Stochastic Network Utility Maximization

6.263/16.37 - Data Communication Networks

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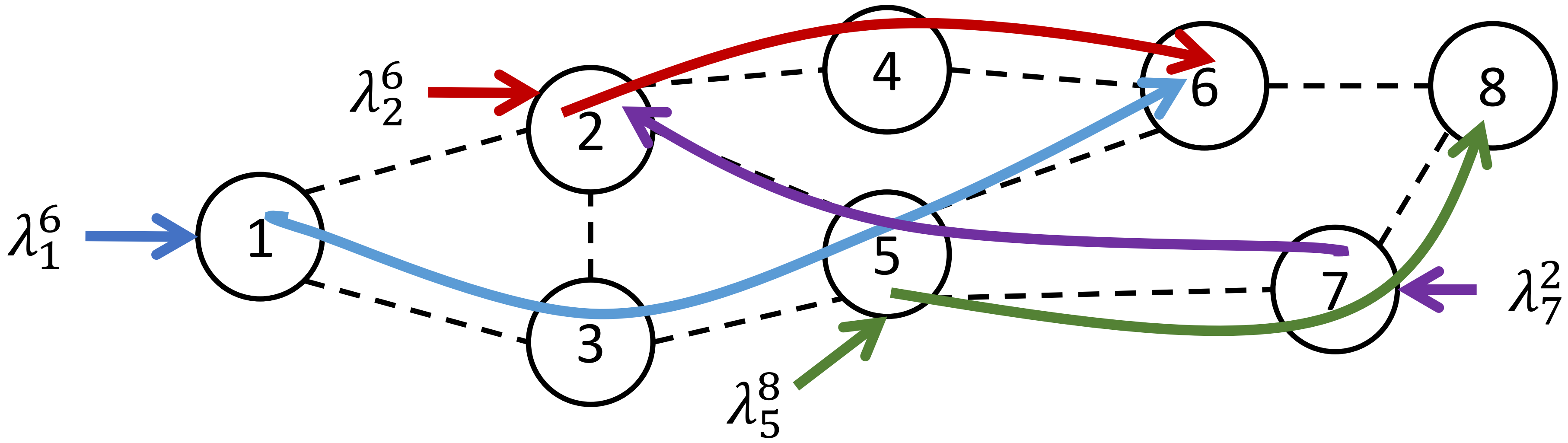
Cambridge, December 07, 2018

Outline

- Multi-commodity flow problem
- Recap from previous lectures
  - Network of Queues
  - Capacity Region and Stationary Randomized Policy
- Overloaded system
  - Problem Statement
  - Utility Function
  - Drift-Plus Penalty Algorithm (Admission + Routing + Scheduling)
  - Performance Analysis and Optimality Results

*slides adapted from Chih-Ping Li's lecture, with additional material from "Resource Allocation and Cross-Layer Control in Wireless Networks" by L. Georgiadis, M. J. Neely and L. Tassiulas and from M. J. Neely's PhD thesis.

Multi-Commodity Flow Problem



- Discrete-time system with slot index t. Network with N nodes and K commodities.
- Each node $n \in \{1,2,\ldots,N\}$ keeps per-commodity queues;
- Commodity-c packets are addressed to node c;
- λ_n^c time-average arrival rate of commodity-c packets at node n;
- **Goal is to find a network control algorithm that supports the arrivals λ_n^c (when possible).**

Network of Queues

- Let $Q_n^c(t)$ be the number of commodity-c packets enqueued at node n at the beginning of time-slot t. Then, according to Lindley recursion:

$$Q_n^c(t+1) \leq \max\left\{Q_n^c(t) - \sum_{j=1}^{N} \mu_{n,j}^c(t)\,;0\right\} + \sum_{i=1}^{N} \mu_{i,n}^c(t) + A_n^c(t)$$

where:

- $A_n^c(t) \geq 0$ is the number of **exogenous** packet arrivals at the end of slot t;
- $\mu_{n,j}^c(t) \geq 0$ is the **offered** transmission opportunity to comm.-c over (n,j) during slot t;
- $\mu_{n,j}(t) = \sum_{c=1}^{K} \mu_{n,j}^c(t)$ is the total transmission opportunity over (n,j) during slot t;
- $0 \leq f_{n,j}^c(t) \leq \mu_{n,j}^c(t)$ is the # of packet **transmissions** of comm.-c over (n,j) during slot t;
- $\mathcal{C}_{n,j}$ is the **capacity** constraint of link (n,j).

Network of Queues

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$$Q_n^c(t+1) = Q_n^c(t) - \sum_{j=1}^{N} f_{n,j}^c(t) + \sum_{i=1}^{N} f_{i,n}^c(t) + A_n^c(t)$$

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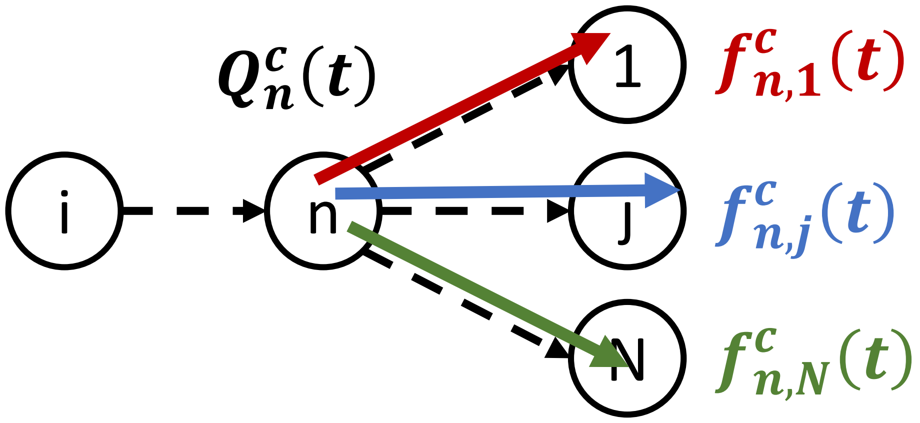
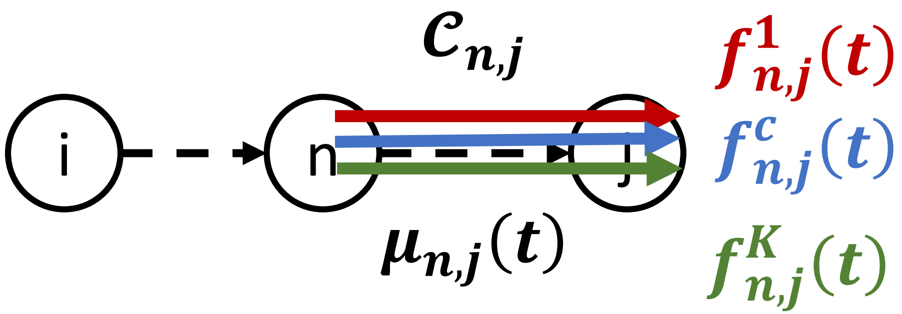
Constraints:

$$\sum_{j=1}^{N} f_{n,j}^c(t) \le Q_n^c(t), \forall n, c, t \qquad \sum_{c=1}^{K} f_{n,j}^c(t) \le \mu_{n,j}(t) \le \mathcal{C}_{n,j}, \forall n, j, t$$





Network of Queues

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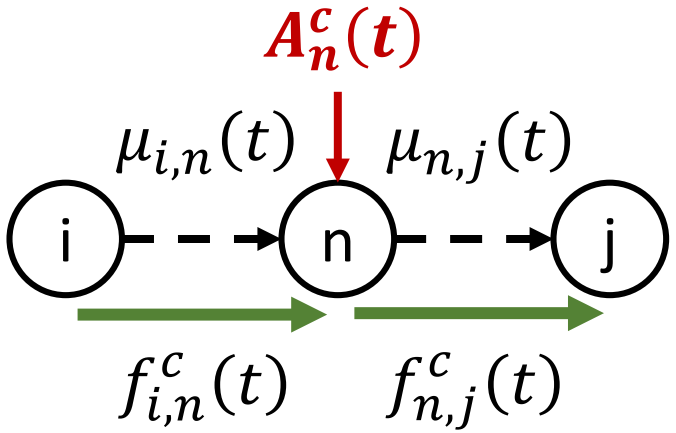
Assumptions:

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} A_n^c(\tau) = \lambda_n^c \quad \text{w.p.}\,1$$

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} f_{n,j}^c(\tau) = \bar{f}_{n,j}^c \quad \text{w.p.}\,1$$

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \mu_{n,j}^c(\tau) = \bar{\mu}_{n,j}^c \quad \text{w.p.}\,1$$



Network of Queues – Stability is the objective

- **Rate stability**: $\lim_{t\to\infty} \frac{Q_n^c(t)}{t} = 0 \quad \text{w.p.}\,1$

Arrival Rate = Departure ≤ Dep. Opport.

  - Queue $Q_n^c(t)$ is rate stable if and only if $\sum_{i=1}^{N} \bar{f}_{i,n}^c + \lambda_n^c = \sum_{j=1}^{N} \bar{f}_{n,j}^c \le \sum_{j=1}^{N} \bar{\mu}_{n,j}^c$
  - If *Arrival* > *Departure*, then: $\lim_{t\to\infty} \frac{Q_n^c(t)}{t} = \sum_{i=1}^{N} \bar{f}_{i,n}^c + \lambda_n^c - \sum_{j=1}^{N} \bar{f}_{n,j}^c \quad \text{w.p.}\,1$

- **Strong stability:** $\lim_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}[Q_n^c(\tau)] < \infty$ [bounded time-average queue backlog]
  - If $Q_n^c(t)$ is strongly stable and there exists $C > 0$ such that

$$arr(t) - dep\_opp(t) \le C \quad \text{w.p.}\,1, \forall t \quad \text{or} \quad dep\_opp(t) - arr(t) \le C \quad \text{w.p.}\,1, \forall t$$

Then $Q_n^c(t)$ is also Rate Stable.

Network of Queues – Action and Outcome

- **Network State**: $s(t) \in \mathcal{S} \leftarrow$ channels, topology, packet arrivals,… [uncontrollable]
  - Assumption: s(t) evolves according to an irreducible MC with finite states such that :

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \mathfrak{I}ndicator_{\{s(\tau)=s\}} = \pi_s \quad \text{w.p.1.} \ ; \ \forall s \in \mathcal{S}$$

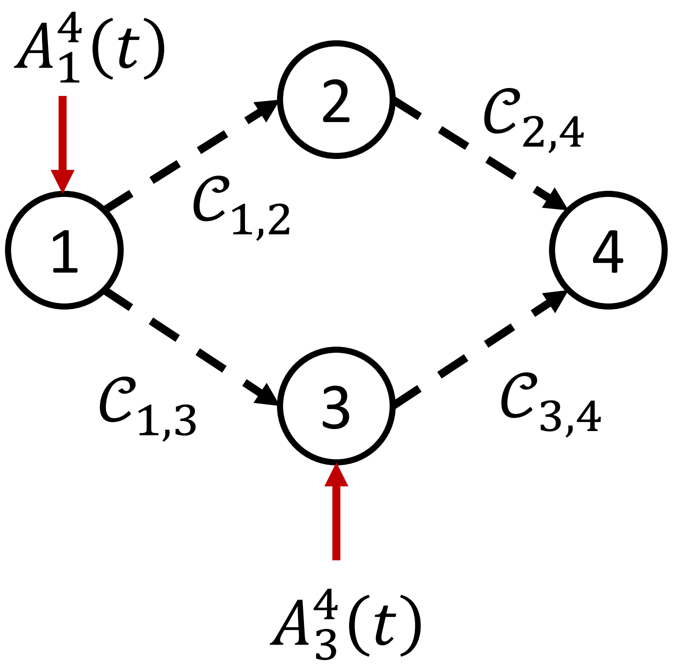
- **Action**: $I(t) \in \mathcal{I}_{s(t)} \leftarrow$ controls $\mu_{i,j}(t)$ and $f_{i,j}^{c}(t)$ using knowledge of the network state and (possibly) other information such as current queue backlog. $I(t)$ satisfies constraints from the state space $\mathcal{I}_{s(t)}$.

- **Outcome**: the result of any given state action pair $(s(t), I(t))$ are: 1) $A_n^c(t)$ for every node n and; 2) $\mu_{i,j}(t)$ and $f_{i,j}^{c}(t)$ for every link (i,j) and commodity c.
  - Define the (total) link transmission rate **matrix** as $\boldsymbol{U}\big(s(t), I(t)\big) = \left[\mu_{i,j}(t)\right]_{i,j}$.

Capacity Region – Definition

Definition: Capacity Region Λ is the closure of the set of all arrival rate matrices $(\lambda_n^c)_{n,c}$ that can be stabilized by some (possibly unknown) network control algorithm.

Example: wireline network with capacity $\mathcal{C}_{n,j}$. State is $s(t) = \text{all channels are ON}, \forall t$.



The capacity region Λ is given by the set of all λ_1^4, λ_3^4 for which there exists flow variables $\bar{f}_{n,j}^c$ satisfying:

Capacity: $\sum_{c=1}^{K} \bar{f}_{i,j}^c \leq \mathcal{C}_{i,j}, \forall i, j;$

Conservation: $\lambda_n^c = \sum_{j=1}^{N} \bar{f}_{n,j}^c - \sum_{i=1}^{N} \bar{f}_{i,n}^c, \forall n, c;$

Notice similarity between flow conservation and the necessary and sufficient conditions for rate stability.

Capacity Region – Definition

Definition: Capacity Region Λ is the closure of the set of all arrival rate matrices $(\lambda_n^c)_{n,c}$ that can be stabilized by some (possibly unknown) network control algorithm.

Supported transmission rate matrices: $\mathcal{Cl}(\Gamma)$

- Consider a fixed state $s \in \mathcal{S}$ and define the set A_s of all possible transmission rate matrices:

$$A_s := \left\{ \boldsymbol{U}(s, I) = \left[\mu_{i,j}(t)\right]_{i,j} \middle| \, I \in \mathcal{I}_s \right\}$$

- Consider the Stationary Randomized policy which selects $I(t) = I$ w.p. $p_s(I) \in (0,1]$.
  - This stationary policy attains $\mathbb{E}\left[\left[\mu_{i,j}(t)\right]_{i,j} \middle| s(t) = s\right] \in ConvHull\{A_s\}$.
  - By appropriately selecting $p_s(I)$, any point in $ConvHull\{A_s\}$ can be achieved.
- The long-term link transmission rate matrix achieved by the class of Stationary Randomized policies is defined as $\Gamma := \sum_{s \in \mathcal{S}} \pi_s ConvHull\{A_s\}$ and its closure is denoted $\mathcal{Cl}(\Gamma)$.

Capacity Region – Definition

Definition: Capacity Region Λ is the closure of the set of all arrival rate matrices $(\lambda_n^c)_{n,c}$ that can be stabilized by some (possibly unknown) network control algorithm.

<u>Illustration:</u>

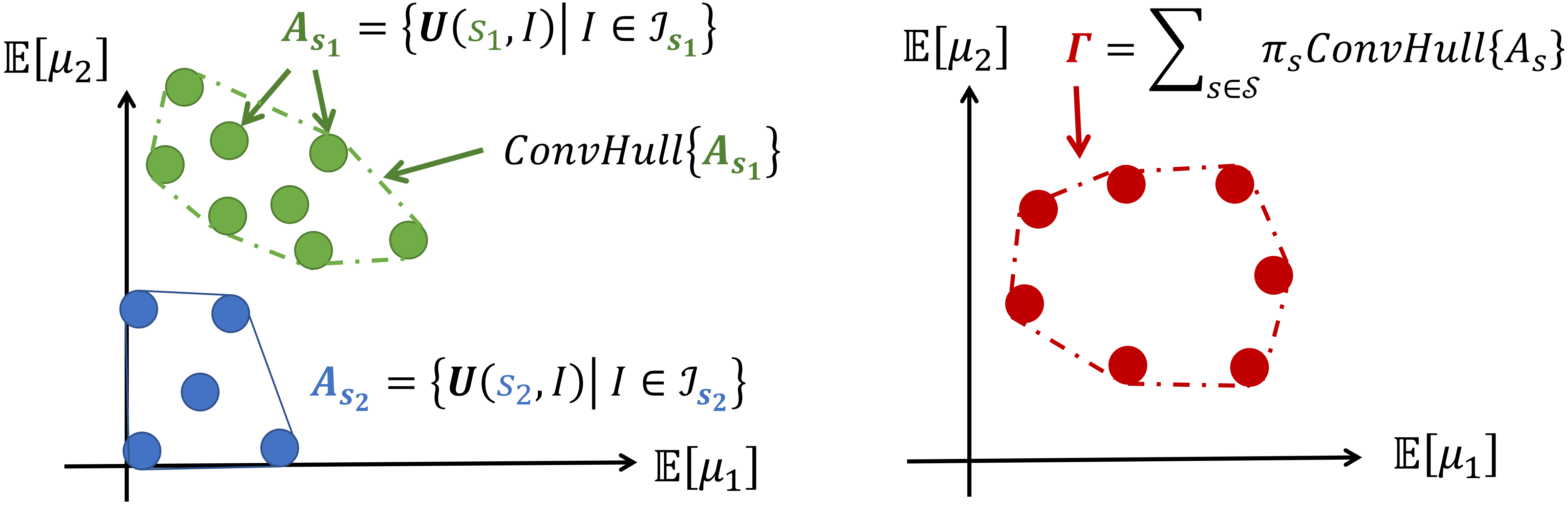


*figure adapted from the book "Resource Allocation and Cross-Layer Control in Wireless Networks"

Capacity Region – Theorem

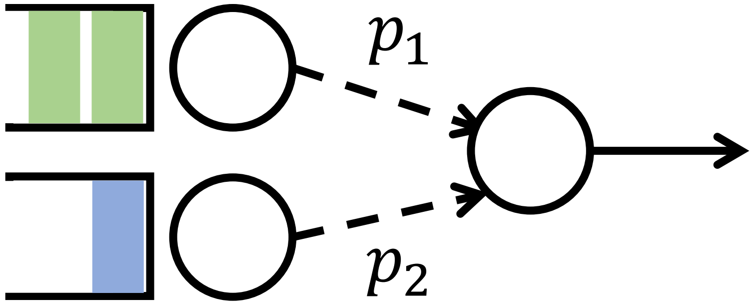
Definition: Capacity Region Λ is the closure of the set of all arrival rate matrices $(\lambda_n^c)_{n,c}$ that can be stabilized by some (possibly unknown) network control algorithm.

Theorem: the capacity region Λ is given by the set of arrival rate matrices $(\lambda_n^c)_{n,c}$ for which there exists a transmission rate matrix $\mathbb{E}\left[\mu_{i,j}(t)\right]_{i,j} = \left[\bar{\mu}_{i,j}\right]_{i,j} \in \mathcal{C}l(\Gamma)$ together with multi-commodity flow variables $\bar{f}_{i,j}^c$ that satisfy the following routing feasibility constraints:

- Flow capacity: $\sum_{c=1}^{K} \bar{f}_{i,j}^c \leq \bar{\mu}_{i,j} \leq \mathcal{C}_{i,j}, \forall i, j;$
- Flow conservation: $\lambda_n^c = \sum_{j=1}^{N} \bar{f}_{n,j}^c - \sum_{i=1}^{N} \bar{f}_{i,n}^c, \forall n, c;$
- Flow are efficient: $\bar{f}_{i,i}^c = 0, \bar{f}_{c,i}^c = 0;$ [no flow from a node to itself or from the destination]
- Flows are non-negative: $\bar{f}_{i,j}^c \geq 0, \forall i, j, c;$
- Routing constraints: $\bar{f}_{i,j}^c = 0, \forall (i, j, c)$ for which flow is not allowed.

Capacity Region – Example On/Off downlink

- i.i.d. Bernoulli channel state with $\mathbb{P}\{s_1(t) = ON\} = p_1$ and $\mathbb{P}\{s_2(t) = ON\} = p_2$
- Every time-slot t, the controller observes the channels $s_n(t)$ and serves at most one packet from one of the queues: $\mu_n(t) \in \{0,1\}$ such that $\mu_1(t) + \mu_2(t) \leq 1, \forall t$.



State	Probability	Transmission Rates
(OFF,OFF)	$(1\text{-}p_1)(1\text{-}p_2)$	(0,0)
(ON,OFF)	$p_1(1\text{-}p_2)$	(0,0), (1,0)
(OFF,ON)	$(1\text{-}p_1)p_2$	(0,0), (0,1)
(ON,ON)	p_1p_2	(0,0), (1,0), (0,1)

- Long-term link transmission rate matrix:

$$\boldsymbol{\Gamma} = (1-p_1)(1-p_2)\{(\mathbf{0},\mathbf{0})\} + p_1(1-p_2)\boldsymbol{ConvHull}\{(\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{0})\} +$$

$$+(1-p_1)p_2\boldsymbol{ConvHull}\{(\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{1})\} + p_1p_2\boldsymbol{ConvHull}\{(\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{1}),(\mathbf{1},\mathbf{0})\}$$

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- Every time-slot t, the controller observes the channels $s_n(t)$ and serves at most one packet from one of the queues: $\mu_n(t) \in \{0,1\}$ such that $\mu_1(t) + \mu_2(t) \leq 1, \forall t$.

- Long-term link transmission rate matrix:

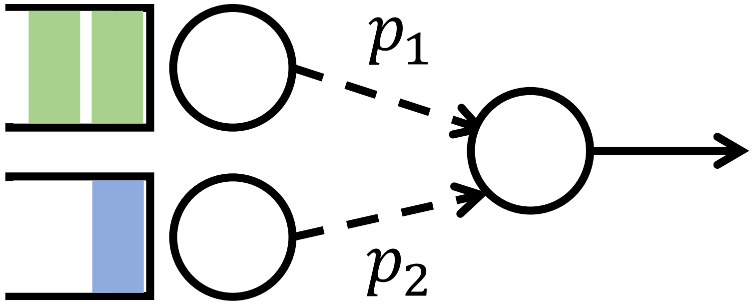
$$\boldsymbol{\Gamma} = (1-p_1)(1-p_2)\{(\mathbf{0},\mathbf{0})\} + p_1(1-p_2)\boldsymbol{ConvHull}\{(\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{0})\} + \\ +(1-p_1)p_2\boldsymbol{ConvHull}\{(\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{1})\} + p_1p_2\boldsymbol{ConvHull}\{(\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{1}),(\mathbf{1},\mathbf{0})\}$$

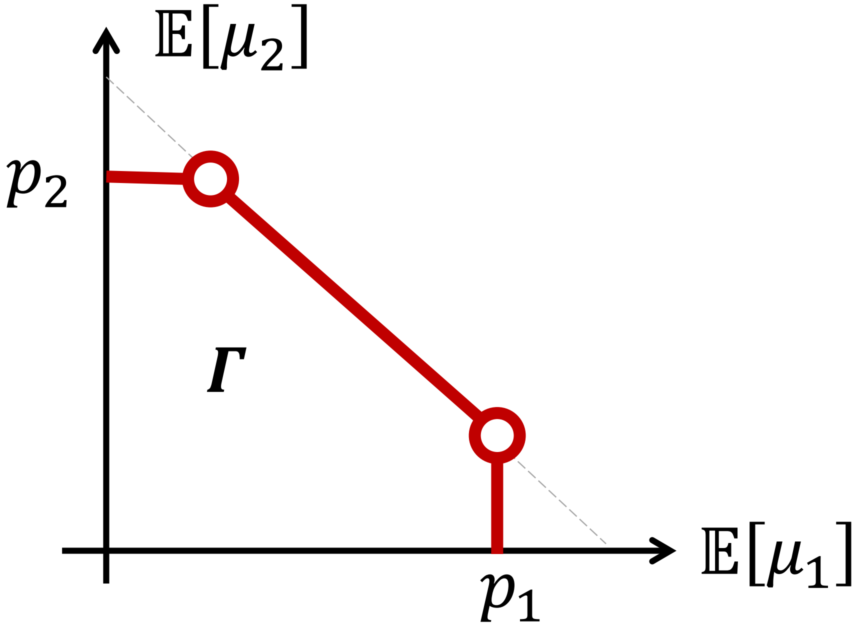
$$\boldsymbol{\Gamma} = (1-p_1)(1-p_2)(\mathbf{0},\mathbf{0}) + p_1(1-p_2)(\boldsymbol{q_1},\mathbf{0}) + (1-p_1)p_2(\mathbf{0},\boldsymbol{q_2}) + p_1p_2(\mathbf{1}-\boldsymbol{q_3},\boldsymbol{q_3})$$

$$\boldsymbol{\Gamma} = (p_1(1-p_2)\boldsymbol{q_1} + p_1p_2(\mathbf{1}-\boldsymbol{q_3}),(1-p_1)p_2\boldsymbol{q_2} + p_1p_2\boldsymbol{q_3}), \text{for } q \in [0,1]$$

Capacity Region – Example On/Off downlink

- i.i.d. Bernoulli channel state with $\mathbb{P}\{s_1(t) = ON\} = p_1$ and $\mathbb{P}\{s_2(t) = ON\} = p_2$
- Every time-slot t, the controller observes the channels $s_n(t)$ and serves at most one packet from one of the queues: $\mu_n(t) \in \{0,1\}$ such that $\mu_1(t) + \mu_2(t) \leq 1, \forall t$.



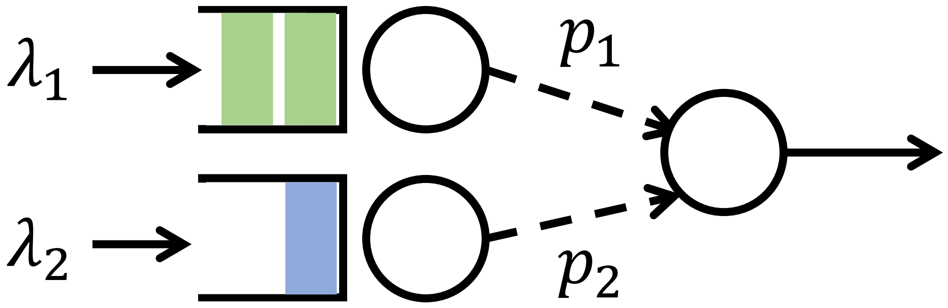


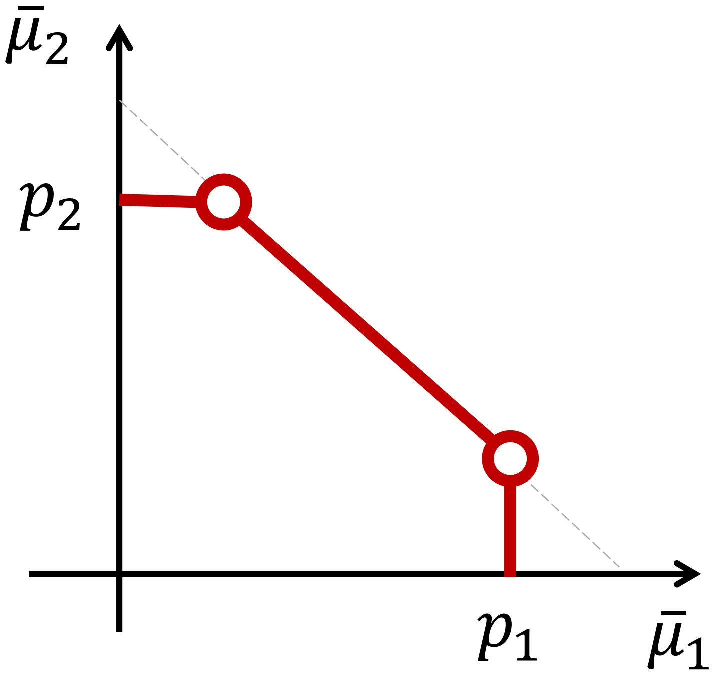
- Long-term link transmission rate matrix:

$$\boldsymbol{\Gamma} = \left(p_1(1-p_2)\boldsymbol{q_1} + p_1 p_2 (\boldsymbol{1} - \boldsymbol{q_3}),\ (1-p_1)p_2\boldsymbol{q_2} + p_1 p_2 \boldsymbol{q_3}\right)$$

Capacity Region – Example On/Off downlink

- **i.i.d. Bernoulli packet arrivals with $\mathbb{P}\{A_1(t) = 1\} = \lambda_1$ and $\mathbb{P}\{A_2(t) = 1\} = \lambda_2$.**
- i.i.d. Bernoulli channel state with $\mathbb{P}\{s_1(t) = ON\} = p_1$ and $\mathbb{P}\{s_2(t) = ON\} = p_2$
- Every time-slot t, the controller observes the channels $s_n(t)$ and serves at most one packet from one of the queues: $\mu_n(t) \in \{0,1\}$ such that $\mu_1(t) + \mu_2(t) \leq 1, \forall t$.





- **We already know the link transmission rate matrix Γ.**
- Flow conservation + Flow capacity yields:
  - $\lambda_n \leq \bar{f}_n \leq \bar{\mu}_n = \mathbb{E}[\mu_n(t)],\ n \in \{1,2\}$.
- **Conclusion: $\Gamma = \Delta$.**

Capacity Region – Randomized policy

Corollary: consider a **Stationary Randomized policy** that observes $s(t) = s$ and select a control $I(t) = I$ according to $p_s(I)$. Notice that $p_s(I)$ disregards $Q_n^c(t)$. If an arrival rate matrix $(\lambda_n^c)_{n,c}$ is interior to Λ, then *there is a randomized policy that stabilizes the system.*

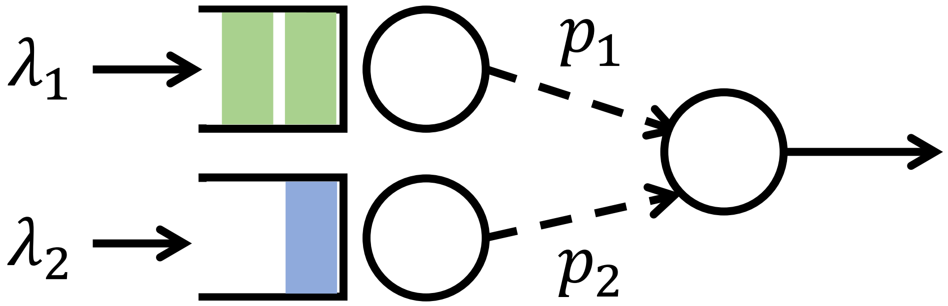
Interpretation: the randomized policy manages packet flows as a “continuous fluid”.

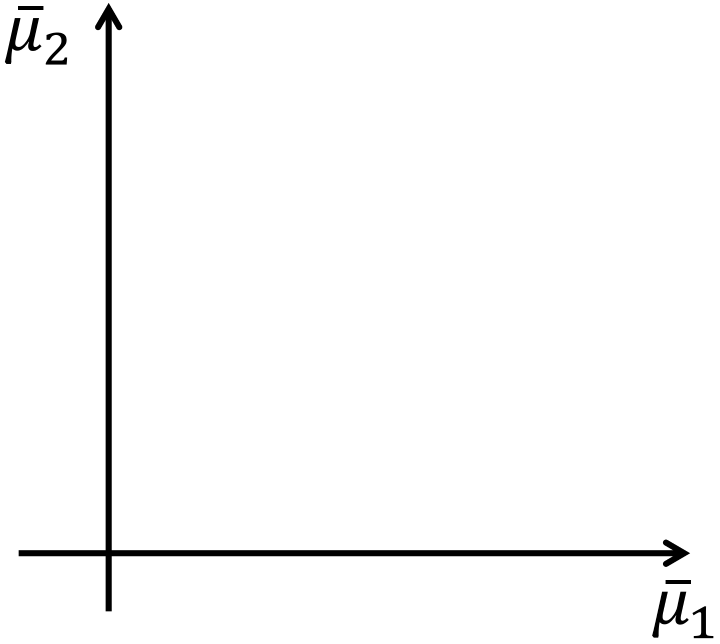
- it schedules links randomly - according to $p_s(I)$ - in order to attain the target time-average packet transmission rates $\left[\bar{\mu}_{i,j}\right]_{i,j}$.
- then, it splits the total rate $\bar{\mu}_{i,j}$ among commodities c, such that the time-average rates $\bar{\mu}_{i,j}^c$ accommodate all flows that pass through link (i,j), namely $\bar{f}_{i,j}^c \leq \bar{\mu}_{i,j}^c, \forall c$.
- this way, it can support all flows and $\lambda_n^c = \sum_{j=1}^N \bar{f}_{n,j}^c - \sum_{i=1}^N \bar{f}_{i,n}^c$

Question: is the randomized policy work-conserving? How can it be throughput optimal? What is its drawback?

Question: any work-conserving policy can stabilize the system?

- Strict priority policy: transmits 1 while $Q_1(t) > 0$ and $s_1(t) = ON$. Transmits 2 otherwise.
- Analysis of transmission rate:
  - Queue 1 has strict priority $\rightarrow \mathbb{E}[\mu_1(t)] = \bar{\mu}_1 = ?$.
  - Queue 2 is served when Queue 1 is not served and $s_2(t) = ON \rightarrow \bar{\mu}_2 = ?$
  - Policy is throughput-optimal?





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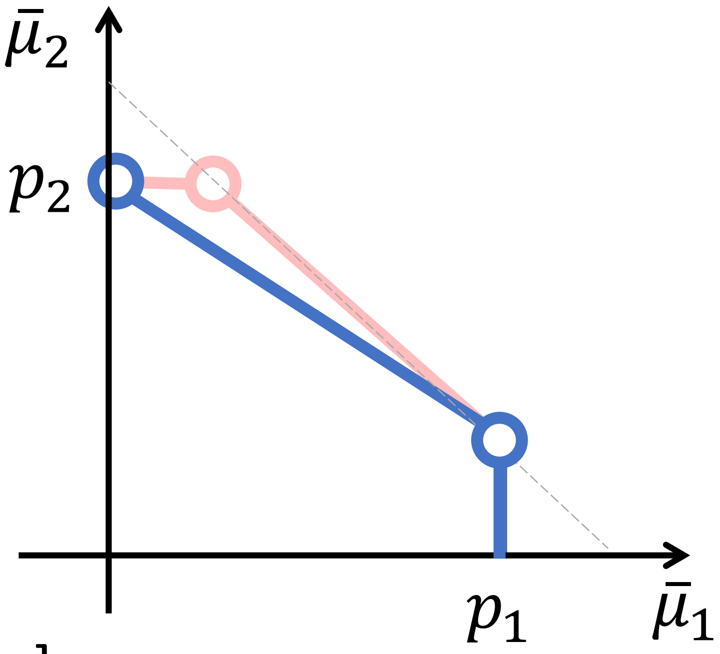
- Analysis of transmission rate:
  - Queue 1 has strict priority → $\mathbb{E}[\mu_1(t)] = \bar{\mu}_1 = \min\{\lambda_1, p_1\}$.
  - Queue 2 is served when Queue 1 is not served and $s_2(t) = ON \rightarrow \bar{\mu}_2 = (1 - \bar{\mu}_1)p_2$
  - Policy is NOT throughput-optimal. See graph →



- Strict priority policy serves Queue 2 only when Queue 1 is empty. What happens if, by then, channel 2 is OFF? The transmission opportunity is lost, since $Q_1(t) = 0$. Policy does not benefit from ***multi-user diversity gain*****.**

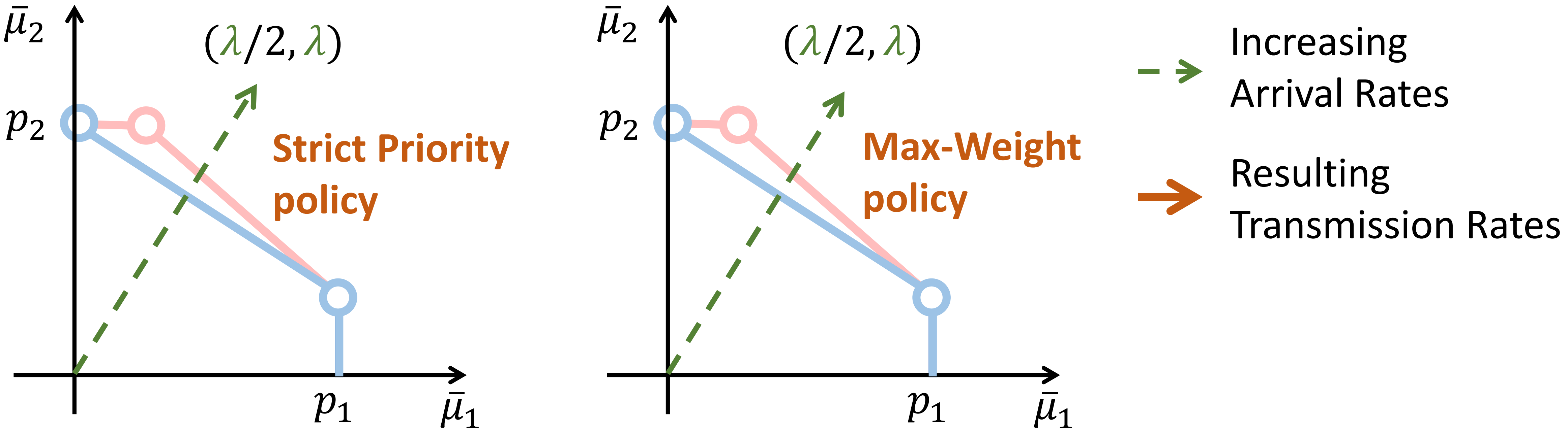
- Max-Weight policy: transmits queue with $s_n(t) = ON$ and largest backlog $Q_n(t)$.
- Max-Weight is throughput-optimal. [to be proven in this lecture] Balances between exploring good channel conditions and serving the largest queue.

Outline

- Multi-commodity flow problem
- Recap from previous lectures
- **Overloaded system**
  - Problem Statement
  - Utility Function
  - Drift-Plus Penalty Algorithm
    - Admission Control
    - Routing Policy
    - Scheduling Policy
  - Performance Analysis and Optimality Results

Overloaded system – Example

- Let $p_1 = 0.5,\ p_2 = 0.6$ and $(\lambda_1, \lambda_2) = (\lambda/2, \lambda)$ for increasing values of $\lambda \geq 0$.
- Outside of the capacity region Λ, both $Q_1(t)$ and $Q_2(t)$ are unstable.



- Strict Priority policy always serves $Q_1(t)$ first. What happens when $\lambda \to \infty$.
- How does the Max-Weight policy behaves?
- Which one is “better”? What would be the desirable outcome?

Overloaded system – Example

- Let $p_1 = 0.5,\ p_2 = 0.6$ and $(\lambda_1, \lambda_2) = (\lambda/2, \lambda)$ for increasing values of $\lambda \geq 0$.
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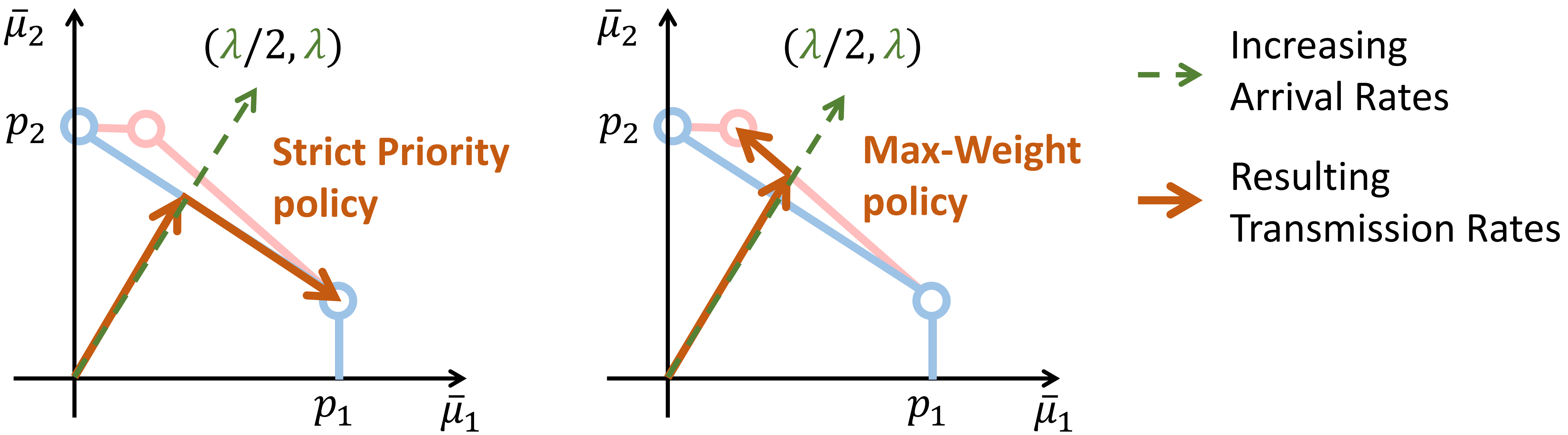
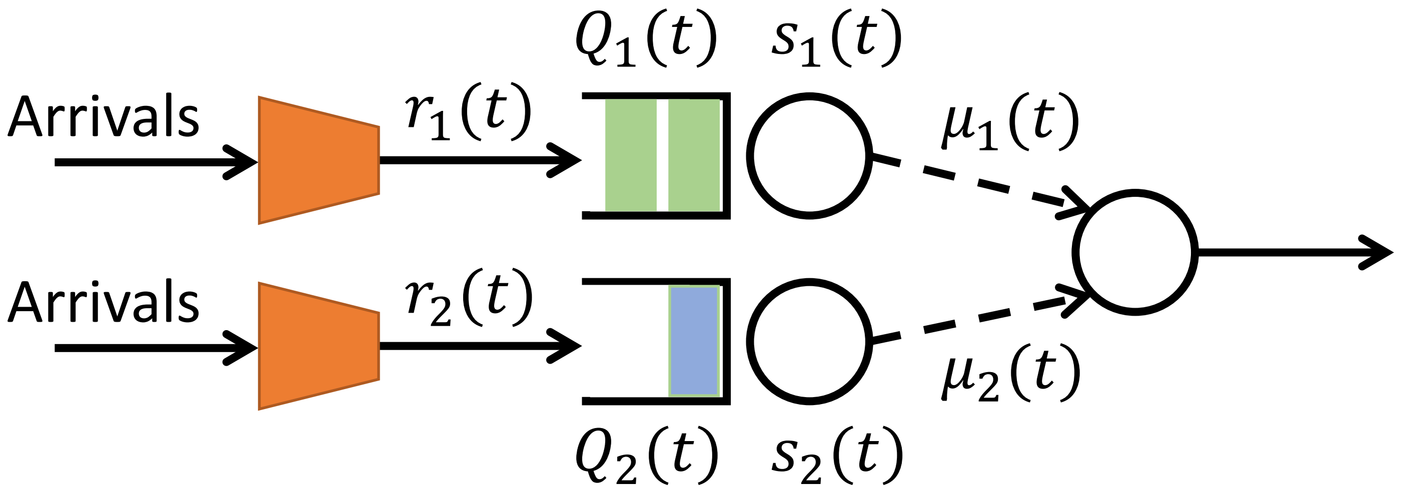


- <u>Strict Priority policy</u> always serves $Q_1(t)$ first. That is why it goes to the RHS when $\lambda \to \infty$.
- <u>Max-Weight policy</u> serves $Q_2(t)$ first because $Q_2(t) > Q_1(t)$ as $t \to \infty$ and $\lambda \to \infty$.
- Which one is "better"? What would be the desirable outcome?

Overloaded system – Definition

- Assume that arrival rates λ_n^c are **infinitely large**. Then all queues $Q_n^c(t)$ are unstable.
- **Admission Control.** Let $r_n^c(t)$ be the number of packets admitted to $Q_n^c(t)$ at time t.
  - Assume that admission is bounded $\sum_{c=1}^{K} r_n^c(t) \leq R_n^{max}, \forall t, n$;
  - Define the time-average admission rate as $\bar{r}_n^c := \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \mathbb{E}\left[r_n^c(\tau)\right], \forall n, c$;



- <u>From another perspective</u>: now we can control packet arrivals to the queues $r_n^c(t)$. Can we utilize admission control to achieve a "desired network behavior"?

Overloaded system – Utility Function

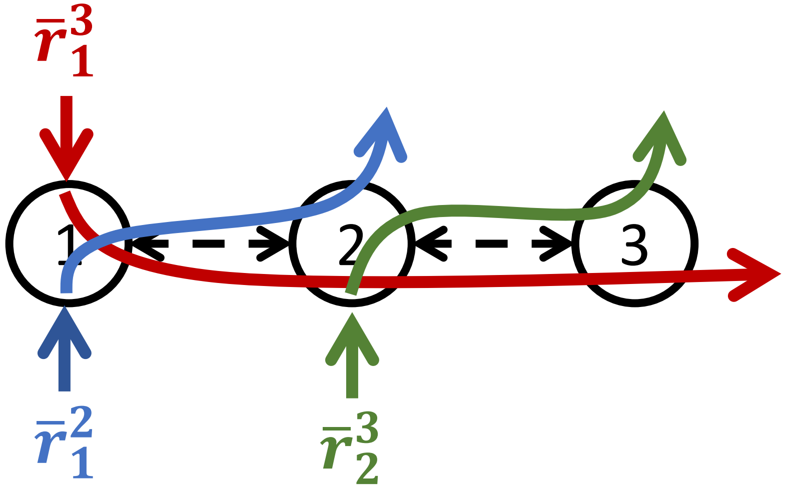
- Let $g_n^c(r)$ be **strictly concave, non-decreasing and continuously differentiable.**
  - Capture satisfaction/utility attained from sending commodity-c at a time-average rate r.
  - Can be used to achieve fairness across commodities and nodes.
  - Good model for elastic flows (e.g. file download). Not good for inelastic flow or for flows with an intrinsic rate (e.g. real-time video).

- Example: consider a network with 3 nodes, 3 flows and equal link capacities of 1. What is the transmission rate distribution that maximizes:

  1) $\bar{r}_1^2 + \bar{r}_1^3 + \bar{r}_2^3$? $(\bar{r}_1^2; \bar{r}_1^3; \bar{r}_2^3) = (1; 0; 1)$

  2) $log(\bar{r}_1^2) + log(\bar{r}_1^3) + log(\bar{r}_2^3)$? *Ans.* $\left(\frac{2}{3}; \frac{1}{3}; \frac{2}{3}\right)$

  3) *Max-min fairness ? Ans.* $\left(\frac{1}{2}; \frac{1}{2}; \frac{1}{2}\right)$



Overloaded system – Goal

- Recall that $\mu_{n,j}^{c}(t)$ is the **offered** transmission opportunity over link (n, j) to commodity-c. Notice that: $\sum_{c=1}^{K} \mu_{n,j}^{c}(t) = \mu_{n,j}(t)$ is the total **offered** transmission opportunity.

- With controlled packet arrivals to the queues $r_n^c(t)$, we have

$$Q_n^c(t+1) \leq \max\left\{Q_n^c(t) - \sum_{j=1}^{N} \mu_{n,j}^{c}(t)\,;0\right\} + \sum_{i=1}^{N} \mu_{i,n}^{c}(t) + r_n^c(t)$$

- **<u>Goal</u>** is to design admission, routing and scheduling algorithms that solve the optimal sum utility problem:

$$\max \sum_{n=1}^{N} \sum_{c=1}^{K} g_n^c(\bar{r}_n^c) \quad s.t.: (\bar{r}_n^c)_{n,c} \in \Lambda \;\text{ and }\; \bar{r}_n^c \geq 0, \forall n, c$$

- Is it possible to use Stationary Randomized Policy? Is it a practical policy?

Lyapunov Optimization

- Lindley recursion: $Q_n^c(t+1) \leq \max\left\{Q_n^c(t) - \sum_{j=1}^{N} \mu_{n,j}^c(t)\,; 0\right\} + \sum_{i=1}^{N} \mu_{i,n}^c(t) + r_n^c(t)$
- Lyapunov Function: $L(t) = \frac{1}{2}\sum_{n=1}^{N}\sum_{c=1}^{K}\left(Q_n^c(t)\right)^2$
- One-slot Lyapunov Drift: $\Delta\big(Q(t)\big) = \mathbb{E}[L(t+1) - L(t)|Q(t)]$
- Drift-Plus Penalty (DPP) Function: $\Delta\big(Q(t)\big) + V\,\mathbb{E}\left[-\sum_{n=1}^{N}\sum_{c=1}^{K} g_n^c\big(r_n^c(t)\big)\,\middle|\,Q(t)\right]$
- Next, we obtain an **upper bound** to the DPP Function and derive an **algorithm that minimizes** this upper bound. By minimizing the upper bound, we aim to achieve low sum of queues backlogs $Q_n^c(t)$ and high sum of utility functions $g_n^c(\bar{r}_n^c)$. **The DPP algorithm is throughput optimal and ensures that utility is arbitrarily close to optimal.**

- Manipulating Lindley Recursion

$$\left(Q_n^c(t+1)\right)^2 \leq \boldsymbol{max}\left\{\boldsymbol{Q_n^c(t)} - \sum_{j=1}^{N} \boldsymbol{\mu_{n,j}^c(t)}\,;\boldsymbol{0}\right\}^{\boldsymbol{2}} + \left(\sum_{i=1}^{N} \mu_{i,n}^c(t) + r_n^c(t)\right)^2 +$$

$$+2\left(\sum_{i=1}^{N} \mu_{i,n}^c(t) + r_n^c(t)\right)\boldsymbol{max}\left\{\boldsymbol{Q_n^c(t)} - \sum_{j=1}^{N} \boldsymbol{\mu_{n,j}^c(t)}\,;\boldsymbol{0}\right\}$$

$$\left(Q_n^c(t+1)\right)^2 - \left(\boldsymbol{Q_n^c(t)}\right)^{\boldsymbol{2}} \leq -\boldsymbol{2Q_n^c(t)}\sum_{j=1}^{N} \boldsymbol{\mu_{n,j}^c(t)} + \left(\sum_{j=1}^{N} \boldsymbol{\mu_{n,j}^c(t)}\right)^{\boldsymbol{2}} +$$

$$+\left(\sum_{i=1}^{N} \mu_{i,n}^c(t) + r_n^c(t)\right)^2 + 2\left(\sum_{i=1}^{N} \mu_{i,n}^c(t) + r_n^c(t)\right)\boldsymbol{Q_n^c(t)}$$

$$\left(Q_n^c(t+1)\right)^2 - \left(Q_n^c(t)\right)^2 \leq -2Q_n^c(t)\left[\sum_{j=1}^{N} \boldsymbol{\mu}_{n,j}^{c}(\boldsymbol{t}) - \sum_{i=1}^{N} \boldsymbol{\mu}_{i,n}^{c}(\boldsymbol{t}) - \boldsymbol{r}_{n}^{c}(\boldsymbol{t})\right] + \\ + \left(\sum_{j=1}^{N} \boldsymbol{\mu}_{n,j}^{c}(\boldsymbol{t})\right)^2 + \left(\sum_{i=1}^{N} \boldsymbol{\mu}_{i,n}^{c}(\boldsymbol{t}) + \boldsymbol{r}_{n}^{c}(\boldsymbol{t})\right)^2$$

- Substituting into the Lyapunov Drift: $\Delta\left(Q(t)\right) = \frac{1}{2}\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[\left(Q_n^c(t+1)\right)^2 - \left(Q_n^c(t)\right)^2 \middle| Q(t)\right]$

$$\Delta(t) \leq -\sum_{n=1}^{N}\sum_{c=1}^{K} Q_n^c(t)\mathbb{E}\left[\sum_{j=1}^{N} \boldsymbol{\mu}_{n,j}^{c}(\boldsymbol{t}) - \sum_{i=1}^{N} \boldsymbol{\mu}_{i,n}^{c}(\boldsymbol{t}) - \boldsymbol{r}_{n}^{c}(\boldsymbol{t}) \middle| Q(t)\right] + B$$

where $B \geq \frac{1}{2}\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[\left(\sum_{j=1}^{N} \boldsymbol{\mu}_{n,j}^{c}(\boldsymbol{t})\right)^2 + \left(\sum_{i=1}^{N} \boldsymbol{\mu}_{i,n}^{c}(\boldsymbol{t}) + \boldsymbol{r}_{n}^{c}(\boldsymbol{t})\right)^2 \middle| Q(t)\right]$

- Assuming that second moments are all bounded, B is a constant.

- Drift-Plus Penalty: consider the expression $\Delta(t) - V\,\mathbb{E}\left[\sum_{n=1}^{N}\sum_{c=1}^{K} g_n^c\left(\boldsymbol{r_n^c(t)}\right) \middle| Q(t)\right]$

$$\Delta(t) - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right] \leq B - \sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[Q_n^c(t)\sum_{j=1}^{N}\boldsymbol{\mu_{n,j}^c(t)} - Q_n^c(t)\sum_{i=1}^{N}\boldsymbol{\mu_{i,n}^c(t)} + \right.$$

$$\left. -Q_n^c(t)\boldsymbol{r_n^c(t)} + V g_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right]$$

$$\Delta(t) - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right] \leq B - \sum_{n=1}^{N}\sum_{c=1}^{K}Q_n^c(t)\mathbb{E}\left[\sum_{j=1}^{N}\boldsymbol{\mu_{n,j}^c(t)} - \sum_{i=1}^{N}\boldsymbol{\mu_{i,n}^c(t)}\middle|Q(t)\right] +$$

$$-\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[-Q_n^c(t)\boldsymbol{r_n^c(t)} + V g_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right]$$

- **DPP algorithm minimizes the upper bound on the RHS at every slot *t*.** The minimization can be separated into **two sub-problems**: i) routing and scheduling $\boldsymbol{\mu_{i,j}^C(t)}$; and ii) admission control $\boldsymbol{r_n^c(t)}$.

Drift-Plus Penalty

The DPP minimization can be separated into two sub-problems:

- Routing & Scheduling: $$\max_{\underline{\mu}(t)\in\Gamma}\left\{\sum_{n=1}^{N}\sum_{c=1}^{K} Q_n^c(t)\mathbb{E}\left[\sum_{j=1}^{N}\boldsymbol{\mu_{n,j}^c(t)}-\sum_{i=1}^{N}\boldsymbol{\mu_{i,n}^c(t)}\,\middle|Q(t)\right]\right\}$$

- Admission Control: $$\max_{r}\left\{\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[-Q_n^c(t)\boldsymbol{r_n^c(t)}+V g_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right]\right\}$$
$$s.t.: \sum_{c=1}^{K}\boldsymbol{r_n^c(t)}\leq R_n^{max},\forall n$$
$$\boldsymbol{r_n^c(t)}\geq 0,\forall n,c$$

Drift-Plus Penalty – Routing & Scheduling

- Routing & Scheduling

$$\max_{\underline{\mu}(t)\in\Gamma}\left\{\sum_{n=1}^{N}\sum_{c=1}^{K} Q_n^c(t)\mathbb{E}\left[\sum_{j=1}^{N}\boldsymbol{\mu_{n,j}^c(t)} - \sum_{i=1}^{N}\boldsymbol{\mu_{i,n}^c(t)}\,\middle|\,Q(t)\right]\right\}$$

$$\max_{\underline{\mu}(t)\in\Gamma}\left\{\sum_{n=1}^{N}\sum_{c=1}^{K}\sum_{j=1}^{N}\mathbb{E}\left[Q_n^c(t)\boldsymbol{\mu_{n,j}^c(t)}\middle|Q(t)\right] - \sum_{n=1}^{N}\sum_{c=1}^{K}\sum_{i=1}^{N}\mathbb{E}\left[Q_n^c(t)\boldsymbol{\mu_{i,n}^c(t)}\middle|Q(t)\right]\right\}$$

$$\max_{\underline{\mu}(t)\in\Gamma}\left\{\sum_{i=1}^{N}\sum_{c=1}^{K}\sum_{j=1}^{N}\mathbb{E}\left[Q_i^c(t)\boldsymbol{\mu_{i,j}^c(t)}\middle|Q(t)\right] - \sum_{j=1}^{N}\sum_{c=1}^{K}\sum_{i=1}^{N}\mathbb{E}\left[Q_j^c(t)\boldsymbol{\mu_{i,j}^c(t)}\middle|Q(t)\right]\right\}$$

$$\max_{\underline{\mu}(t)\in\Gamma}\left\{\sum_{i=1}^{N}\sum_{c=1}^{K}\sum_{j=1}^{N}\left(Q_i^c(t) - Q_j^c(t)\right)\mathbb{E}\left[\boldsymbol{\mu_{i,j}^c(t)}\middle|Q(t)\right]\right\}$$

Drift-Plus Penalty – Routing & Scheduling

Routing & Scheduling
$$\max_{\underline{\mu}(t)\in\Gamma}\left\{\sum_{i=1}^{N}\sum_{c=1}^{K}\sum_{j=1}^{N}\left(Q_i^c(t)-Q_j^c(t)\right)\mathbb{E}\left[\boldsymbol{\mu_{i,j}^c(t)}\middle|Q(t)\right]\right\}$$

Solution [**Backpressure** – presented in previous lectures]:

- Routing: at time t and for every link (i,j), select the commodity with highest differential backlog, namely $c_{i,j}^* = \text{argmax}\{Q_i^c(t) - Q_j^c(t)\}$
- Scheduling: for a given state $s(t) = s$, select action $I(t) = I$ such that the set of transmission rates $\boldsymbol{U}(s, I) = [\boldsymbol{\mu_{i,j}(t)}]_{i,j}$ yields maximum sum:
$$\sum_{(i,j)}\left(Q_i^{c_{i,j}^*}(t) - Q_j^{c_{i,j}^*}(t)\right)\boldsymbol{\mu_{i,j}(t)}$$

Notice that full rate is allocated to commodity $c_{i,j}^*$, namely $\boldsymbol{\mu_{i,j}(t)} = \boldsymbol{\mu_{i,j}^{c_{i,j}^*}(t)}$.

Drift-Plus Penalty – Admission Control

- Admission Control

$$\max_{r}\left\{\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[-Q_n^c(t)\boldsymbol{r_n^c(t)}+Vg_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right]\right\}\quad s.t.:\sum_{c=1}^{K}\boldsymbol{r_n^c(t)}\le R_n^{max},\forall n$$

$$\boldsymbol{r_n^c(t)}\ge 0,\forall n,c$$

- Maximization is **separable into a per-node problem**. At time t, each node n should select the set of values $\boldsymbol{r_n^c(t)}, \forall c$, that solve the problem:

$$\max_{r_n^c(t)}\left\{\sum_{c=1}^{K}\left[Vg_n^c\left(\boldsymbol{r_n^c(t)}\right)-Q_n^c(t)\boldsymbol{r_n^c(t)}\right]\right\}\quad s.t.:\sum_{c=1}^{K}\boldsymbol{r_n^c(t)}\le R_n^{max}$$

$$\boldsymbol{r_n^c(t)}\ge 0,\forall n,c$$

Each node solves the problem independently of other nodes. The objective function is concave and constraints are linear. **How to solve?**

Drift-Plus Penalty – KKT Conditions

- Lagrangean:

$$\mathcal{L}(\boldsymbol{r_n^c(t)}, \eta, \gamma^c) = \sum_{c=1}^{K}\left[V g_n^c\left(\boldsymbol{r_n^c(t)}\right) - Q_n^c(t)\boldsymbol{r_n^c(t)}\right] - \eta\left(\sum_{c=1}^{K}\boldsymbol{r_n^c(t)} - R_n^{max}\right) + \sum_{c=1}^{K}\gamma^c\boldsymbol{r_n^c(t)}$$

where η and γ^c are non-negative KKT multipliers. The KKT Conditions can be written as:

(Stationarity) $\nabla_{\boldsymbol{r_n^c(t)}}\mathcal{L}(.) = V\left(g_n^c\left(\boldsymbol{r_n^c(t)}\right)\right)' - Q_n^c(t) - \eta + \gamma^c = 0$

(Complementary Slackness) $\eta\left(\sum_{c=1}^{K}\boldsymbol{r_n^c(t)} - R_n^{max}\right) = 0$ and $\gamma^c\boldsymbol{r_n^c(t)} = 0, \forall c$

(Primal/Dual Feasibility) $\sum_{c=1}^{K}\boldsymbol{r_n^c(t)} \leq R_n^{max}$ and $\boldsymbol{r_n^c(t)} \geq 0, \forall c$ and $\eta \geq 0$ and $\gamma^c \geq 0, \forall c$

Drift-Plus Penalty – Solution

- From Stationarity: $\nabla_{\boldsymbol{r_n^c(t)}}\mathcal{L}(.) = 0 \;\Rightarrow\; \left(g_n^c\left(\boldsymbol{r_n^c(t)}\right)\right)' = \frac{Q_n^c(t) + \eta - \gamma^c}{V}$ (Eq.1)
- From Complementary Slackness, if $\gamma^c \neq 0$, then $\boldsymbol{r_n^c(t)} = 0$.
- Initially, assume that $\gamma^c = 0, \forall c$. Then we know that $\left(g_n^c\left(\boldsymbol{r_n^c(t)}\right)\right)' = \frac{Q_n^c(t) + \eta}{V}, \forall c$
- Notice that $\left(g_n^c(r)\right)'$ is non-increasing, invertible and **that ↑ $\boldsymbol{\eta}$ always leads to ↓ $\boldsymbol{r}$**.
- Algorithm:
  1) Initialization: $\eta = 0$
  2) Find $\left(\boldsymbol{r_n^c(t)}\right)_{c=1}^{K}$ associated with η using (Eq.1). If $\boldsymbol{r_n^c(t)} < 0$, then set $\boldsymbol{r_n^c(t)} = 0$.
  3) If $\sum_{c=1}^{K} \boldsymbol{r_n^c(t)} \leq R_n^{max}$ then the **unique** solution $\left(\boldsymbol{r_n^c(t)}\right)_{c=1}^{K}$ was found.
  4) Otherwise, increase η slightly and go back to step 2.

Drift-Plus Penalty – Solution (Example)

Example: consider the utility function $g_n^c(\boldsymbol{r_n^c(t)}) = \log(\boldsymbol{r_n^c(t)})$.

- Then $\left(g_n^c(\boldsymbol{r_n^c(t)})\right)' = \dfrac{1}{\boldsymbol{r_n^c(t)}} = \dfrac{Q_n^c(t) + \eta - \gamma^c}{V} \rightarrow \boldsymbol{r_n^c(t)} = \dfrac{V}{Q_n^c(t) + \eta - \gamma^c}, \forall n, c, t$

- *Assuming* $\eta = \gamma^c = 0$ *and that* R_n^{max} *is large enough, the solution is:*

$$\boldsymbol{r_n^c(t)} = \frac{V}{Q_n^c(t)}, \forall n, c, t$$

- Notice that:
  - A larger backlog on the queue ↑ $\mathrm{Q_n^c(t)}$ leads to less packets being admitted to the queue ↓ $\mathbf{r_n^c(t)}$, what makes sense.
  - Larger V implies in larger $\mathbf{r_n^c(t)}$ which, in turn, implies in more network congestion.

Drift-Plus Penalty Algorithm

At every time t, the DPP algorithm runs three steps:

- Routing: for every link (i, j), select the commodity with highest differential backlog, namely

$$c_{i,j}^{*} = \operatorname{argmax}\{Q_i^c(t) - Q_j^c(t)\}$$

- Scheduling: for a given state $s(t) = s$, select action $I(t) = I$ such that the set of transmission rates $\boldsymbol{U}(s, I) = [\boldsymbol{\mu_{i,j}(t)}]_{i,j}$ yields maximum sum:

$$\sum_{(i,j)} \left(Q_i^{c_{i,j}^{*}}(t) - Q_j^{c_{i,j}^{*}}(t) \right) \boldsymbol{\mu_{i,j}(t)}$$

- Admission control: each node n uses its own KKT Conditions to attain the values of $\boldsymbol{r_n^c(t)}$ which solve:

$$\max_{\boldsymbol{r_n^c(t)}} \left\{ \sum_{c=1}^{K} \left[V g_n^c\left(\boldsymbol{r_n^c(t)}\right) - Q_n^c(t) \boldsymbol{r_n^c(t)} \right] \right\} \quad s.t.: \sum_{c=1}^{K} \boldsymbol{r_n^c(t)} \leq R_n^{max} ; \; \boldsymbol{r_n^c(t)} \geq 0, \forall n, c$$

Drift-Plus Penalty Algorithm

- **Admission Control**. The parameter V captures the emphasis on utility maximization. If V is large, then the admitted rates tend to be large, thus increasing the utility, but at the same time increasing the network delay caused by congestion. Notice that admission control on node n only requires information available locally.

- **Routing & Scheduling** is done using the Backpressure policy (already discussed in previous lectures). Recall that routing requires no pre-specified paths, since paths are learned dynamically. Moreover, this algorithm does not need information about arrival rates or channel state statistics.

- Next, we show that the **DPP Algorithm is throughput optimal and ensures that utility is arbitrarily close to optimal.**

Performance Analysis – Randomized Policy

- Prior to analyzing the performance of the DPP algorithm, we assess the **Stationary Randomized Policy** associated with our original utility maximization problem:

$$\max \sum_{n=1}^{N} \sum_{c=1}^{K} g_n^c(\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}}) \quad s.t.: (\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}})_{n,c} \in \Lambda \text{ and } \bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}} \geq 0, \forall n, c$$

- Denote $(\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*})_{n,c}$ as the optimal solution to the utility maximization. A simple admission control algorithm that achieves the optimal solution is: $\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*} = r_n^c(t), \forall t.$

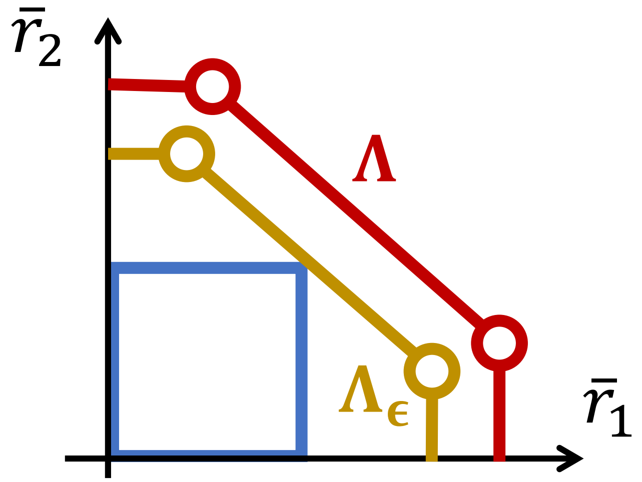
- We know from our discussion of the capacity region that for $(\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*})_{n,c} \in \Lambda,$ there exists flow variables $\bar{f}_{i,j}^c$ such that $\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*} + \sum_{i=1}^{N} \bar{f}_{i,n}^c = \sum_{j=1}^{N} \bar{f}_{n,j}^c$ and a **Stationary Randomized Policy** with time-average packet transmission rates $\bar{\boldsymbol{\mu}}_{\boldsymbol{i,j}}^{\boldsymbol{c}}$ such that $\bar{f}_{i,j}^c \leq \bar{\boldsymbol{\mu}}_{\boldsymbol{i,j}}^{\boldsymbol{c}}, \forall i, j, c.$

- Consider the **Stationary Randomized Policy with rates** $\bar{\boldsymbol{\mu}}_{\boldsymbol{i,j}}^{\boldsymbol{c}} = \bar{f}_{i,j}^c, \forall i, j, c.$

Performance Analysis – Near-optimal solution

- **Near-optimal solution.** Let $\epsilon > 0$ and define the set $\Lambda_\epsilon = \{\bar{r}_n^c \mid (\bar{r}_n^c + \epsilon) \in \Lambda\}$.
- Consider the near-optimal solution $\left(\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}(\boldsymbol{\epsilon})\right)_{n,c}$ to the utility maximization problem:

$$\max \sum_{n=1}^{N} \sum_{c=1}^{K} g_n^c(\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}}) \quad s.t.: (\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}})_{n,c} \in \Lambda_\epsilon \text{ and } \bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}} \geq 0, \forall n, c$$



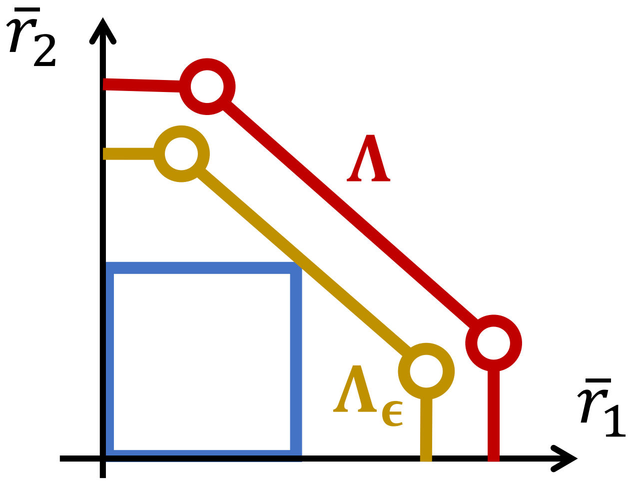
- **Lemma:** if $g_n^c(r)$ are non-negative and concave, and if there is a scalar $r_{min} > 0$ such that a hypercube with edge size r_{min} can fit into Λ, then:

$$\sum_{n=1}^{N} \sum_{c=1}^{K} g_n^c\left(\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}(\boldsymbol{\epsilon})\right) \to \sum_{n=1}^{N} \sum_{c=1}^{K} g_n^c(\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}) \quad \text{as} \quad \epsilon \to 0$$

Performance Analysis – Randomized Policy

- **Near-optimal solution.** Let $\epsilon > 0$ and define the set $\Lambda_\epsilon = \{\bar{r}_n^c \mid (\bar{r}_n^c + \epsilon) \in \Lambda\}$.
- Consider the near-optimal solution $\left(\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}(\boldsymbol{\epsilon})\right)_{n,c}$ to the utility maximization problem:

$$\max \sum_{n=1}^{N} \sum_{c=1}^{K} g_n^c(\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}}) \quad s.t.: (\bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}})_{n,c} \in \Lambda_\epsilon \text{ and } \bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}} \geq 0, \forall n, c$$



- Consider the **Stationary Randomized Policy:** $\bar{r}_n^c(t) = \bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}(\boldsymbol{\epsilon}), \forall n, c, t$ and $\bar{\boldsymbol{\mu}}_{\boldsymbol{i,j}}^{\boldsymbol{c}} = \bar{f}_{i,j}^c, \forall i, j, c$. Recall that $\sum_{i=1}^N \bar{f}_{i,n}^c + \bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*} = \sum_{j=1}^N \bar{f}_{n,j}^c$. Hence, it follows that:

$$\sum_{i=1}^{N} \bar{\boldsymbol{\mu}}_{\boldsymbol{i,n}}^{\boldsymbol{c}} + \bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}(\boldsymbol{\epsilon}) + \epsilon \leq \sum_{j=1}^{N} \bar{\boldsymbol{\mu}}_{\boldsymbol{n,j}}^{\boldsymbol{c}} \Rightarrow \sum_{j=1}^{N} \bar{\boldsymbol{\mu}}_{\boldsymbol{n,j}}^{\boldsymbol{c}} - \sum_{i=1}^{N} \bar{\boldsymbol{\mu}}_{\boldsymbol{i,n}}^{\boldsymbol{c}} - \bar{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}(\boldsymbol{\epsilon}) \geq \epsilon > 0 \quad \text{(Eq2)}$$

- Next, we compare the drift of the DPP algorithm with the drift of the Randomized Policy.

Performance Analysis – DPP Algorithm

- Recall the expression for the Drift-Plus Penalty:

$$\Delta(t) - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right] \leq B - \sum_{n=1}^{N}\sum_{c=1}^{K} Q_n^c(t)\mathbb{E}\left[\sum_{j=1}^{N}\boldsymbol{\mu_{n,j}^c(t)} - \sum_{i=1}^{N}\boldsymbol{\mu_{i,n}^c(t)}\,\middle|Q(t)\right] + -\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[-Q_n^c(t)\boldsymbol{r_n^c(t)} + V g_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right]$$

- By definition, this **upper bound is minimizes by the DPP Algorithm**. Hence, the Stationary Randomized Policy achieves a larger upper bound. Substituting $\boldsymbol{\bar{\mu}_{i,j}^c}$ and $\boldsymbol{\bar{r}_n^{c*}(\epsilon)}$:

$$\Delta(t) - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right] \leq B - \sum_{n=1}^{N}\sum_{c=1}^{K} Q_n^c(t)\left[\sum_{j=1}^{N}\boldsymbol{\bar{\mu}_{n,j}^c} - \sum_{i=1}^{N}\boldsymbol{\bar{\mu}_{n,j}^c}\right] + -\sum_{n=1}^{N}\sum_{c=1}^{K}\left[-Q_n^c(t)\boldsymbol{\bar{r}_n^{c*}(\epsilon)} + V g_n^c\left(\boldsymbol{\bar{r}_n^{c*}(\epsilon)}\right)\right]$$

- By rearranging the upper bound and utilizing (Eq.2), we have:

$$\Delta(t) - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{r_n^c(t)}\right)\middle|Q(t)\right] \leq B - \epsilon\sum_{n=1}^{N}\sum_{c=1}^{K}Q_n^c(t) - \sum_{n=1}^{N}\sum_{c=1}^{K}\left[V g_n^c\left(\boldsymbol{\bar{r}_n^{c*}(\epsilon)}\right)\right]$$

- Taking the expectation w.r.t $Q_n^c(t)$ and using the definition of Lyapunov Drift:

$$\mathbb{E}[L(t+1)] - \mathbb{E}[L(t)] - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{r_n^c(t)}\right)\right] \leq B - \epsilon\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}[Q_n^c(t)] - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{\bar{r}_n^{c*}(\epsilon)}\right)\right]$$

- Summing over $t \in \{0,2,\dots,T-1\}$ and dividing by T gives:

$$\frac{\mathbb{E}[L(T)]}{T} - \frac{\mathbb{E}[L(0)]}{T} - \frac{V}{T}\sum_{t=0}^{T-1}\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{r_n^c(t)}\right)\right] \leq$$

$$\leq B - \frac{\epsilon}{T}\sum_{t=0}^{T-1}\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}[Q_n^c(t)] - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{\bar{r}_n^{c*}(\epsilon)}\right)\right]$$

- Rearranging the expression and knowing that $\mathbb{E}[L(T)]/T$ is non-negative:

$$\frac{\epsilon}{T}\sum_{t=0}^{T-1}\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}[Q_n^c(t)] - \frac{V}{T}\sum_{t=0}^{T-1}\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\boldsymbol{r}_{\boldsymbol{n}}^{\boldsymbol{c}}(\boldsymbol{t})\right)\right] \le B - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\overline{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}(\boldsymbol{\epsilon})\right)\right] + \frac{\mathbb{E}[L(0)]}{T}$$

- Taking the limit $T \to \infty$ and assuming that $\mathbb{E}[L(0)]$ is finite gives:

$$\sum_{n=1}^{N}\sum_{c=1}^{K}\lim_{T\to\infty}\left[\frac{\epsilon}{T}\sum_{t=0}^{T-1}\mathbb{E}[Q_n^c(t)]\right] - \sum_{n=1}^{N}\sum_{c=1}^{K}\lim_{T\to\infty}\left[\frac{V}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[g_n^c\left(\boldsymbol{r}_{\boldsymbol{n}}^{\boldsymbol{c}}(\boldsymbol{t})\right)\right]\right] \le B - V\sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\overline{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}(\boldsymbol{\epsilon})\right)\right]$$

- **Conclusion 1:**

$$\sum_{n=1}^{N}\sum_{c=1}^{K}\lim_{T\to\infty}\left[\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}[Q_n^c(t)]\right] \le \frac{B}{\epsilon} + \frac{V}{\epsilon}\sum_{n=1}^{N}\sum_{c=1}^{K}\left\{\lim_{T\to\infty}\left[\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[g_n^c\left(\boldsymbol{r}_{\boldsymbol{n}}^{\boldsymbol{c}}(\boldsymbol{t})\right)\right]\right] - \mathbb{E}\left[g_n^c\left(\overline{\boldsymbol{r}}_{\boldsymbol{n}}^{\boldsymbol{c}*}(\boldsymbol{\epsilon})\right)\right]\right\} < \infty$$

- **All queues $\boldsymbol{Q_n^c(t)}$ are strongly stable. DPP algorithm is throughput optimal.**

- On the other hand: $$\sum_{n=1}^{N}\sum_{c=1}^{K}\lim_{T\to\infty}\left[\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[g_n^c\left(\boldsymbol{r}_n^c(\boldsymbol{t})\right)\right]\right] \geq \sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\bar{\boldsymbol{r}}_n^{c*}(\boldsymbol{\epsilon})\right)\right] - \frac{B}{V}$$

- By Jensen's inequality and by concavity and continuity of $g_n^c\left(\boldsymbol{r}_n^c(\boldsymbol{t})\right)$, we have:

$$\sum_{n=1}^{N}\sum_{c=1}^{K}\lim_{T\to\infty} g_n^c\left(\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}[\boldsymbol{r}_n^c(\boldsymbol{t})]\right) \geq \sum_{n=1}^{N}\sum_{c=1}^{K}\lim_{T\to\infty}\left[\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[g_n^c\left(\boldsymbol{r}_n^c(\boldsymbol{t})\right)\right]\right] \geq \sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\bar{\boldsymbol{r}}_n^{c*}(\boldsymbol{\epsilon})\right)\right] - \frac{B}{V}$$

$$\sum_{n=1}^{N}\sum_{c=1}^{K} g_n^c\left(\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}[\boldsymbol{r}_n^c(\boldsymbol{t})]\right) \geq \sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c\left(\bar{\boldsymbol{r}}_n^{c*}(\boldsymbol{\epsilon})\right)\right] - \frac{B}{V}$$

- **Conclusion 2:** For $\epsilon \to 0$: $$\sum_{n=1}^{N}\sum_{c=1}^{K} g_n^c(\bar{\boldsymbol{r}}_n^c) \geq \sum_{n=1}^{N}\sum_{c=1}^{K}\mathbb{E}\left[g_n^c(\bar{\boldsymbol{r}}_n^{c*})\right] - \frac{B}{V}$$

- **Larger values of V take the DPP Algorithm arbitrarily close to the optimal utility.**

Topics covered

- Definition of the Multi-commodity flow problem and discussion about Queue Stability, Capacity Region and Stationary Randomized policies.
- Discussion about Utility Function and Fairness.
- Development of the Drift-Plus Penalty algorithm for an overloaded system. In particular, we described an Admission Control Algorithm, a Routing Policy and a Scheduling Policy,
- Performance analysis of the Drift-Plus Penalty algorithm. Under mild assumptions, it was shown to stabilize all queues in the network and at the same time achieve utility which is arbitrarily close to the optimal.

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