Index Policies: Gittins and Whittle Indices

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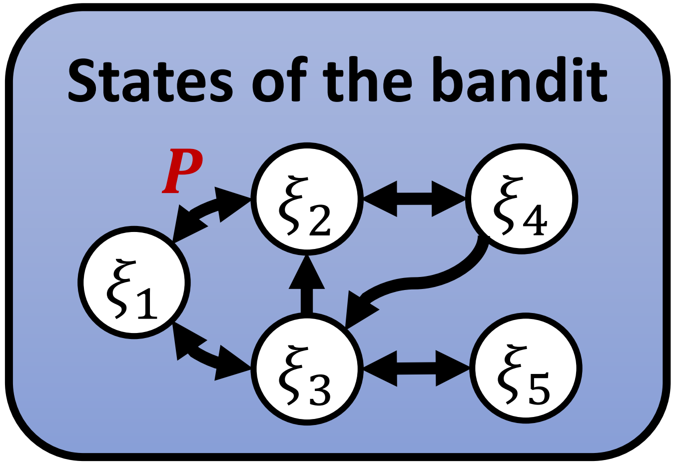
Outline

- Introduction
  - Markov Bandit Process, Objective Function, Examples
- Gittins Index
  - Index Theorem, Derivation of Gittins Index, Examples
- Whittle Index
  - Three optimization problems, Indexability, Whittle Index
  - Application in the Age of Information minimization problem

Markov Bandit Process

- MDP on a countable state space, where $\xi(t) \in \{\xi_1, \dots, \xi_K\}$ is the state of the bandit at the discrete decision time $t \in \{0,1,2,\dots\}$.
- Controls applied at decision time t :
  - $u(t) = 0$ freezes the process and gives no reward;
  - $u(t) = 1$ continues the process and gives instantaneous reward $a^t r(\xi(t))$.

State Transitions are instantaneous with $P(\xi'|\xi)$ **when $\boldsymbol{u(t) = 1}$.**



$a \in (0,1)$ is the discount factor

$r(.) > 0$ is the bounded reward

Simple Family of Alternative Bandit Processes

- n Markov Bandit Processes with state space $\vec{E} = E_1 \times E_2 \times \cdots \times E_n$.
  - Notice that $|\vec{E}|$ is exponential on the number of bandits.
- Control $\boldsymbol{u(t)} = \mathbf{1}$ **is applied to a single bandit $\boldsymbol{i_t}$** at each decision time t.
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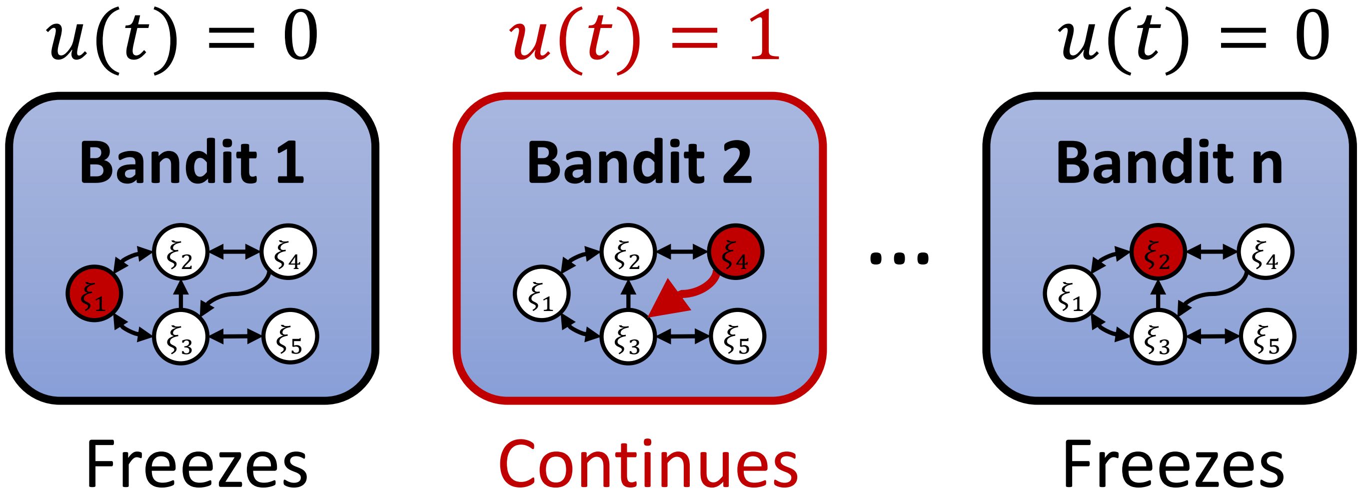

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- Control $\boldsymbol{u(t)} = \mathbf{1}$ **is applied to a single bandit $\boldsymbol{i_t}$** at each decision time t.
  - Control $u(t) = 0$ is applied to all **other bandits.**
- Sequence of selected bandits $\{\boldsymbol{i_1}, \boldsymbol{i_2}, \ldots\}$.
- State of the selected bandit $\boldsymbol{i_t}$ at each decision time t: $\xi_{\boldsymbol{i_t}}(t) = \xi_{\boldsymbol{i_t}}$.
- Reward accrued from the selected bandit: $a^t r_{i_t}\left(\xi_{i_t}\right)$.
- Transition probability $P_{i_t}(\xi' | \xi_{i_t})$. **All other bandits remain in the same state.**

Objective Function

- Problem: sequentially allocate effort between different processes to maximize the **infinite-horizon expected discounted sum of rewards**.

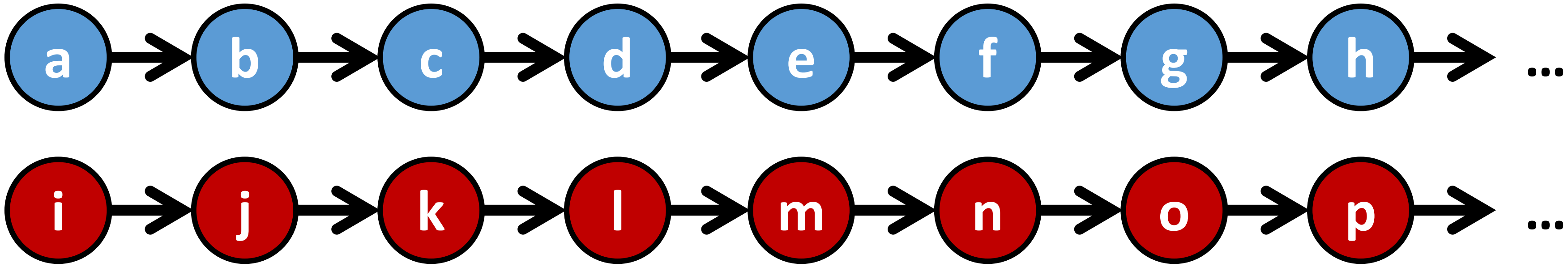
- Maximize:

$$J_\pi(\vec{\xi}\,) = \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T-1} a^t r_{i_t}(\xi_{i_t}) \middle| \vec{\xi}(0) = \vec{\xi}\,\right]$$

- **At time t, we know** the state $\vec{\xi} = [\xi_1, \dots, \xi_n]$, the probabilities $P_i(\xi'|\xi_i)$, the discount factor a and the reward function $r_i(.)$ for each bandit.

Example 1

- Consider 2 bandits, each evolving according to a deterministic state sequence.



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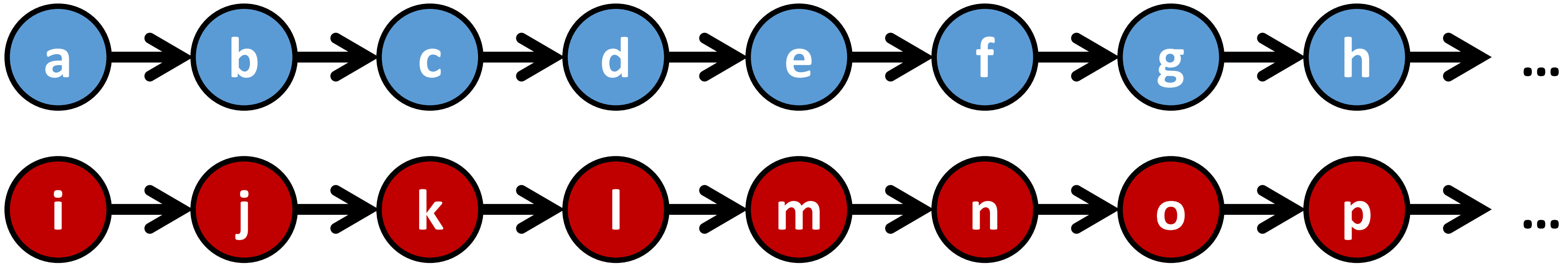

- Let the sequences provide the rewards below:
  - Bandit 1 : { 10 , 9 , 8 , 7 , 6 , 0 , 0 , 0 , … }
  - Bandit 2 : { 5 , 4 , 3 , 2 , 1 , 0 , 0 , 0 , … }

- What is the policy that maximizes $\lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T-1} a^t r_{i_t}(\xi_{i_t})\right]$?

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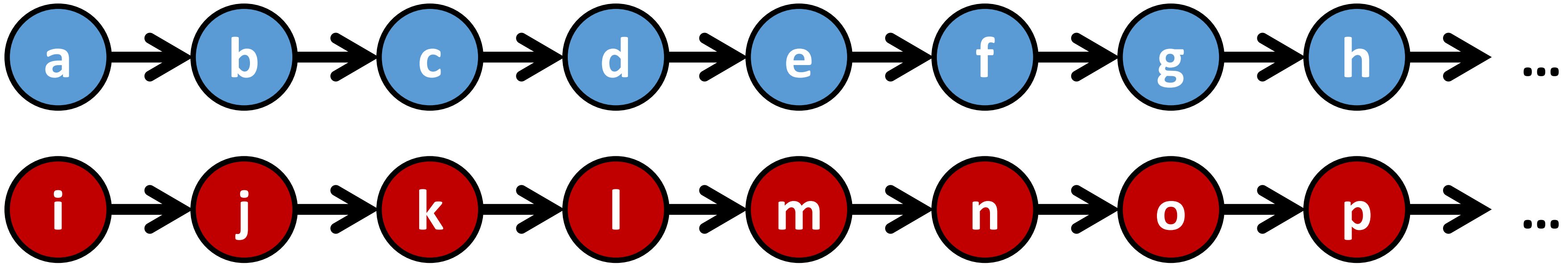

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$$10a^0 + 9a^1 + 8a^2 + 7a^3 + 6a^4 + 5a^5 + \cdots$$

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Policy 1: $10a^0 + ?a^1 + ?a^2 + ?a^3 + ?a^4 + ?a^5 + ?a^6 + \cdots \qquad (a = 0.1)$

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"Future is (almost) as important as the present"

Policy 2: $10a^0 + 2a^1 + 8a^2 + 7a^3 + 6a^4 + 5a^5 + 4a^6 + \cdots$ $(a = 0.9)$

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"Future is somewhat important"

Policy 3: $10a^0 + 5a^1 + 2a^2 + 8a^3 + 7a^4 + 6a^5 + 4a^6 + \cdots \quad (a = 0.5)$

Gittins Index

Multi Armed Bandit Problem

(open problem for almost 40 years)

Index Policy

- Objective is to Maximize:

$$J_\pi(\vec{\xi}\,) = \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T-1} a^t r_{i_t}(\xi_{i_t}) \middle| \vec{\xi}(0) = \vec{\xi}\,\right]$$

- **Index Theorem**: Optimal policy for this problem **is an Index policy.**

- **Index policy:** there exists a function $v_i(\xi_i)$, computed **separately for each bandit**, such that, for every state $\vec{\xi}$, the optimal policy continues the bandit:

$$i_t = \underset{i\in\{1,\ldots,n\}}{\operatorname{argmax}} \{v_i(\xi_i)\}$$

Notice that computing the index is simple, for it only depends on the parameters associated with a single bandit. **But how such function should be designed?**

Derivation of the Index

- How to design a function $v_i(\xi_i)$ that encodes the **value of choosing bandit i** ?
  - Value: present reward + future expected rewards
  - How to consider future reward? Future reward is the expected value of choosing bandit i forever? Or up until a given horizon? How to characterize this horizon?

Derivation of the Index – Single bandit with charge

- Consider a **single** bandit i with a "**playing charge**" of λ.
- Optimal Policy is a **stopping rule**.
  - if at time τ it is optimal to stop, at time $\tau + 1$ it is also optimal to stop.

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$$J(\xi_i) = \max_{\pi} J_\pi(\xi_i) = \sup_{\tau > 0} \mathbb{E}\left[\sum_{t=0}^{\tau-1} a^t [r_i(\xi_i(t)) - \lambda] \middle| \xi_i(0) = \xi_i\right]$$

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- For every ξ_i, there is a λ such that there is a null reward for playing:

$$J(\xi_i) = \mathbf{0}$$

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- Notice that $J(\xi_i)$ is convex and decreasing on λ. Thus, it has a **single root** which is the Gittins Index, $v_i(\xi_i)$, given by:

$$v_i(\xi_i) = \sup_{\tau>0} \frac{\mathbb{E}\left[\sum_{t=0}^{\tau-1} a^t \, r_i(\xi_i(t)) \mid \xi_i(0) = \xi_i \right]}{\mathbb{E}\left[\sum_{t=0}^{\tau-1} a^t \mid \xi_i(0) = \xi_i \right]}$$

Details

- This $v_i(\xi_i)$ is called the **fair charge** during state ξ_i.
- **This is the charge that makes it equally desirable to play and to stop.**

Gittins Index

- Going back to the Simple Family of Alternative Bandit Processes with **n bandits** and **no playing charge**. The Gittins index associated with bandit i in state ξ_i is

$$v_i(\xi_i) = \sup_{\tau>0} \frac{\mathbb{E}\,[\, \sum_{t=0}^{\tau-1} a^t\, r_i(\xi_i(t)) \mid \xi_i(0) = \xi_i \,]}{\mathbb{E}\,[\, \sum_{t=0}^{\tau-1} a^t \mid \xi_i(0) = \xi_i \,]}$$

where τ is the stopping-time.

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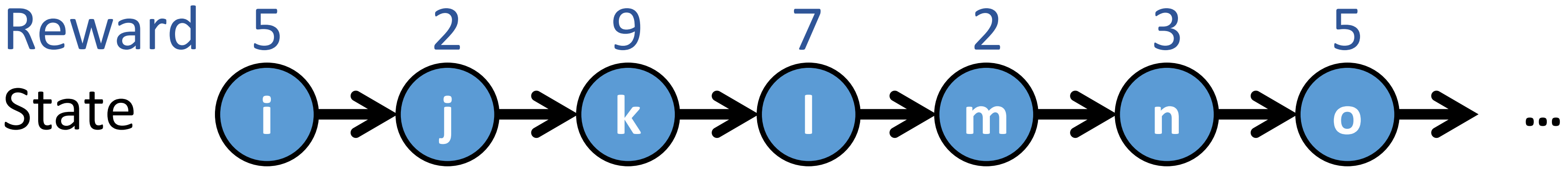

- For $a = 0.5$: “Future is somewhat important”

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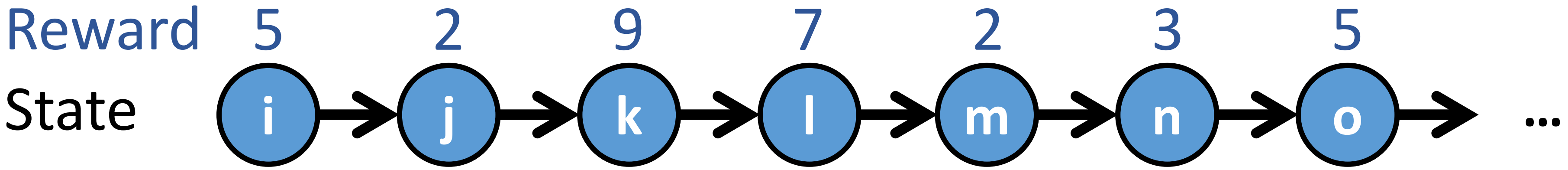

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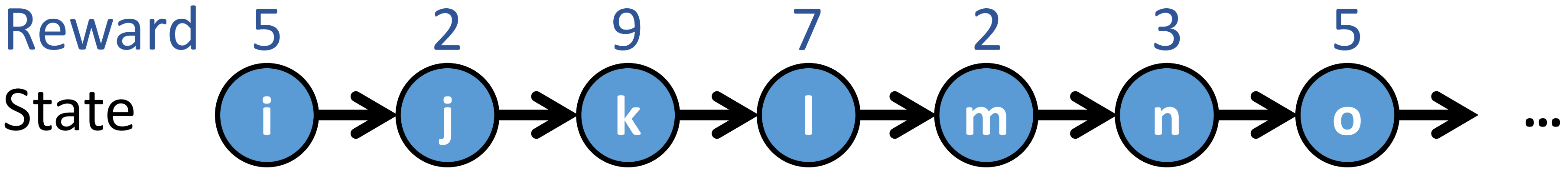

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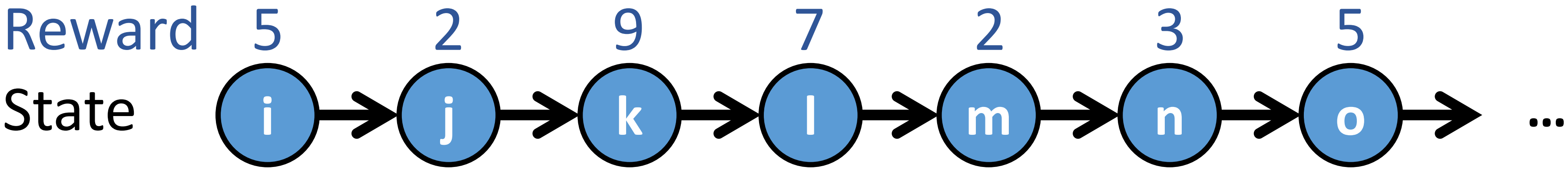

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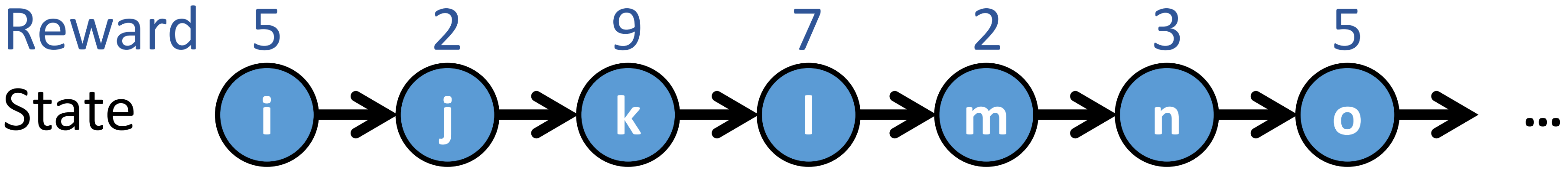

- For $a = 1$: "Future is as important as the present"

$\boldsymbol{\tau}$	1	2	3	**4**	5	6	7
$\boldsymbol{v_i(\xi_i, \tau)}$	5.00	3.50	5.33	**5.75**	5.00	4.67	4.71

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- $v_i(\xi_i)$ a maximum reward per unit time (maximum “reward density”).
- Interpretation from [1]: “greatest **per period rent** that one would be willing to pay for ownership of the rewards arising from the bandit as it is continued for one or more periods.”

[1] J. Gittins, K. Glazebrook and R. Weber, *Multi-armed Bandit Allocation Indices*, 2 Ed., 2011.

Gittins Index

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- Interpretation from [1]: "greatest **per period rent** that one would be willing to pay for ownership of the rewards arising from the bandit as it is continued for one or more periods."
- **GITTINS INDEX POLICY** chooses the bandit with highest $v_i(\xi_i)$ at every decision time t.

Remarks

- In supplemental slides we have the proof that the Gittins Index Policy is optimal. (adapted from [4]).

- This proof is instructive because: 1) provides insight into why the Gittins Index Policy is optimal; and 2) provides insight into why it is NOT optimal for the **restless** case;

- Main ideas in the proof:
  - We always choose the bandit with larger current reward density value.
  - There is no “opportunity cost” since other bandits are frozen.

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Breaks down when bandits are restless, as we see next…

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Whittle Index

Restless Multi Armed Bandit Problem

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- Whittle **extends the notion of index to restless bandits.**
- Generalizations in comparison to the MAB problem:

1. At each time t, exactly **m out of n** bandits are given the action $u = 1$
   Formally, $u_i(t) \in \{0,1\}, \forall i, t$ and $\sum_{i=1}^{n} u_i(t) = m, \forall t$

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2. Action $u = 0$ **no longer freezes the bandit**.
   They **evolve** (possibly) in a distinct way than when $u = 1$.
   They **accrue reward** (possibly) in a distinct way than when $u = 1$.

   Use cases: work / rest and high speed / low speed.

Three Optimization Problems

- **[Original].** Original Problem: **maximize** $\lim_{T\to\infty} \mathbb{E}[\sum_{t=0}^{T-1} a^t \sum_{i=1}^{n} r_i(\xi_i, u_i)]$

  s.t. $\sum_{i=1}^{n} u_i(t) = m, \forall t$

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- **[Relaxed].** Problem with Relaxed activation constraint.

$$\sum_{t=0}^{\infty} a^t \sum_{i=1}^{n} u_i(t) = m/(1-a)$$

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- **[Lagrange]**. The Lagrange Dual Function is given by:

$$\mathcal{L}(\lambda) = \textbf{maximize} \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T-1} a^t \sum_{i=1}^{n} \left(r_i(\xi_i, u_i) - \lambda u_i(t)\right)\right] + \lambda(m/(1-a))$$

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Decoupling the [Lagrange] Problem

- **[Lagrange]**. The Lagrange Dual Function is given by:

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$$\text{s.t. } u_i(t) \in \{0,1\}, \forall i$$

- Notice that we can decouple this problem and neglect the last term (constant). Then, for a fixed $\lambda \geq 0$ and for each bandit, we have:

[Decoupled Problem]

$$\textbf{maximize} \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T-1} a^t \big(r_i(\xi_i, u_i) - \lambda u_i(t)\big)\right]$$

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[Similar to Gittins!]

Solution to the Decoupled Problem

- Main difference when compared to the MAB problem is that **passive bandits may change state and accrue reward**. Thus, the optimal policy for the Decoupled Problem may NOT be a stopping rule.

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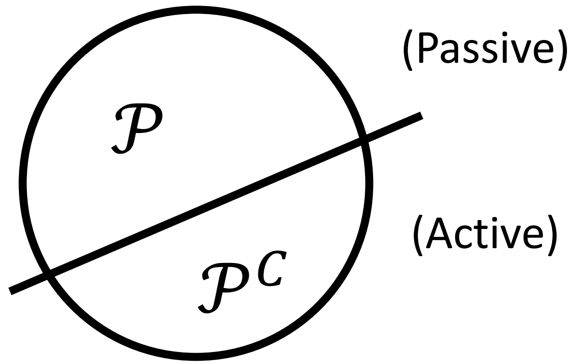
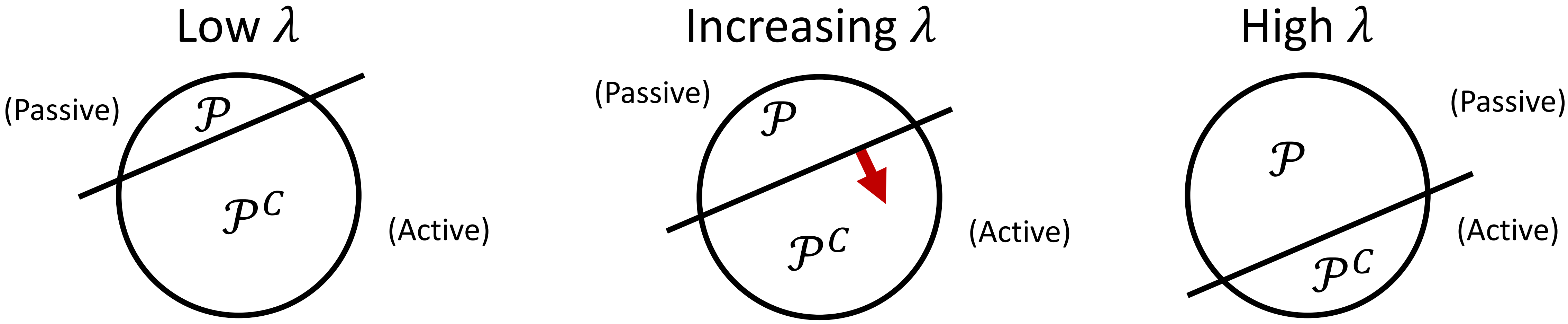
- In general, the optimal policy divides the state space into two subsets:
    - Let $\mathcal{P}(\lambda)$ be the set of ALL states for which it is **optimal to idle** when the playing charge is λ.
    - The set $\mathcal{P}(\lambda)$ is characterized by the solution of the Decoupled Problem.
    - **<u>Optimal Policy</u>**: play, if $\xi_i \in \mathcal{P}^C(\lambda)$; stop, otherwise.

State Space with λ



Indexability

- **<u>Definition of Indexability</u>**: The Decoupled Problem associated with bandit i is *indexable* if $\mathcal{P}(\lambda)$ **increases monotonically from $\emptyset$ to the entire state space** as λ increases from 0 to $+\infty$. The RMAB problem is *indexable* if the Decoupled Problem is *indexable* for all bandits.



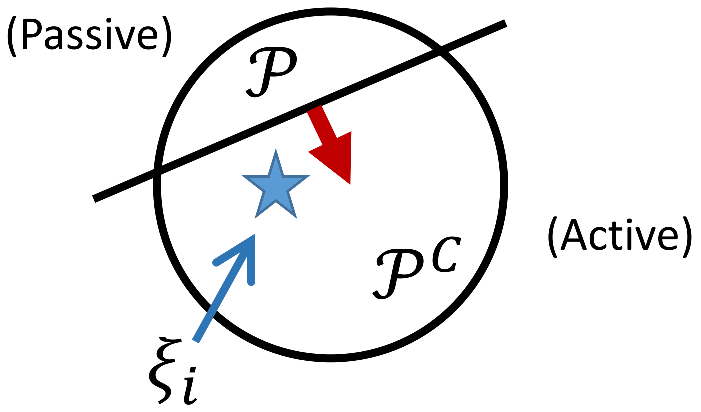
- Means that if a bandit is rested with λ, it should also be rested when $\lambda' > \lambda$.

Whittle Index

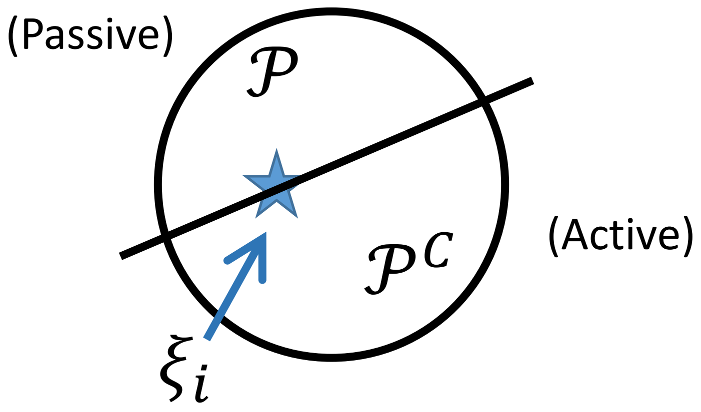
- **<u>Definition of Index</u>**: Consider the Decoupled Problem and denote by $v_i(\xi_i)$ the Whittle Index in state ξ_i. Given *indexability*, $v_i(\xi_i)$ is the **infimum playing charge λ that makes it equally desirable to play and to stop** in state ξ_i.

- Recall that this definition of index is the same as for Gittins. (slide 20)

Increase λ



Until λ' is such that



→ **Then $v_i(\xi_i) = \lambda'$**

Whittle Index Policy

- Going back to our [**Original**] problem:
  - At each time t, exactly **m out of n** bandits are given the action $u = 1$
  - There is no "playing charge" λ.
- The Whittle Index Policy is one that, at every decision time t, **selects the m bandits with higher values of $v_i(\xi_i)$.**
- The **Index Policy is a low-complexity heuristic** that has been extensively used in the literature and is known to have a strong performance in a range of applications.

Whittle Index Policy

- The **challenge** associated with this approach is that the Index Policy is only defined for problems that are *indexable*, a condition that is often difficult to establish. Moreover, it is often hard to find a closed-form expression to $v_i(\xi_i)$.
- Notice that if our RMAB problem is actually a MAB, then **Whittle ≡ Gittins**. Thus, in this case, Whittle is optimal.

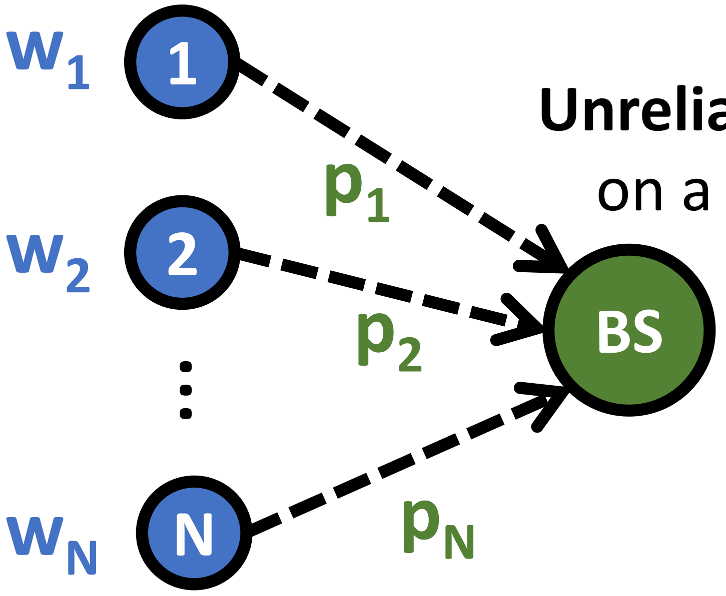
Application of Whittle Index

Age-of-Information Minimization Problem

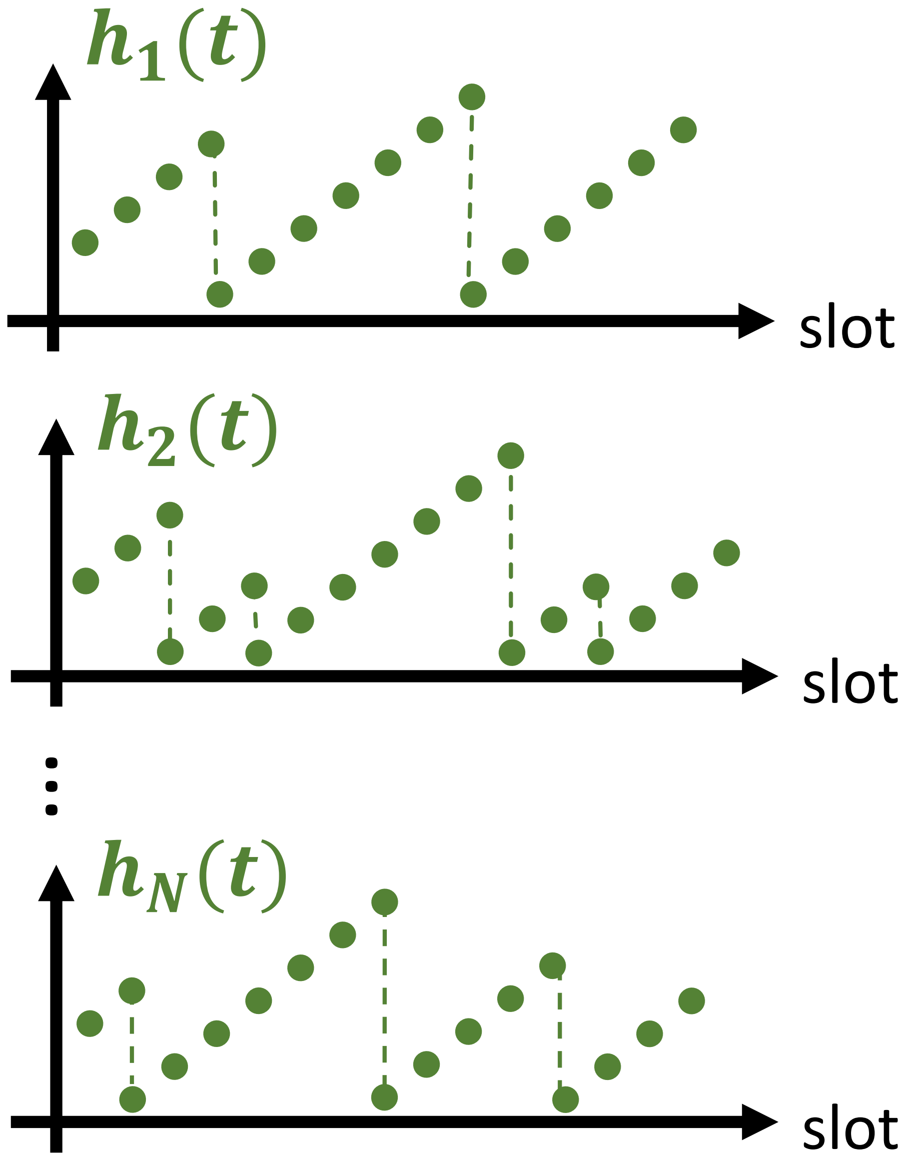
[7] I. Kadota, “Age-of-Information in Wireless Networks: Theory and Implementation”, PhD thesis, 2020.

System Model

Sources (or Bandits) always have packets to transmit



Unreliable transmissions on a **shared** wireless channel



Weight $w_i > 0$ represents **priority** of source i

Probability $p_i \in (0,1]$ represents **quality of the link**

Original Problem

<u>Goal</u>: find a **transmission scheduling policy $\boldsymbol{\pi}^*$** that minimizes

$$\min_{\pi \in \Pi} \left\{ \lim_{T \to \infty} \frac{1}{TN} \sum_{t=1}^{T} \sum_{i=1}^{N} \boldsymbol{w_i} \mathbb{E}[\boldsymbol{h_i^{\pi}(t)}] \right\}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} u_i^{\pi}(t) = 1, \forall t$$

$$u_i^{\pi}(t) \in \{0,1\}, \forall i$$

Relaxed Problem

<u>Goal</u>: find a **transmission scheduling policy $\boldsymbol{\pi}^*$** that minimizes

$$\min_{\pi \in \Pi} \left\{ \lim_{T \to \infty} \frac{1}{TN} \sum_{t=1}^{T} \sum_{i=1}^{N} \boldsymbol{w_i} \mathbb{E}[\boldsymbol{h_i^{\pi}(t)}] \right\}$$

$$\text{s.t.} \ \frac{1}{TN} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathbb{E}[u_i^{\pi}(t)] \leq \frac{1}{N}$$

$$u_i^{\pi}(t) \in \{0,1\}, \forall i$$

Lagrange Dual Function

<u>Goal</u>: find a **transmission scheduling policy $\boldsymbol{\pi}^*$** that minimizes

$$\mathcal{L}(\lambda) = \min_{\pi \in \Pi} \left\{ \lim_{T \to \infty} \frac{1}{TN} \sum_{t=1}^{T} \sum_{i=1}^{N} \left(\boldsymbol{w_i} \mathbb{E}[\boldsymbol{h_i^{\pi}(t)}] + \lambda \mathbb{E}[u_i^{\pi}(t)] \right) \right\} - \frac{\lambda}{N}$$

$$\text{s.t. } u_i^{\pi}(t) \in \{0,1\}, \forall i$$

Notice that the problem can be decoupled…

Decoupled Problem

Goal: find a **transmission scheduling policy $\boldsymbol{\pi}^*$** that minimizes

$$\min_{\pi\in\Pi}\left\{\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left(\boldsymbol{w_i}\mathbb{E}[\boldsymbol{h_i^{\pi}(t)}]+\lambda\mathbb{E}[u_i^{\pi}(t)]\right)\right\}$$

$$\text{s.t. } u_i^{\pi}(t)\in\{0,1\},\forall i$$

$$\lambda\geq 0$$

Optimal policy?

Decoupled Problem

<u>Goal</u>: find a **transmission scheduling policy $\boldsymbol{\pi}^*$** that minimizes

$$\min_{\pi\in\Pi}\left\{\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left(\boldsymbol{w_i}\mathbb{E}[\boldsymbol{h_i^\pi(t)}]+\lambda\mathbb{E}[u_i^\pi(t)]\right)\right\}$$

$$\text{s.t. } u_i^\pi(t)\in\{0,1\},\forall i$$

$$\lambda\geq 0$$

The optimal policy π^* has a threshold structure, namely

transmits when $h_i^\pi(t)\geq H$; and

idles when $h_i^\pi(t)\leq H-1$

Solution to the Decoupled Problem

- The stationary scheduling policy that solves the Decoupled Problem is a threshold policy that, in each decision time t:
  - transmits when $h_i^\pi(t) \geq H$; and
  - idles when $h_i^\pi(t) \leq H - 1$,

  where

$$H = \left\lfloor \frac{3}{2} - \frac{1}{p_i} + \sqrt{\left(\frac{1}{p_i} - \frac{1}{2}\right)^2 + \frac{2\lambda}{w_i p_i}} \right\rfloor$$

Indexability

- For a given value of $\lambda \geq 0$, the set $\mathcal{P}(\lambda)$ of states $h_i^\pi(t)$ in which the threshold policy idles is given by

$$\mathcal{P}(\lambda) = \{h_i^\pi(t) \in \{1,2,3,\dots\} | h_i^\pi(t) \leq H - 1\}$$

where

$$H = \left\lfloor \frac{3}{2} - \frac{1}{p_i} + \sqrt{\left(\frac{1}{p_i} - \frac{1}{2}\right)^2 + \frac{2\lambda}{w_i p_i}} \right\rfloor$$

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where

$$H = \left\lfloor \frac{3}{2} - \frac{1}{p_i} + \sqrt{\left(\frac{1}{p_i} - \frac{1}{2}\right)^2 + \frac{2\lambda}{w_i p_i}} \right\rfloor$$

- Notice that as λ increases from 0 to $+\infty$, the value of H increases from $H = 1$ to $H \rightarrow \infty$ and, thus, $\mathcal{P}(\lambda)$ increases from $\mathcal{P}(\lambda) = \emptyset$ to the entire state space.
- Hence, the Decoupled Problem is indexable for all $i \in \{1,2,\ldots,N\}$.

Whittle's Index

- The index $v_i(h_i^\pi(t))$ is the infimum playing charge λ that makes it equally desirable to play and to stop in state $h_i^\pi(t)$.
- For both scheduling decisions to be equally desirable in state $h_i^\pi(t)$, the threshold should be $H = h_i^\pi(t) + 1$. Hence, by substituting

$$H = \left\lfloor \frac{3}{2} - \frac{1}{p_i} + \sqrt{\left(\frac{1}{p_i} - \frac{1}{2}\right)^2 + \frac{2\lambda}{w_i p_i}} \right\rfloor$$

we obtain the index in closed-form:

$$v_i(h_i^\pi(t)) = \frac{w_i p_i h_i^\pi(t)}{2}\left[h_i^\pi(t) + \frac{2}{p_i} - 1\right]$$

References

[1] J. Gittins, K. Glazebrook and R. Weber, *Multi-armed Bandit Allocation Indices*, 2 Ed., 2011.

[2] R. Weber, *Tutorial on Bandit Processes and Index Policies*, YEQT VII workshop, 2013.

[3] M. Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, 2008.

[4] R. Weber, On the Gittins Index for Multiarmed Bandits, 1992.

[5] R. Weber and Weiss, “On an Index Policy for Restless Bandits”, 1990

[6] P. Whittle, “Restless Bandits: Activity Allocation in a Changing World”, 1981

[7] I. Kadota, “Age-of-Information in Wireless Networks: Theory and Implementation”, PhD thesis, 2020.

Supplementary Slides

General Bandit Process

Bandit Process

- Bandit process is a special type of semi-Markov decision process.
- Continuous time and a succession of (random) decision times $t_1, t_2, t_3, \ldots$
- Same controls applied at decision times
  - $u(t_i) = 0$ freezes the process and gives no reward.
    Time $t_i + \delta$ is another decision time.
  - $u(t_i) = 1$ continues the process and gives instantaneous reward $a^{t_i} r\big(x(t_i)\big)$.
    Time $t_i + s$ is another decision time, where s is drawn from $F(s|y, x)$.

  where $x(t)$ is the current state, y is the next state, $a \in (0,1)$ is the discount factor and r(.) is the positive (and bounded) reward .
- State Transitions are instantaneous with $P(y|x)$.
- Markov bandit process is a Bandit Process with discrete decision times t={0,1,…}

Gittins Index – Proof

Gittins Index – Proof

- Consider a **single** bandit i with a "**playing charge**" of λ.
- Optimal Policy is a **stopping rule**.
  - if at time τ it is optimal to stop, at time $\tau + 1$ it is also optimal to stop.
- **Optimal Reward**:

$$J(\xi_i) = \max_{\pi} J_\pi(\xi_i) = \sup_{\tau > 0} \mathbb{E}\left[\sum_{t=0}^{\tau-1} a^t [r_i(\xi_i(t)) - \lambda] \middle| \xi_i(0) = \xi_i\right]$$

- **Optimal Policy**:

At every decision time, calculate $J(\xi_i)$:

Play, if $J(\xi_i) \geq 0$; Stop, otherwise.

Gittins Index – Proof

- For every ξ_i, there is a λ such that there is a null reward for playing:

$$J(\xi_i) = \sup_{\tau>0} \mathbb{E}\left[\sum_{t=0}^{\tau-1} a^t [r_i(\xi_i(t)) - \lambda] \middle| \xi_i(0) = \xi_i\right] = \mathbf{0}$$

- Notice that $J(\xi_i)$ is convex and decreasing on λ. Thus, it has a **single root** which is the Gittins Index, $v_i(\xi_i)$, given by:

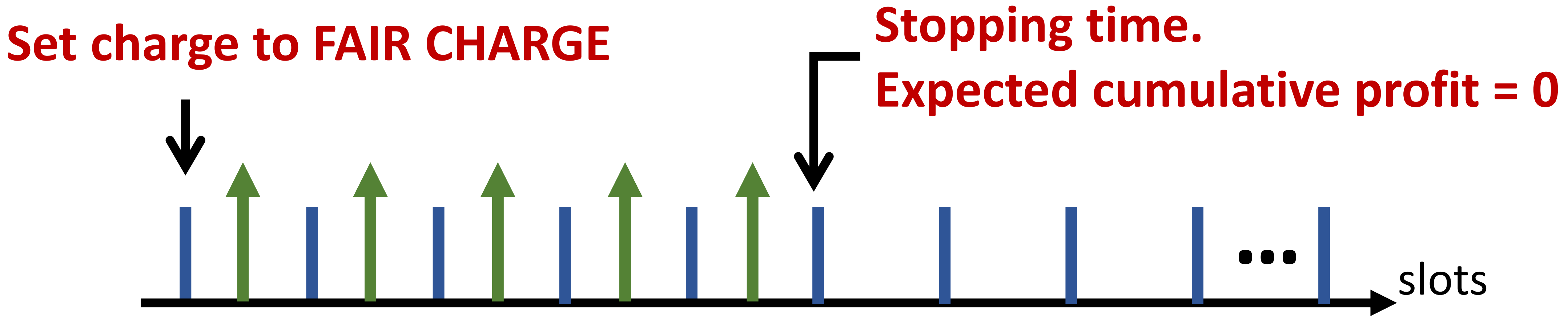
$$v_i(\xi_i) = \sup_{\tau>0} \frac{\mathbb{E}\left[\sum_{t=0}^{\tau-1} a^t \, r_i(\xi_i(t)) \mid \xi_i(0) = \xi_i\right]}{\mathbb{E}\left[\sum_{t=0}^{\tau-1} a^t \mid \xi_i(0) = \xi_i\right]}$$

Details

- This $v_i(\xi_i)$ is called the **fair charge** during state ξ_i.
- **This is the charge that makes it equally desirable to play and to stop.**

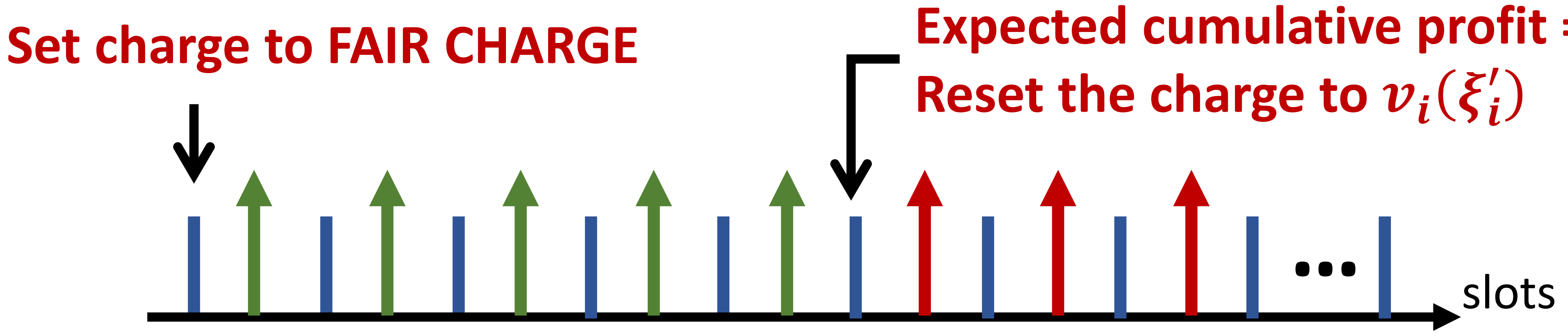
Gittins Index – Proof

- Suppose that at time $t = 0$ we are in state ξ_i with a **fair charge** of $v_i(\xi_i)$.
- If we set $\lambda = v_i(\xi_i)$ and **play bandit i optimally**, we expect 0 profit.
  - Optimal play is not profitable nor loss-making.
- If we deviate from the optimal policy, then we expect loss.
- **What is the optimal policy in this case?** (Stopping rule)



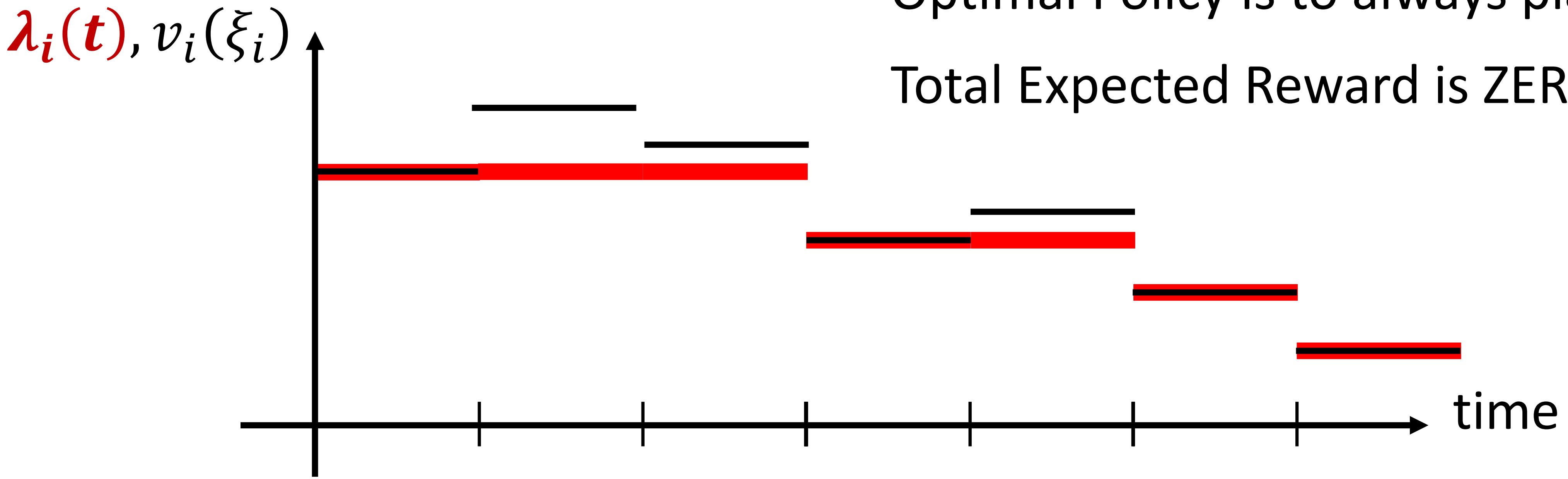
Gittins Index – Proof

- What if **at the stopping time**, we **reset the charge**.
- At the stopping time, instead of stopping, we reset the charge to $v_i(\xi_i')$ and continue playing.
- If we do this **repeatedly**, the expected profit would still be ZERO.
  - The bandit is **continuously playing a fair game** with optimum policy.



Gittins Index – Proof

- Notice that as the game evolves, the charge is reset several times.
- Let $\lambda_i(\mathrm{t})$ be the current fee and $v_i(\xi_i)$ the calculated fair fee.
- $\lambda_i(\mathrm{t})$ is non-increasing and is equal to the minimum fair charge "so far".

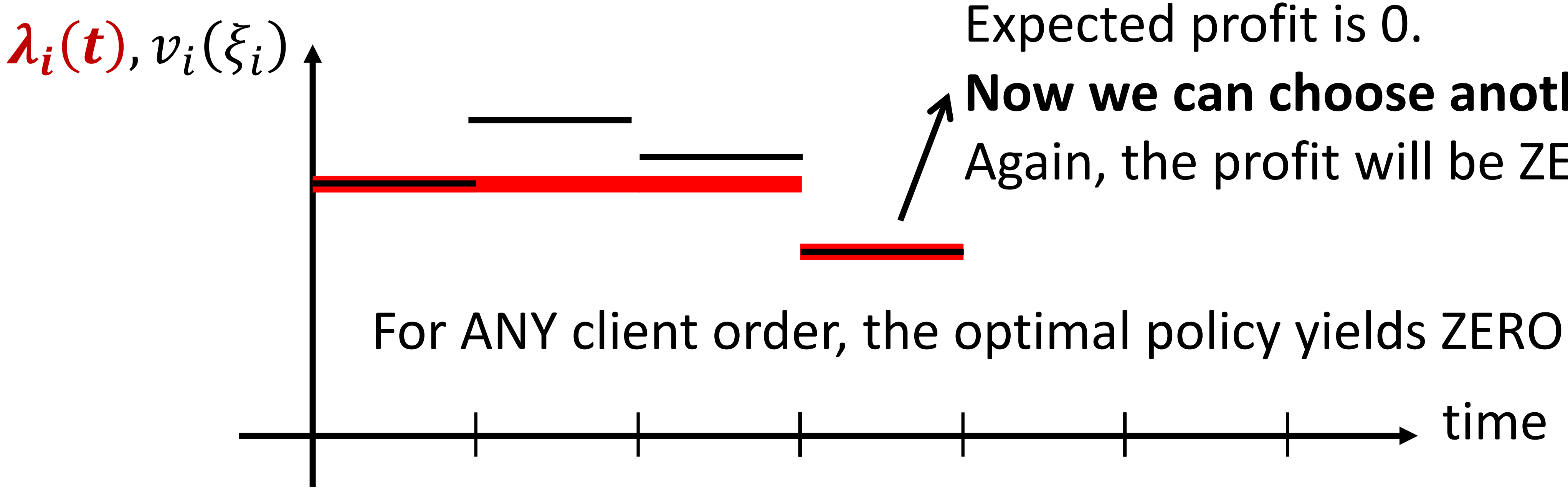


Optimal Policy is to always play.

Total Expected Reward is ZERO.

Gittins Index – Proof

- Consider **n bandits**, each with a different initial state ξ_i.
- We set **each initial charge as $\boldsymbol{\lambda_i = v_i(\xi_i)}$**, $\forall i$ and update them as before.
- Assume we selected bandit i. The optimal policy tells us to play bandit i until **$\boldsymbol{\lambda_i}$ is reset.** If we don't, we will incur in a loss.



Expected profit is 0.
Now we can choose another client.
Again, the profit will be ZERO.

For ANY client order, the optimal policy yields ZERO profit.

Gittins Index – Proof

- Consider the policy that selects the bandit with highest $\lambda_i(t)$ at every slot.
- This policy has NULL profit. And **incurs the HIGHEST sum of discounted charges.**
  - This is because it selects the highest charges first, in a non-increasing order. (recall Example 1 at the beginning of the presentation)
  - Since Profit = Reward – Charges → This policy incurs highest Reward.

- Notice that choosing the bandit with highest $\lambda_i(t)$ is EQUIVALENT to choosing the bandit with highest $\nu_i(\xi_i)$. **Thus the Gittins Index Policy is optimal**. ∎

$J(\xi_i)$ is convex and decreasing on λ

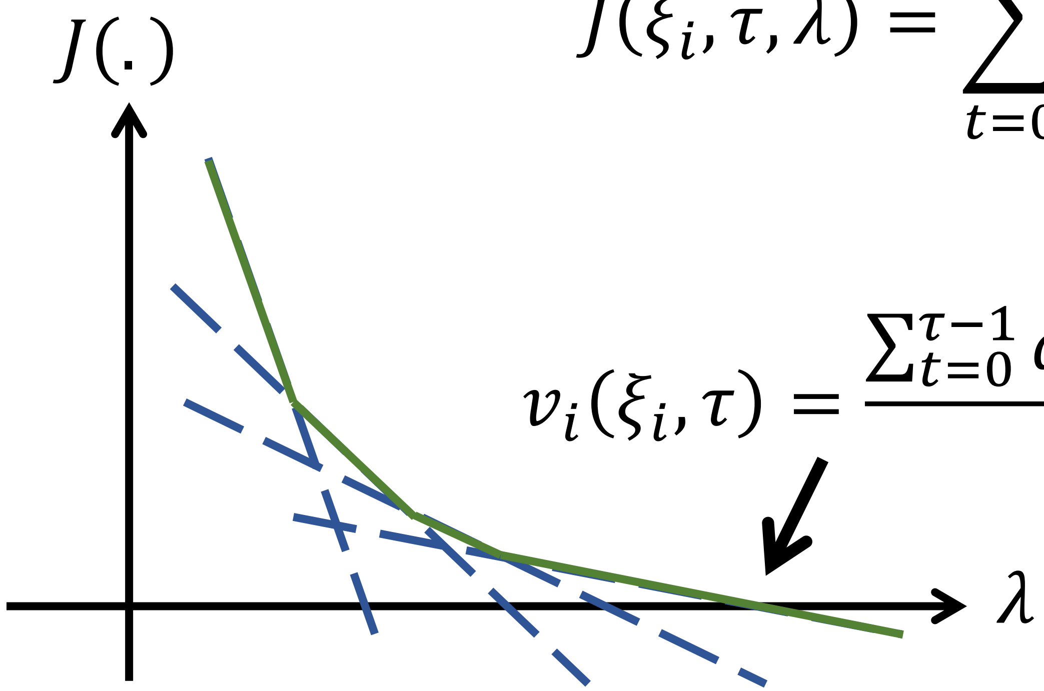
- Equation:

$$J(\xi_i) = \sup_{\tau>0} \mathbb{E}\left[\sum_{t=0}^{\tau-1} a^t [r_i(\xi_i(t)) - \lambda] \middle| \xi_i(0) = \xi_i\right] = 0$$

- For a fixed ξ_i and τ , the function $J(\xi_i, \tau, \lambda)$ is linear and decreasing on λ.

$$J(\xi_i, \tau, \lambda) = \sum_{t=0}^{\tau-1} a^t \mathbb{E}[r_i(\xi_i(t))] - \lambda \sum_{t=0}^{\tau-1} a^t$$

(Dashed blue lines for each τ)



$$v_i(\xi_i, \tau) = \frac{\sum_{t=0}^{\tau-1} a^t \mathbb{E}[r_i(\xi_i(t))]}{\sum_{t=0}^{\tau-1} a^t}$$

The Gittins Index is the highest $v_i(\xi_i, \tau)$

Necessary Conditions and Extensions

Necessary Conditions for Gittins

- Control space is finite
- Infinite Horizon
- Constant exponential discounting
- Single processor/server

[1] J. Gittins, K. Glazebrook and R. Weber, *Multi-armed Bandit Allocation Indices*, 2 Ed., 2011.

Extensions

- Uncountable state space
- Continuous time
- Reward can be unbounded
- Instead of a discounted reward problem, one could formulate the problem as an infinite horizon problem

[1] J. Gittins, K. Glazebrook and R. Weber, *Multi-armed Bandit Allocation Indices*, 2 Ed., 2011.

Asymptotic Optimality

Asymptotic Optimality (for average cost problems)

- **Intuition**: as $n \to \infty$, we expect a weaker coupling among different bandits.
- **Conjecture** [6]: with $m/n = \alpha$ and as $n \to \infty$, the **reward of the optimal policy** is asymptotically the same as the reward achieved by **Whittle's index policy**.
- From [5]: this **conjecture is NOT always satisfied in RMAB**. Using theory of large deviations, [5] derives sufficient conditions for the conjecture to hold. One of which is indexability.
- From [5]: "Evidence so far is that counterexamples to the conjecture are rare and that the degree of sub-optimality is very small. It appears that in most cases the index policy is a very good heuristic."

[5] R. Weber and Weiss, "On an Index Policy for Restless Bandits", 1990
[6] P. Whittle, "Restless Bandits: Activity Allocation in a Changing World", 1981