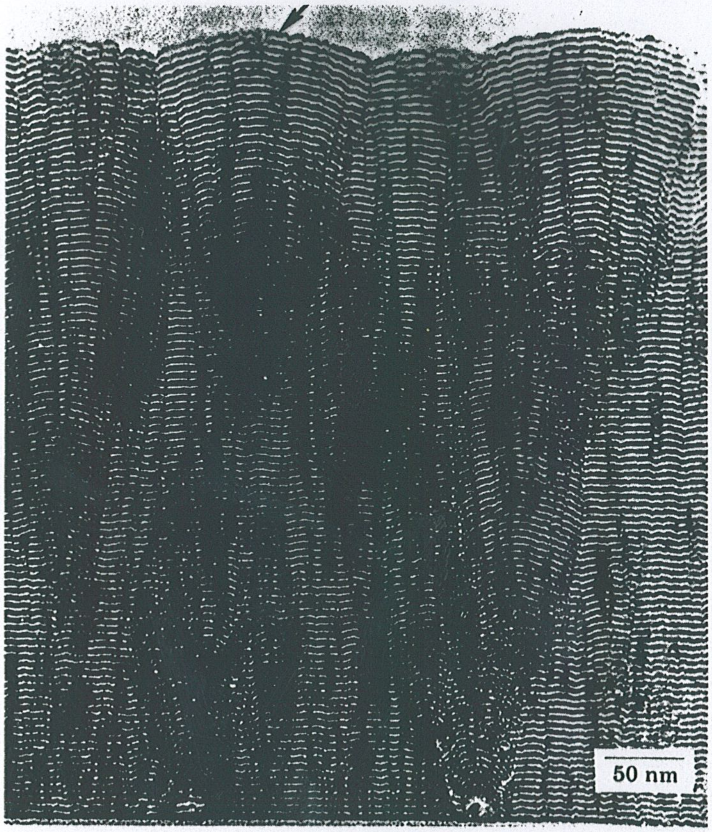
- Numerous Simulations over the past few years have confirmed the scaling picture from Renormalization Group.
- Numerical results in dimensions $d \geqslant 3$ have led to the conjecture : $\begin{cases} \chi \approx \dfrac{2}{d+2} \\ z \approx \dfrac{2d+2}{d+2} \end{cases}$ (Kim, Kosterlitz, 1989)
- Interface width :

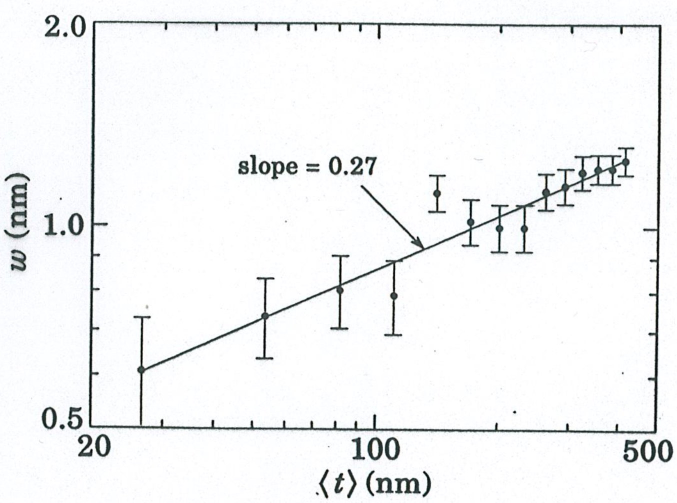
$$W = \left(\langle (\delta h)^2 \rangle \right)^{1/2} = L^{\chi} f\left(\frac{t}{L^z}\right) \sim \begin{cases} t^{\chi/z \equiv \beta} & t \ll L^z \\ L^{\chi} & t \gg L^z \end{cases}$$

Experimental Results $(d = 3)$

- Multilayer decoration of Sputtered NbN/AlN films

D.J. Miller, K.E. Gray, R.T. Kampwirth, + J.M. Murduck, Europhys. Lett. 19, 27 (1992).



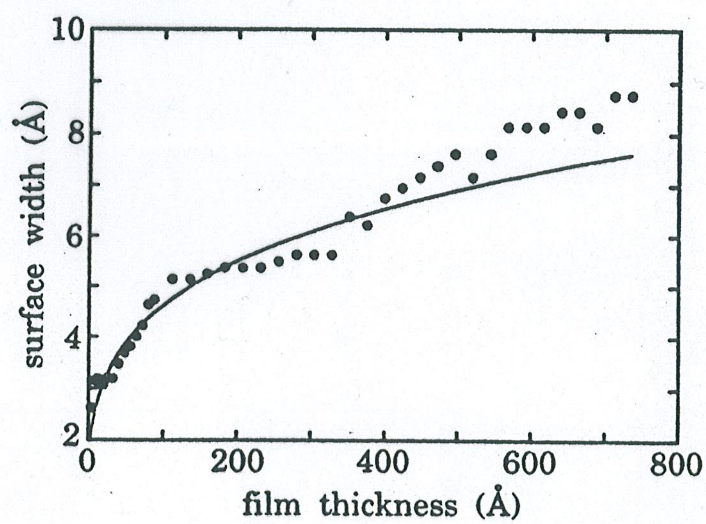


$$\frac{\chi}{z} \approx 0.27$$

- RHEED study of Fe on Si grown by MBE

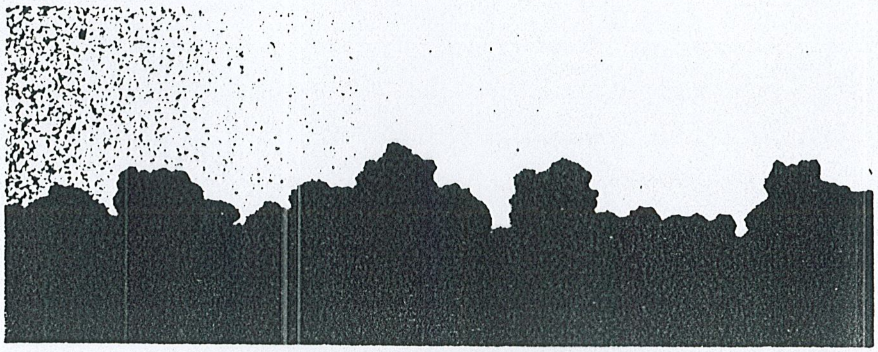
J. Chevrier, V. Le Thanh, R. Buys, + J. Derrien, Europhys. Lett. 16, 737 (1991).

$$\frac{\chi}{z} \approx 0.25$$



Some Experimental Results ($d = 2$)

(1) Growth of Bacterial Colonies (E Coli)



(T. Vicsek, M. Cserző, V.K. Horváth, 1990)

$\boxed{\chi = 0.78 \pm 0.07}$

(2) Viscous Flow in Porous Media

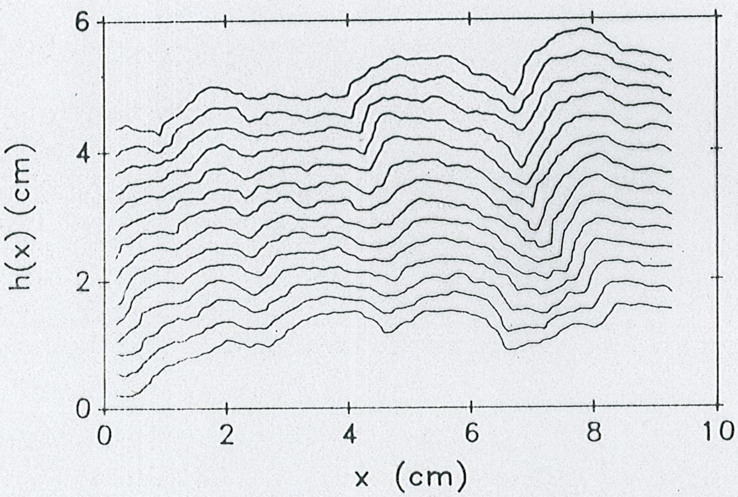


FIG. 1. Plot of fourteen successive interfaces at Ca = 4.93 $\times 10^{-3}$. The time interval between single interfaces is 30 s.

(M.A. Rubio, C.A. Edwards, A. Dougherty, J.P. Gollub, 1990).

$\boxed{\chi = 0.73 \pm 0.03}$

* c.f. theory / simulations $\chi = 1/2$.

Physica A 189 (1992) 383–389
North-Holland

PHYSICA A

Modeling forest fire by a paper-burning experiment, a realization of the interface growth mechanism

Jun Zhang[a], Y.-C. Zhang[b], P. Alstrøm[a] and M.T. Levinsen[a]

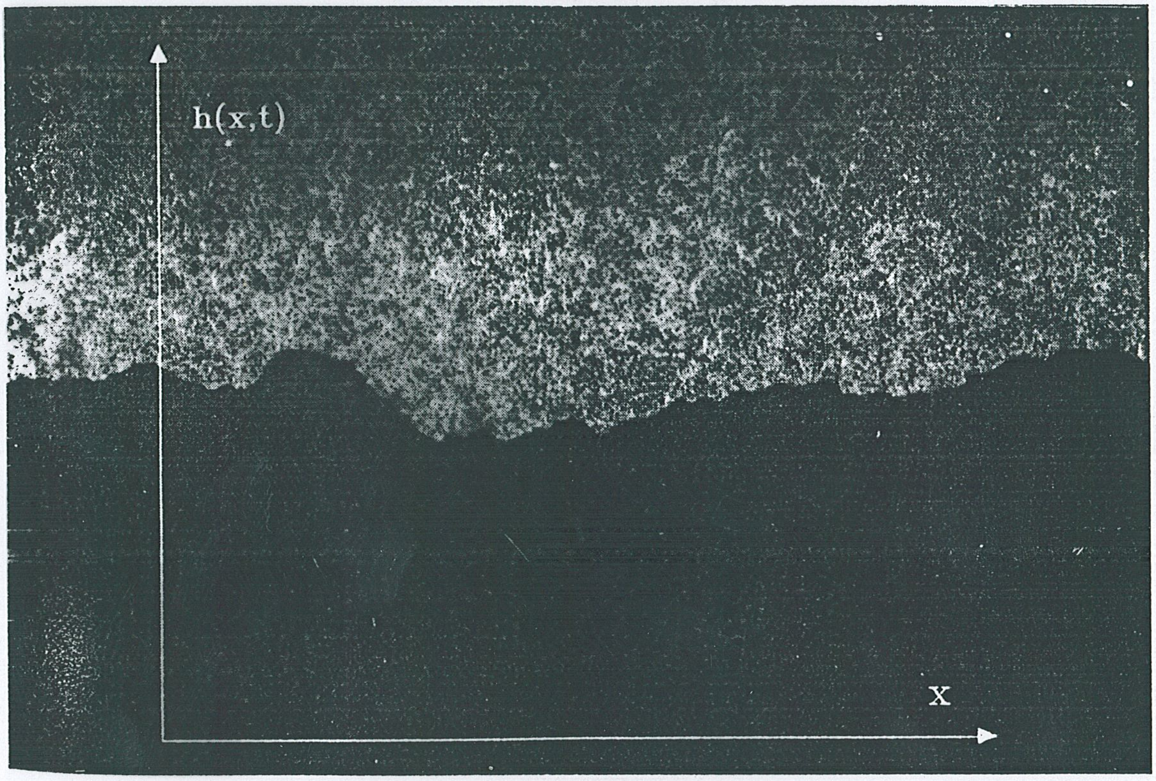


Fig. 1. This photo shows a segment of a piece of burning paper. The transverse size is about 8.5 cm, which is only a small part of the total (~46 cm). Fire is propagating upwards, the smoke indicates slow and laminar air circulation. The actual fire front is a few mm wide, but by using a strong background light the sharp fire line can be identified.

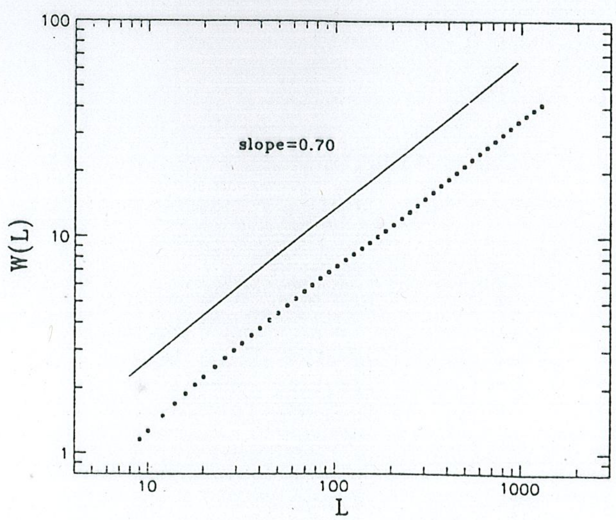


Fig. 2. The mean fire-line width $W(L)$ plotted versus the transverse length L. The data are from a single instantaneous fire line including the segment shown in fig. 1. The average is over different segments by shifting their starting point on the fire line. The data appear to follow a straight line of slope 0.70, over about two decades.

Fractals 1, 67 (93)

SELF-AFFINE RUPTURE LINES IN PAPER SHEETS

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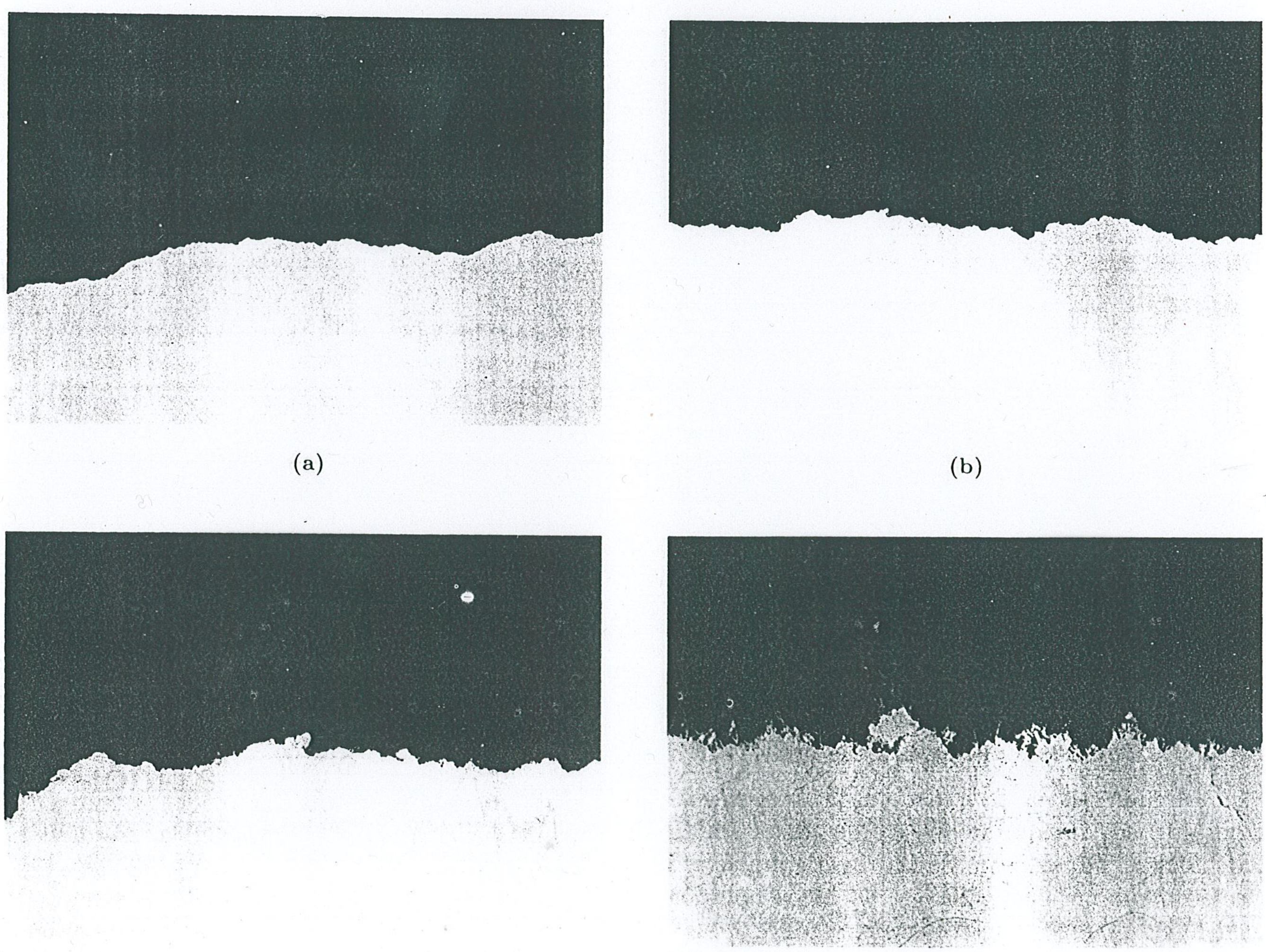
Abstract

Experiments were carried out using a tensile testing machine to study the morphology of rupture lines in paper. Constant velocity strain was applied until the breaking process separated the sample into two pieces. The resulting quasi one-dimensional patterns show scaling correlations usually over two orders of magnitudes. The measured values of the Hurst exponent ζ are in the range $0.63 < \zeta < 0.72$. The relevance of models of statistical physics in the interpretation of the results is discussed.



(a) (b) (c) (d)

Figs. 1.1(a)–(d) and Figs. 1.2(a)–(d) Series of photographs of rupture lines in two kinds of paper sheets. The magnifications from (a) to (d) are: 1:3:9:27; the width of the samples in **1.1a** and **1.2a** is 280 mm.

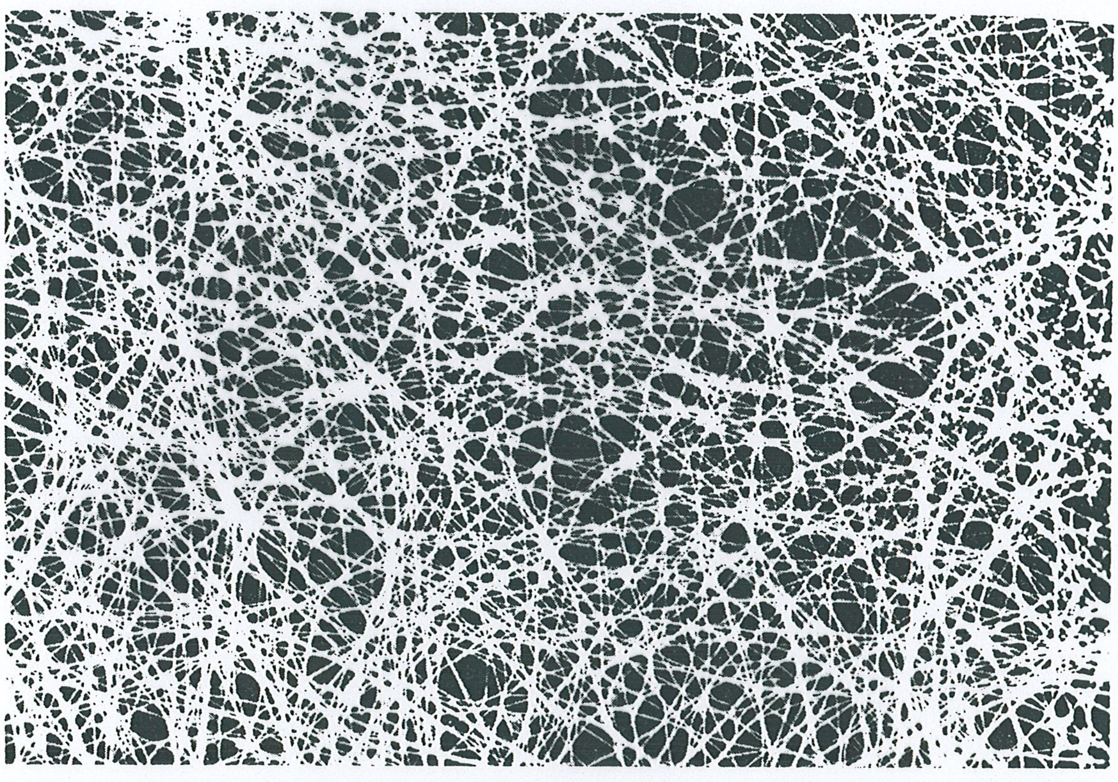
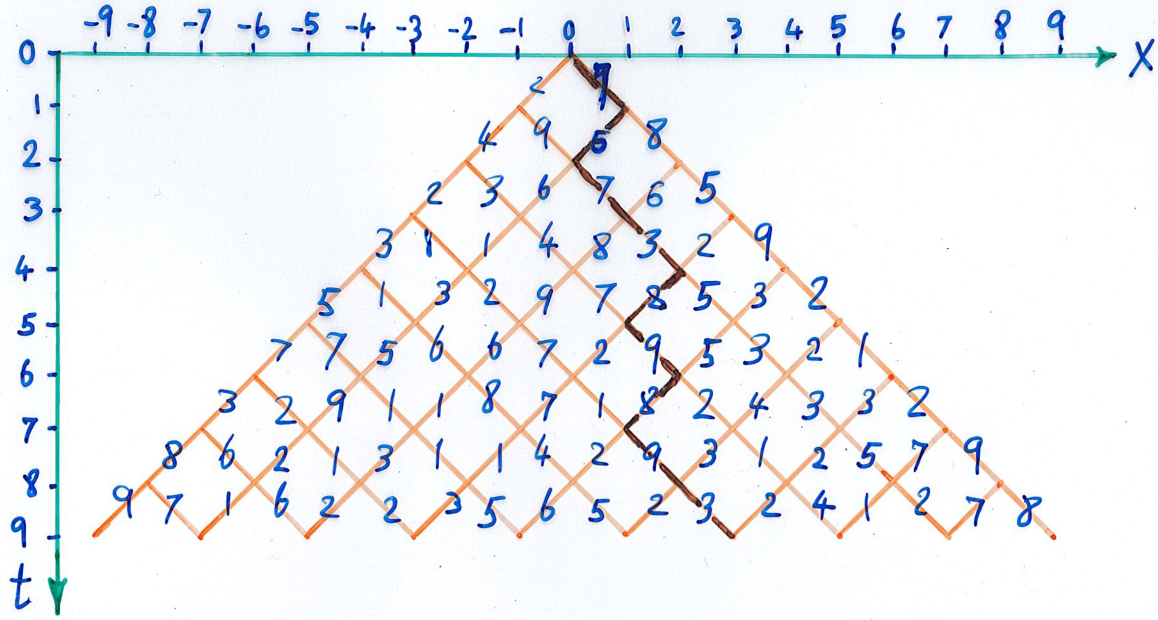


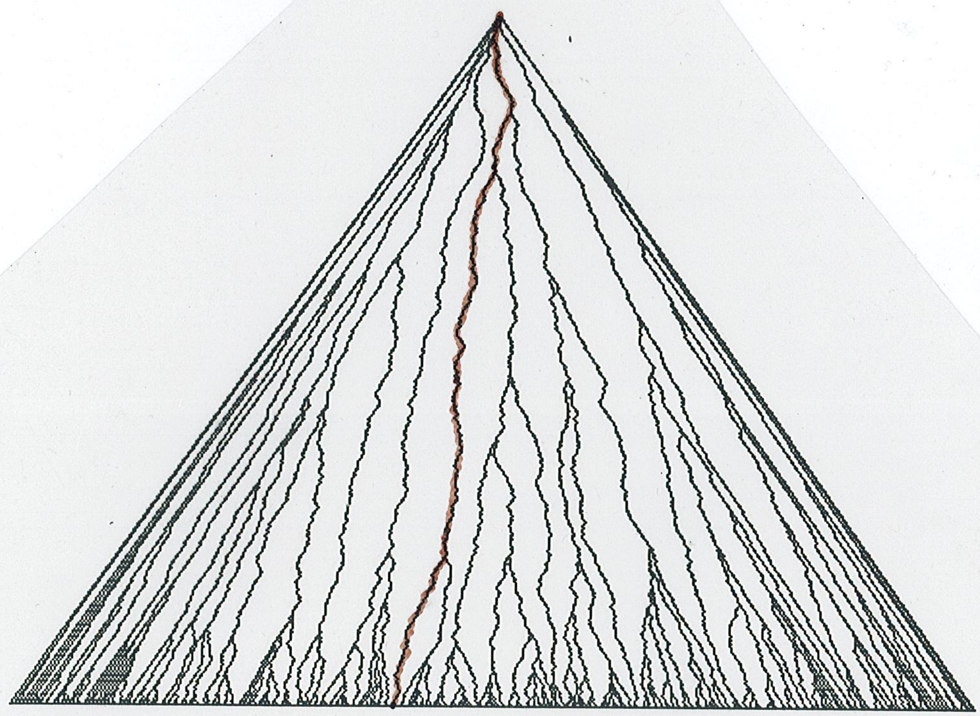
Fig. 3. The apparent uniform paper reveals much of its inhomogeneity at microscopic scales, in this photo at magnification 20. The intrinsic fibre network and uneven KNO_3 concentration on it all contribute to the noise affecting the fire propagation.

Optimal Paths on a Random Energy Landscape



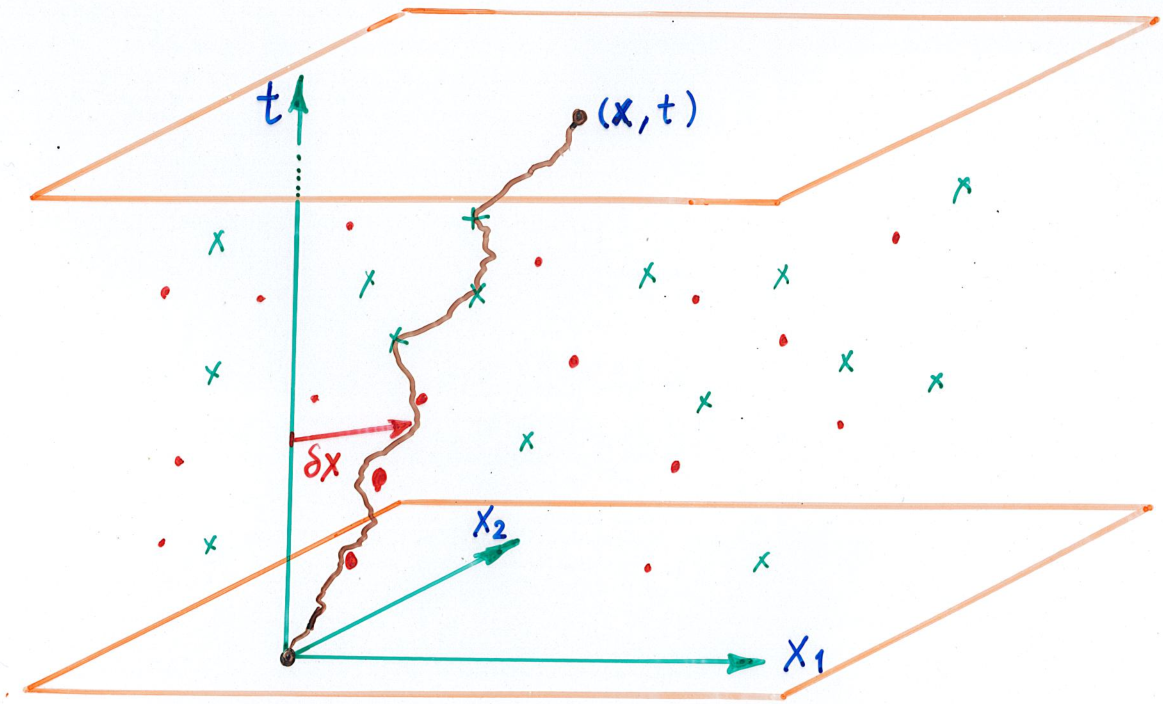
- $E_\Gamma = \sum_{i=1}^{t} V(x_i, i)$

- $t = 500$



- $\overline{|x_{opt}(t)|} \sim t^{2/3}$ v.s. $\sqrt{\langle x^2 \rangle} \sim t^{1/2}$ for Random Walk

Vortex Lines, Finite Temperatures



- Partition Function, at finite temperature $kT = 1/\beta$

$$W(\mathbf{x},t) = \sum_{\text{All } \Gamma \text{ from } (0,0) \text{ to } (\mathbf{x},t)} e^{-\beta E_\Gamma}$$

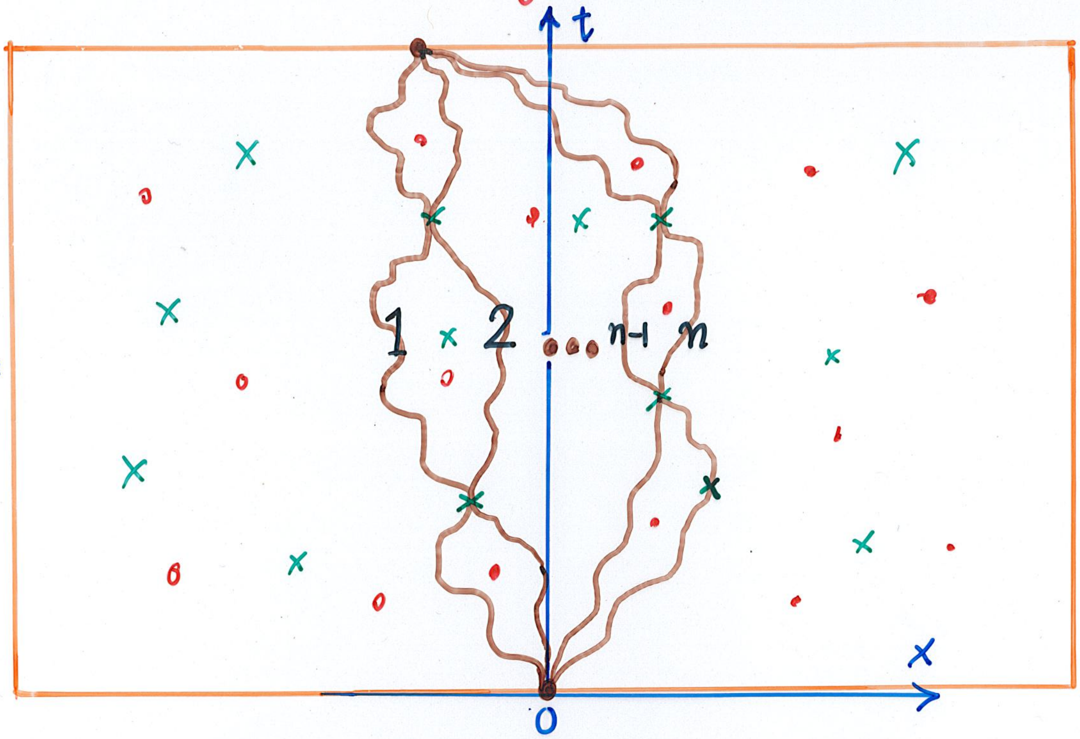
$$\boxed{W(\mathbf{x},t) \approx \int_{(0,0)}^{(\mathbf{x},t)} \mathcal{D}\mathbf{x}'(t') \exp\left\{-\int_0^t dt' \left[\frac{1}{2\nu}\left(\frac{d\mathbf{x}'}{dt'}\right)^2 + \mu(\mathbf{x}'(t'),t')\right]\right\}}$$

$\Rightarrow$ $$\frac{\partial W}{\partial t} = \nu \nabla^2 W + \mu(\mathbf{x},t) W \qquad W(\mathbf{x},0) = \delta^{d'}(\mathbf{x})$$

$\therefore$ Static Problem + Disorder $\Longleftrightarrow$ Stochastic Dynamic Problem

$$\overline{|\delta x|}^{z} \sim t \quad \Rightarrow \quad \overline{|\delta x|} \sim t^{1/z}$$

Moment Analysis ($P(W) \Leftrightarrow \overline{W^n}$)

- $W^n(t) =$ 

- Averaging over a bond crossed by m replicas gives an attraction:

$$\overline{e^{-m\mu(x,t')}} = \left(\overline{e^{-\mu}}\right)^m e^{\frac{m(m-1)\sigma^2}{2}}$$

(For Gaussian distributed μ of width σ)

$\therefore$ $$\overline{W_n(x_1,\dots,x_n;t)} = \int_{(0,0)}^{(x_1,t)} \mathcal{D}x_1(t') \dots \int_{(0,0)}^{(x_n,t)} \mathcal{D}x_n(t') \exp\left\{-\int_0^t dt' \left[\sum_{\alpha=1}^{n} \frac{1}{2\nu}\left(\frac{dx'_\alpha}{dt'}\right)^2 + \sum_{\alpha,\beta=1}^{n} \frac{\sigma^2}{2} \delta\left(x'_\alpha(t') - x'_\beta(t')\right)\right]\right\}$$

$\Rightarrow$ $$\boxed{\frac{\partial \overline{W_n}}{\partial t} = \underbrace{\left[\nu \sum_{\alpha=1}^{n} \nabla_\alpha^2 + \frac{\sigma^2}{2} \sum_{\alpha,\beta=1}^{n} \delta(x_\alpha - x_\beta)\right]}_{-H_n} \overline{W_n}}$$

$-H_n$: Hamiltonian for n attracting Particles

$\therefore$ $\overline{W_n(t)} \underset{t\to\infty}{\approx} \exp\left[-\mathcal{E}_n^0\, t\right] F(x_1,\dots,x_n;t)$; where $\mathcal{E}_n^0$ is the ground state energy of H_n .

Exact Solution in $d' = 1$

- Hamiltonian : $$-H_n = \nu \sum_{\alpha=1}^{n} \frac{\partial^2}{\partial x_\alpha^2} + \frac{\sigma^2}{2} \sum_{\alpha \neq \beta} \delta(x_\alpha - x_\beta)$$
- Ground State wave-function: $$\psi_0(x_1, \dots, x_n) = \exp\left[-\frac{\kappa}{2} \sum_{\alpha \neq \beta} |x_\alpha - x_\beta|\right]$$
- Ground State energy : $$-\varepsilon_n^0 = n(n^2-1) \frac{\nu \kappa^2}{6} \quad \rho$$

$$2\nu\kappa = \sigma^2 \qquad \rho = \frac{\nu\kappa^2}{6} = \frac{\sigma^4}{24\nu}$$

$$\therefore \quad \boxed{\overline{W^n(t)} \underset{t\to\infty}{=} e^{\rho n(n^2-1)t} \; \overline{W_1(t)}^{\,n}}$$

- $\overline{W^n(t)}$ is also a <u>characteristic function</u>, generating moments of $\ln W$

$$\overline{W^n} = \overline{e^{n \ln W}} = e^{n \overline{\ln W} + \frac{n^2}{2} \overline{(\ln W)^2_c} + \frac{n^3}{6} \overline{(\ln W)^3_c} + \cdots}$$

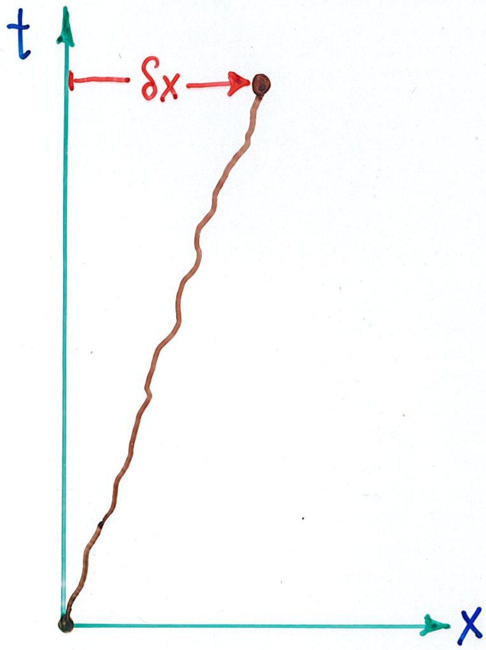
- Matching powers of n

$$\begin{cases} \overline{\ln W(t)} = (f_0 - \rho)t \\ \overline{(\ln W)^2_c} = 0 \quad (\text{to order } t) \\ \overline{(\ln W(t))^3_c} = 6\rho t \end{cases}$$

$W(t)$ is log-"normal" with

$$\delta \ln W \sim t^{1/3}$$

- Transverse Fluctuations



Energy cost from line tension due to increased length $\sim \frac{(\delta x)^2}{t}$

$=$ gain from free-energy fluctuations $\sim t^{1/3}$

$$\frac{\delta x^2}{t} \sim t^{1/3} \Rightarrow \boxed{\delta x \sim t^{2/3}}$$

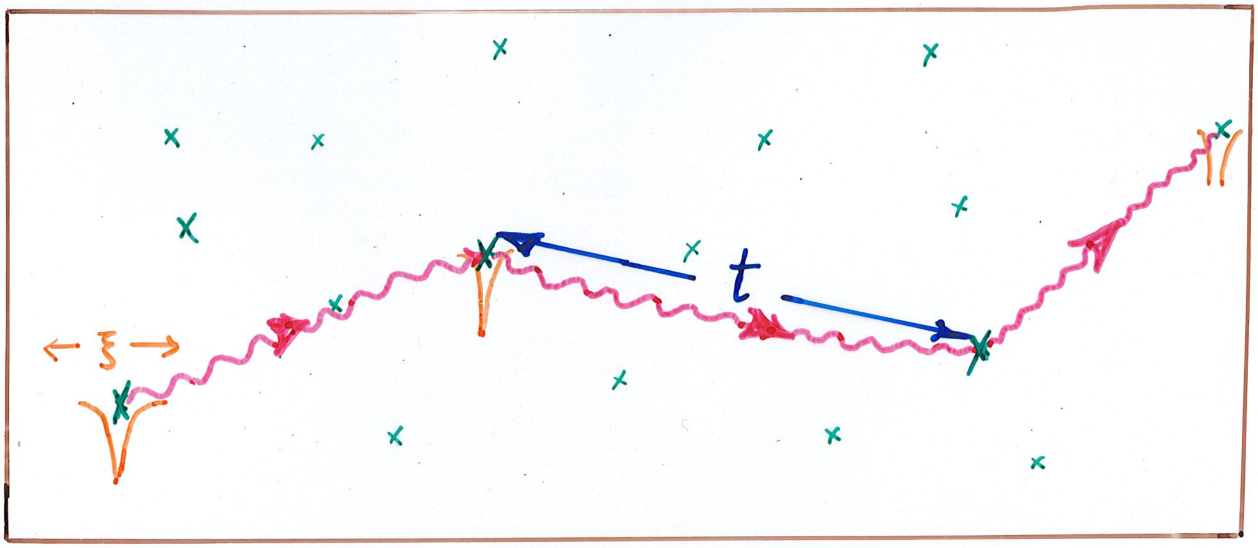
- Higher dimensions

$d' = d-1$	$\sigma^2 = 0$	$\sigma^2 \neq 0$
$d' = 0$	$\varepsilon_n^0 = 0$	$\varepsilon_n^0 = \sigma^2 n(n-1)$
$d' = 1$	$\varepsilon_n^0 = 0$	$\varepsilon_n^0 = \rho\, n(n^2-1)$
$d' = 2$	$\varepsilon_n^0 = 0$	?
$d' = 3$	$\varepsilon_n^0 = 0$	?

- Conjecture based on numerical results $\varepsilon_n^0 = \rho\, n(n^{d'+1} - 1)$

Quantum Interference in Insulators

- In highly doped semiconductors, electronic states are <u>localized</u>.



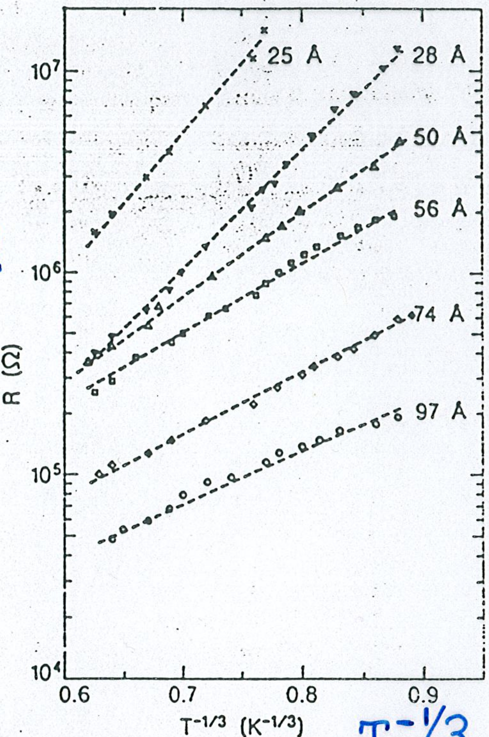
- Finite T conductivity by phonon-assisted Q-tunneling (Mott)
- Tunneling Probability $\sim e^{-t/2\xi}\, e^{-\varepsilon_{if}/kT} \sim \exp\left[-\frac{2t}{\xi} - \frac{c}{kT\,N(\varepsilon_F)\,t^d}\right]$
- Probability is highly peaked at $t^* \sim \left[\frac{c\,\xi}{N(\varepsilon_F)\,kT}\right]^{\frac{1}{d+1}}$
- Conductivity $\boxed{\sigma \sim \exp\left[-\left(\frac{T_0}{T}\right)^{\frac{1}{d+1}}\right]}$ $\qquad T_0 \sim c/kN(\varepsilon_F)\,\xi^d$

$$\therefore \quad \ln\sigma \sim \begin{cases} T^{-1/3} & (d=2) \\ T^{-1/4} & (d=3) \end{cases}$$

Thin Film Experiments

In_2O_{3-x}

ln R



$T^{-1/3}$

FIG. 5. Resistance as a function of temperature for several 100-Å-thick samples. The curves are labeled by the respective ξ values derived through Eq. (1) using $N(0)=10^{32}$ erg^{-1} cm^{-3} (cf. Sec. II) and T_* was taken from the logarithmic slope of the $R(T)$ data.

S2: 1×0.3 mm^2

S1: 2×2 mm^2

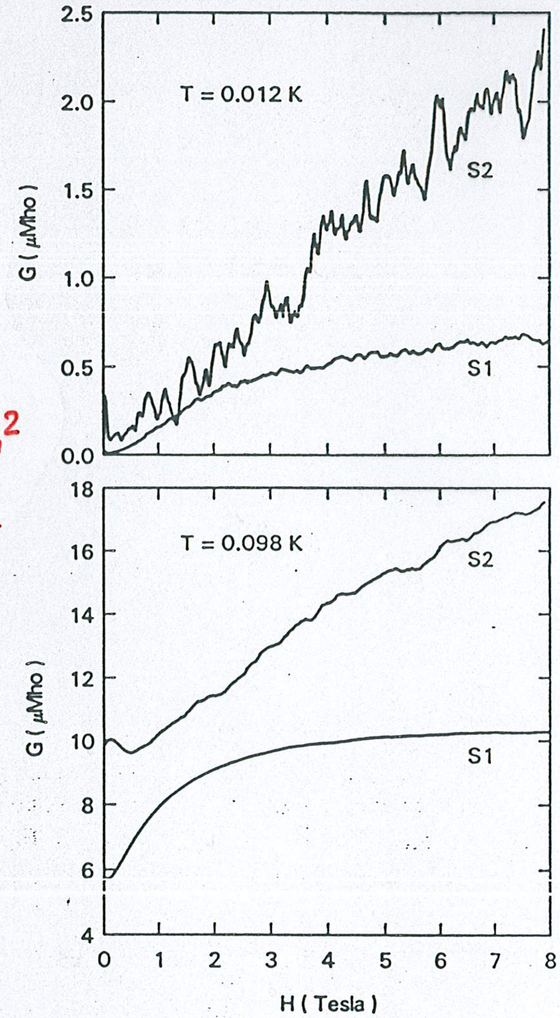
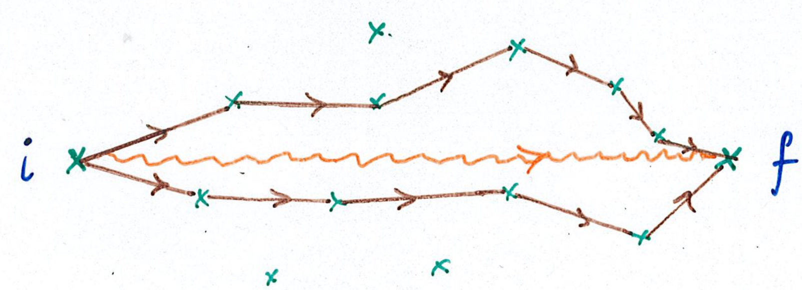


FIG. 2. Magnetoconductance data for S1 and S2 at $T=0.012$ and 0.098 K.

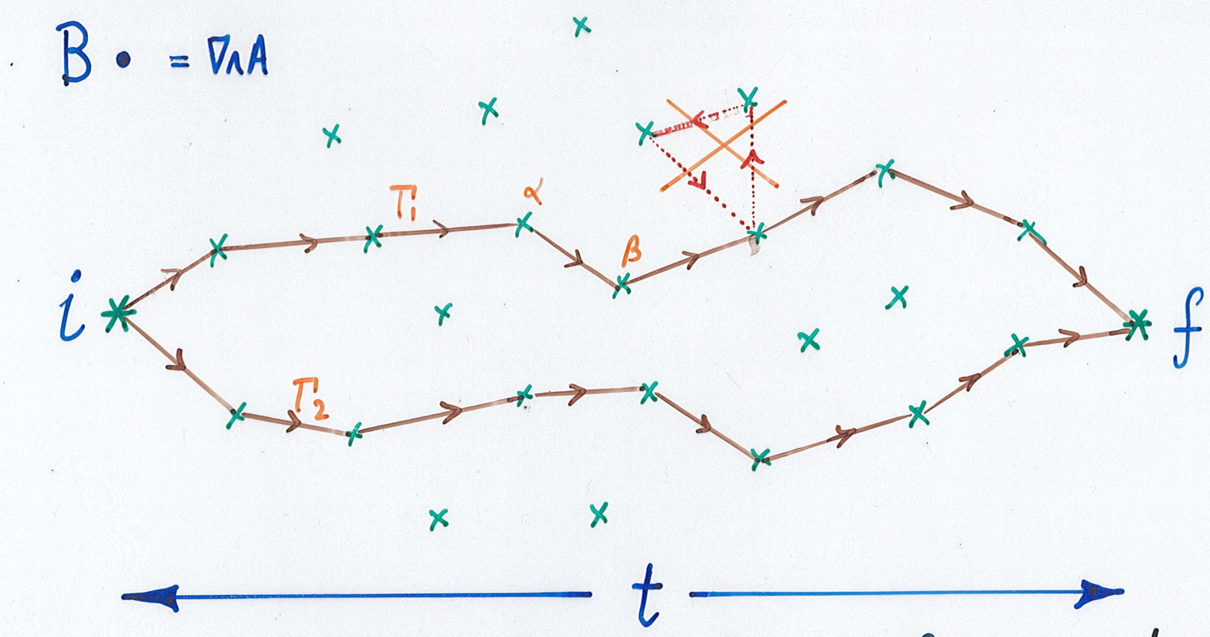
O. Faran + Z. Ovadyahu, PRB 38, 5457 (1988) F.P. Milliken + Z. Ovadyahu (1990)

- Two-dimensional tunneling $T^{-1/3}$
- Reproducible conductance Fluctuations $\delta\sigma \gtrsim 10^2\ e^2/\hbar$
- Positive magnetoconductance.

* Nguyen, Spivak, Shklovski (85) suggested Quantum Interference of tunneling paths:

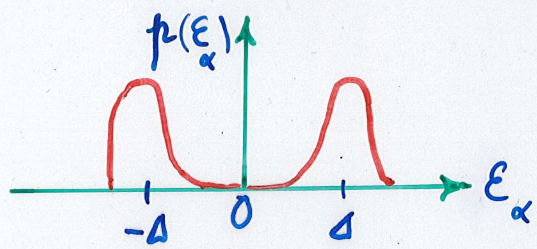


Quantum Interference of Tunneling Paths



- Quantum Hamiltonian for weakly overlapping sites (ala Anderson)

$$H = \sum_{\substack{\alpha \\ \sigma}} \varepsilon_\alpha \, a^{+}_{\alpha,\sigma} a_{\alpha,\sigma} + \sum_{\substack{(\alpha\beta) \\ \sigma\sigma'}} V_{\alpha\beta} \, e^{ieA_{\alpha\beta}} \, a^{+}_{\alpha\sigma} a^{+}_{\beta\sigma'} \, U^{\alpha\beta}_{\sigma\sigma'}$$



$$p(\varepsilon) = p(-\varepsilon) \quad , \quad \frac{V}{\Delta} \ll 1$$

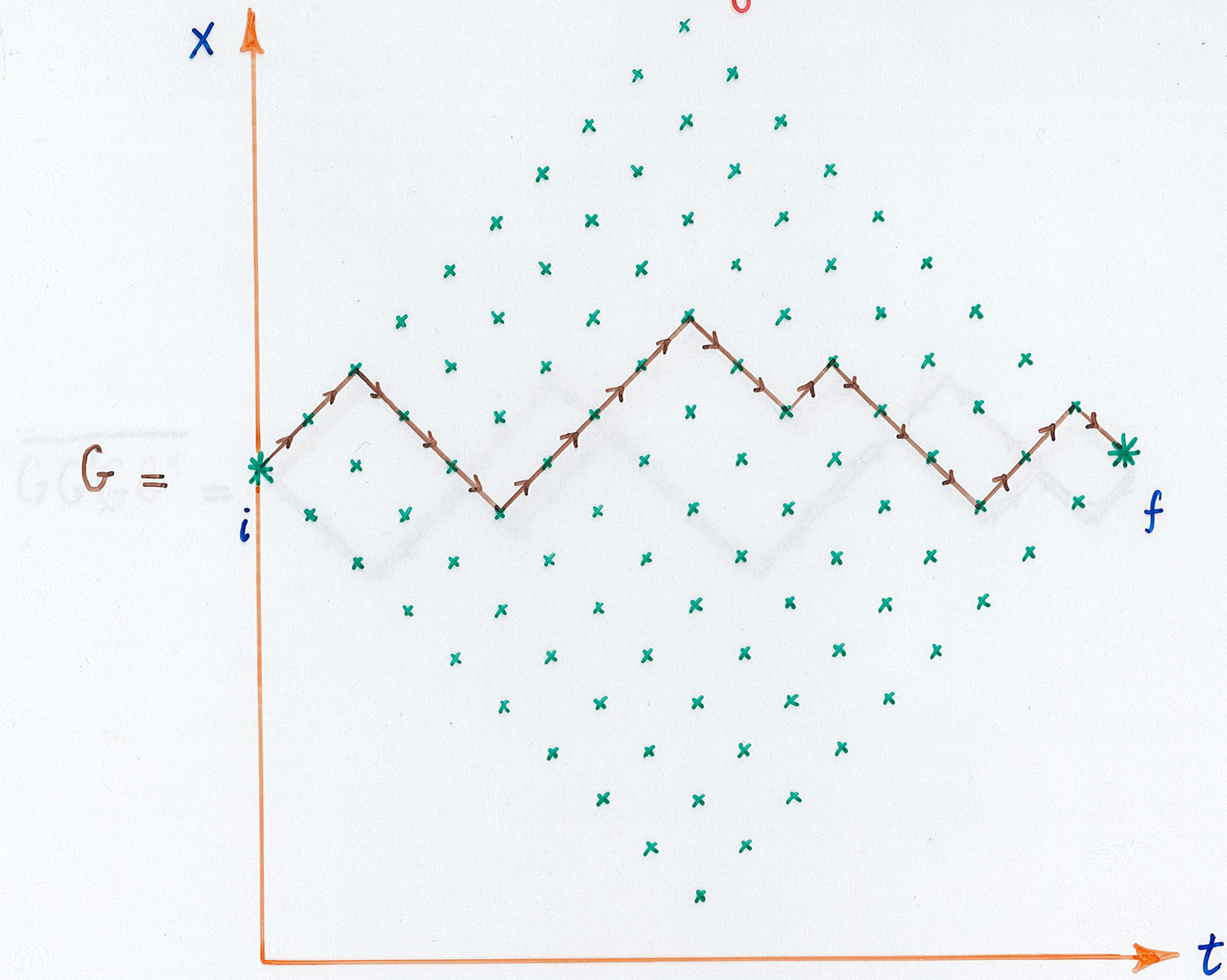
$$(\varepsilon_i = \varepsilon_f = 0)$$

- Perturbative evaluation of overlap (Green's function)

$$G_{if} = \sum_{\Gamma} \prod_{(\alpha\beta)_\Gamma} \frac{V_{\alpha\beta} \, e^{ieA_{\alpha\beta}}}{\varepsilon_\alpha} U^{\alpha\beta} \approx \left(\frac{V}{\Delta}\right)^t \underbrace{\sum_{\Gamma'} \mathrm{sign}(\varepsilon_\alpha) e^{ieA_{\alpha\beta}} U^{\alpha\beta}}_{\text{Directed Sum } J(t)}$$

- For $B, A = 0$ Random Signs
- For $B, A \neq 0$ "Random" Phases
- For $U^{\alpha\beta}_{\sigma\sigma'} \neq 1$ Spin-Orbit (SO) Scattering

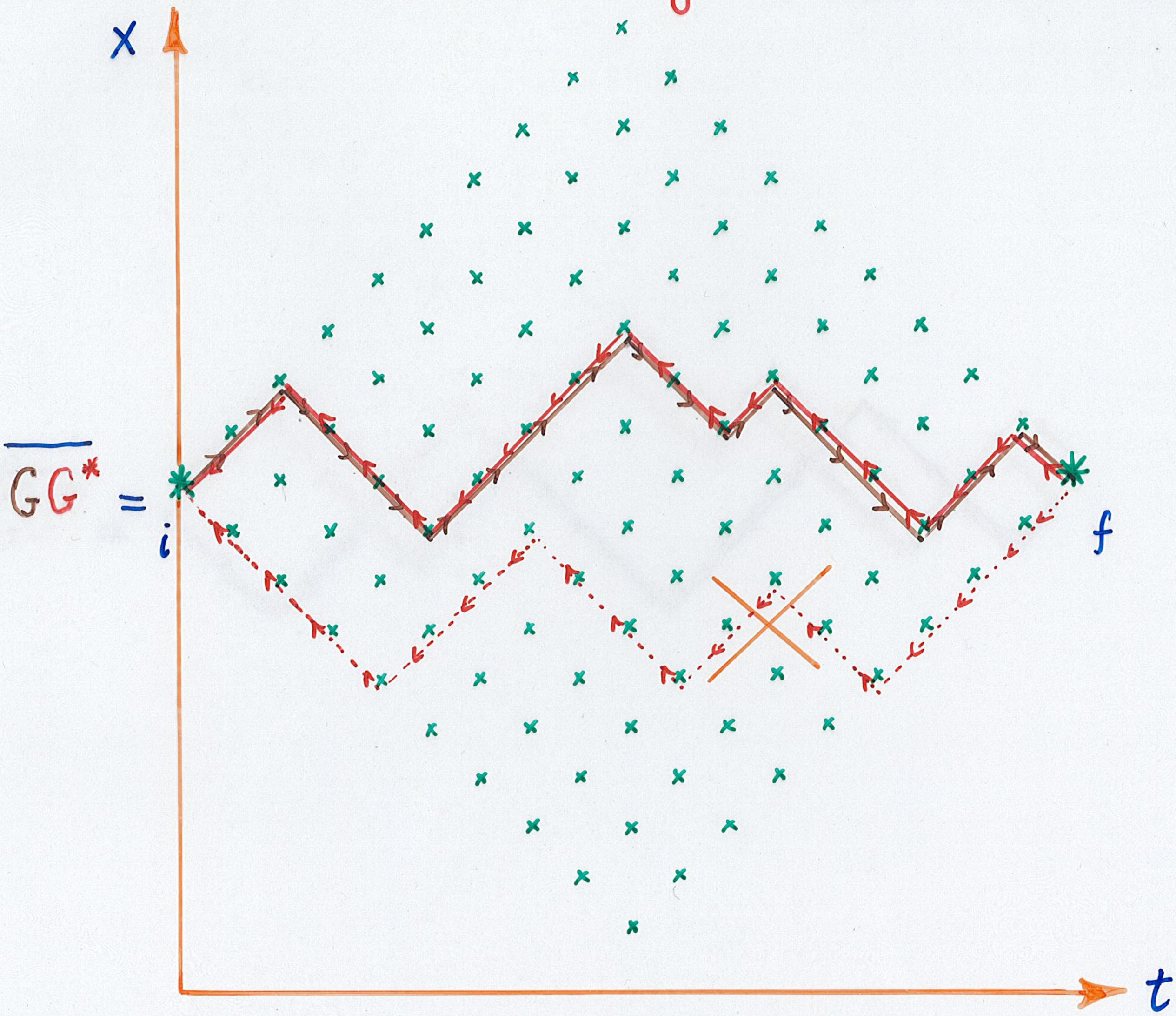
Moment Analysis $P(G) \Leftrightarrow \overline{|G^2|^n}$



- Since $p(\varepsilon) = p(-\varepsilon)$, $P(G) = P(-G)$,

and all odd moments vanish $\overline{G^{2n+1}} = 0$

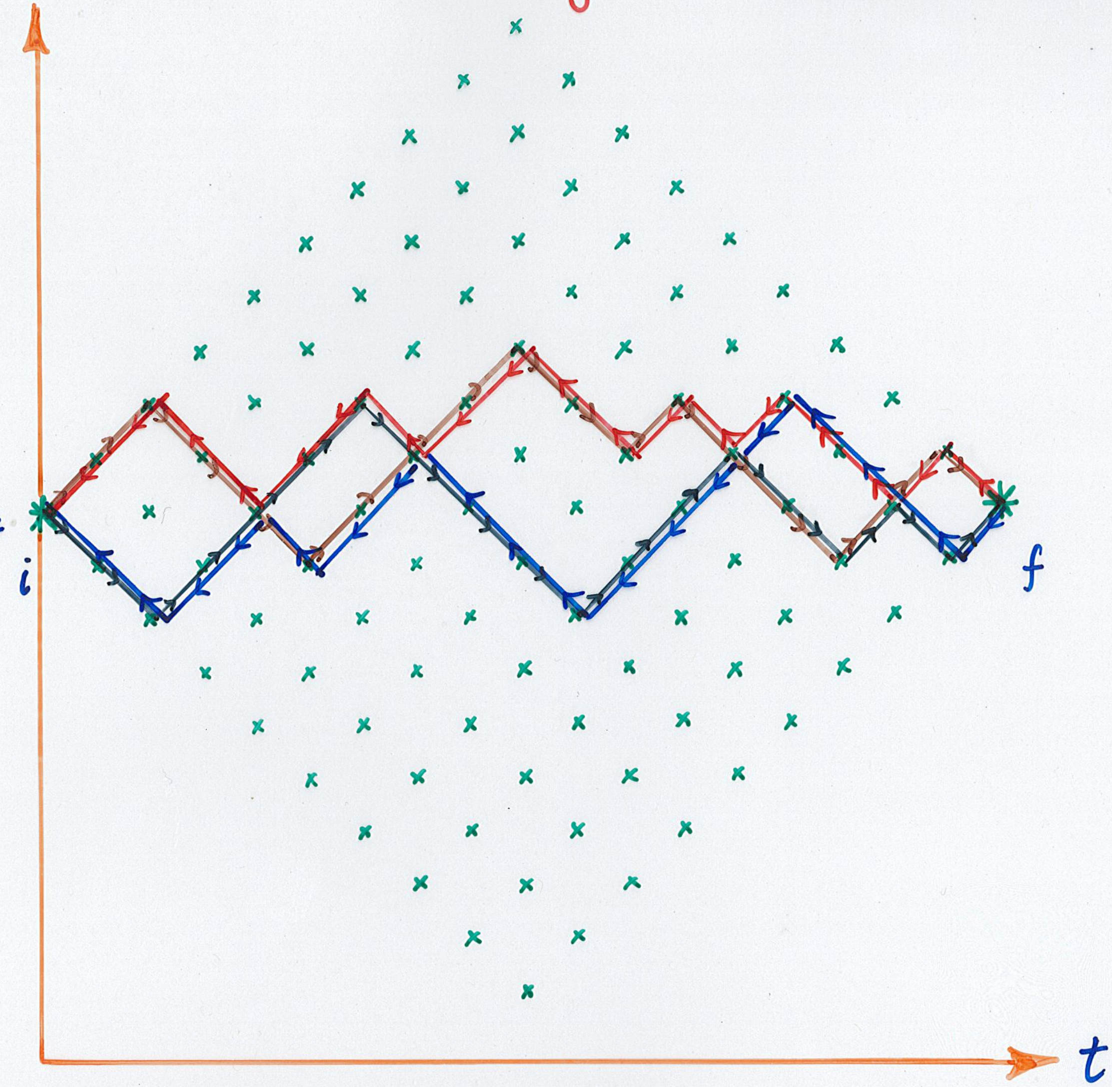
Moment Analysis $P(G) \Leftrightarrow \overline{|G^2|^n}$



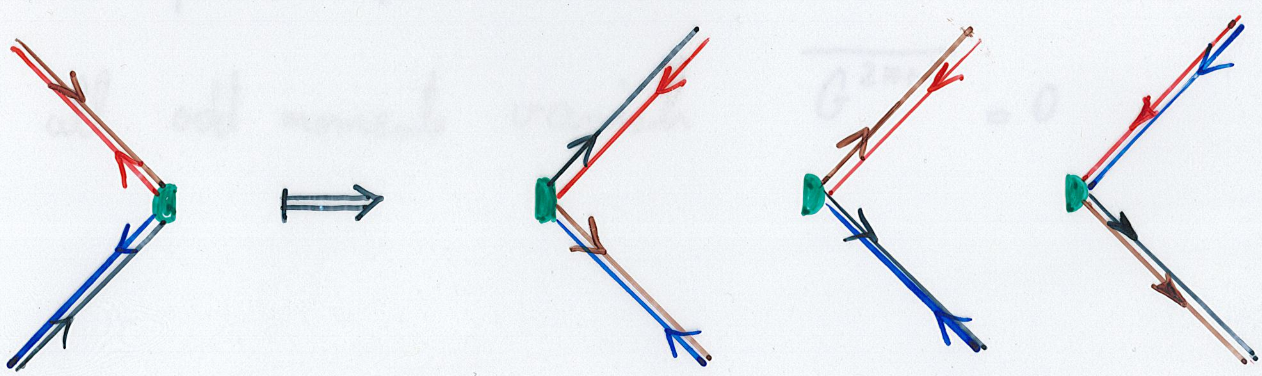
- Averaging over $p(\varepsilon)$ forces a pairing of "time-reversed" paths

Moment Analysis $\quad P(G) \Leftrightarrow \overline{|G^2|^n}$

$\overline{G G^* G G^*} =$



- Exchange Attraction (in zero field, $B=0$) of 3 per crossing

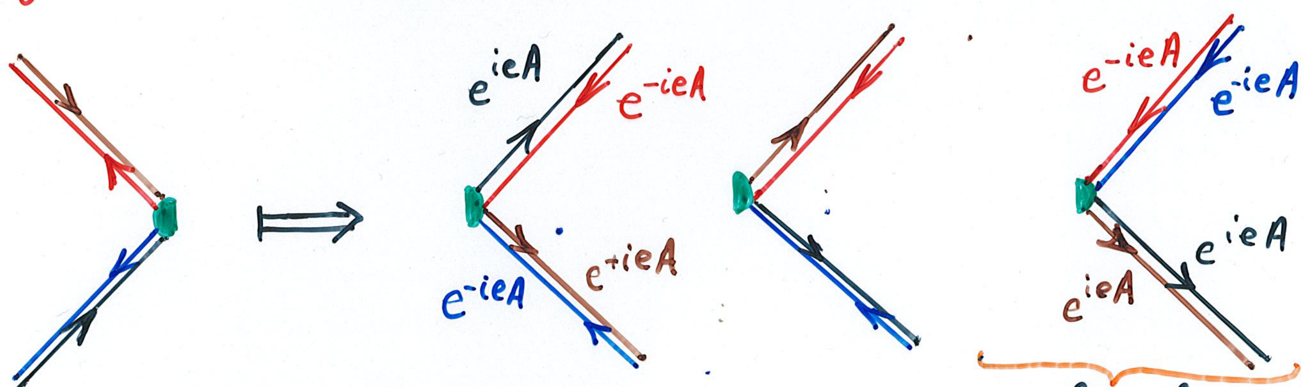


- $\overline{|G(t)|^{2n}}$ = Sum over n attracting paths $= e^{-\frac{2nt}{\xi_0}} \, e^{\rho\, n(n^2-1)t}$

$$\begin{cases} \overline{\ln|G(t)|^2} = \left(-2\xi_0^{-1} - \rho(B)\right)t \triangleq -2t/\xi(B) \\ \delta \ln|G(t)|^2 \sim (\rho t)^{1/3} \qquad (\sim T^{-1/9}) \end{cases}$$

Magnetoconductance

- Exchange Attraction (in finite field $B \neq 0$) approaches 2 per crossing.



Zero after phase average

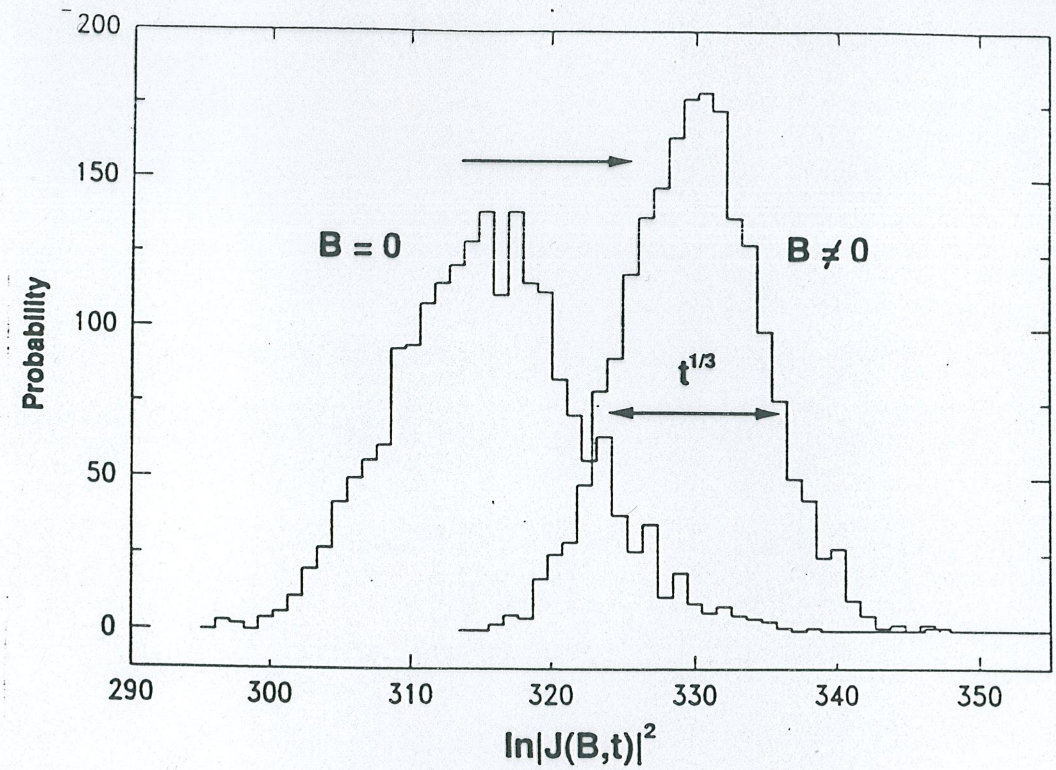
$\overline{e^{2ieA}} \to 0$

- $$\frac{\text{Attraction in } B=0}{\text{Attraction in } B \text{ large}} = \frac{3}{2}$$

$\therefore$ $\rho(B)$ decreases with B

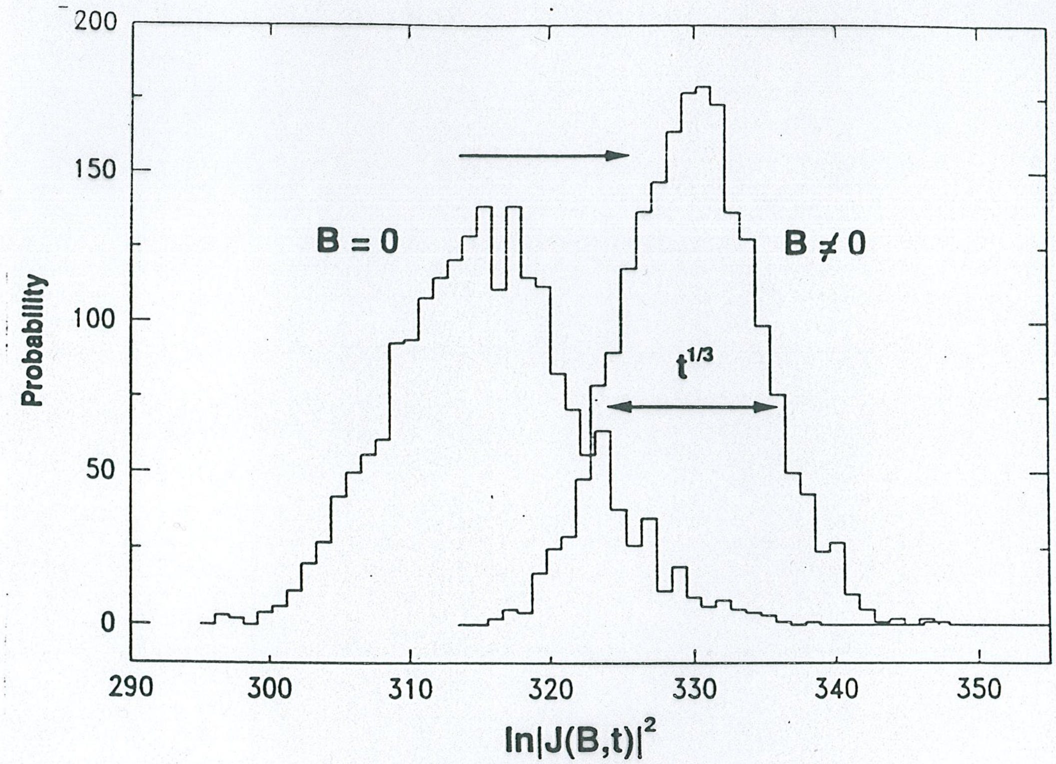
Localization length $\xi(B)$ increases with B

Tunneling is enhanced (positive magnetoconductance), while fluctuations decrease with B



$$\begin{cases} \overline{\ln |G_{if}(B,t)|^2} = -\frac{2t}{\xi_0} - \rho(B)\, t + \ln A(B) \\ \delta \ln |G_{if}(B,t)|^2 \sim [\rho(B)\, t]^{1/3} \end{cases}$$

$(t \sim T^{-1/3})$



- $\begin{cases} \overline{\ln |G_{if}(B,t)|^2} = -\dfrac{2t}{\xi_0} - \rho(B)\,t + \ln A(B) \\ \delta \ln |G_{if}(B,t)|^2 \sim [\rho(B)\,t]^{1/3} \end{cases}$ $\quad (t \sim T^{-1/3})$

* Without Spin-Orbit scattering $\quad \rho(B) = \rho(B=0) - c\,B^{1/2}$ ↓

* With Spin-Orbit scattering $\quad \begin{cases} \rho_{so}(B) = \rho_{so}(0) \\ A(B) \simeq A(0) + B^2 t^3 \quad (B^2 t^3 \lesssim 1) \end{cases}$

- The changes in Exchange Attraction reflect Hamiltonian symmetries

Orthogonal : Unitary : Simplectic $= 3 : 2 : 3/2$

Propagation vs Decay

$$\begin{cases} W(x,t) = \int \mathcal{D}x'(t') \exp\left\{ -\int_0^t dt' \left[\frac{1}{2\nu}\left(\frac{dx'}{dt'}\right)^2 + \mu(x'(t'),t') \right] \right\} \\ \frac{\partial W}{\partial t} = \nu \nabla^2 W + \mu(x,t) W \end{cases}$$

* For statistics of vortex lines, paths μ is real
For Quantum tunneling μ is imaginary.

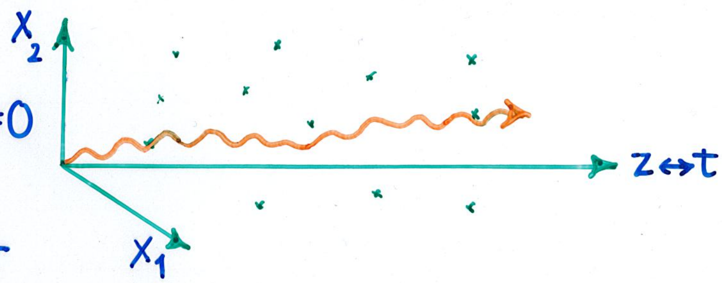
* What if μ and ν are both imaginary? (Feng, Golubovic, Zhang, 1990)

- Schrödinger eq. in a t-dependent random potential

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x,t)\psi$$

- Coherent light popagation in a random medium

$$\left[\nabla^2 + \frac{d^2}{dz^2} + k^2 n(x,y,z)\right]\left(e^{-ik n_0 z} E(x,y,z)\right) = 0$$



$$\cancel{-\frac{\partial^2}{\partial t^2}E} + 2ik n_0 \frac{\partial E}{\partial t} = \nabla^2 E + k^2 \delta n E$$

- However, because of unitarity ($\int d^{d'}x \, |E(x,t)|^2 = 1$),

only "Poisson" fluctuations are allowed :

$$\overline{|E^2|^n} = n! \left(\overline{|E^2|}\right)^n \qquad \text{(R. Dashen, 1979)}$$

- Beam Width $\overline{\langle x^2 \rangle} \sim t$
- Beam Center $\overline{\langle x \rangle^2} \sim t^{\frac{d'-2}{2}}$ $\quad t^{1/4}$ ($d=2$) $\quad \ln t$ ($d=3$)

Summary

- The generalized diffusion equation appears in many contexts :

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t) \Longleftrightarrow \partial_t \vec{v} + \lambda\, \vec{v}\cdot\nabla\vec{v} = \nu \nabla^2 \vec{v} - \nabla \eta$$

$$\partial_t W = \nu \nabla^2 W + \frac{\lambda}{2\nu}\eta W \Longleftrightarrow W = \int \mathcal{D}x(t')\, e^{-\frac{1}{2\nu}\int_0^t dt' \left[\left(\frac{dx}{dt'}\right)^2 + \lambda \eta(x(t'),t')\right]}$$

- Its correlations scale as

$$\langle [h(x,t) - h(0,0)] \rangle \sim \overline{\Delta \ln W(x,t)} \sim |x|^{\chi} f(t/|x|^{z}) \qquad \chi + z = 2$$

- Exact results (in $d' = 0, 1$), and numerical simulations so far , are consistent with $\chi \approx \frac{2}{d'+3}$

- These equations can be used as a <u>starting point</u> to address the scaling of fluctuations in a wide range of phenomena.