Problem Set 5. Solutions.

Problem 1

First note that $f$ is continuous: it is continuous in any non-zero point $x$ since $e^x$ and $\cos(x)$ are continuous. If $\{x_n\}$ is a sequence such that $\lim_{n \to \infty} x_n = 0$, then for any $\varepsilon > 0$ there is $N_1$ such that, if $n > N_1, x_n > 0$ we have $|f(x_n) - 1| = |\cos(x_n) - \cos(0)| < \varepsilon$, and there is $N_2$ such that, if $n > N_2, x_n \leq 0$ we have $|f(x_n) - 1| = |e^{x_n} - 1| < \varepsilon$. Taking $N = \max(N_1, N_2)$ we get that $f$ is continuous at 0.

Recall the definition of the uniformly continuous function: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y, |x - y| < \delta, |f(x) - f(y)| < \varepsilon$. So fix $\varepsilon > 0$. Since $e > 2, e^x < 2^x$ for all $x < 0$, so $e^x$ decreases rapidly as $x$ goes from 0 to $-\infty$. More precisely, we can find $x_0 < 0$ such that for all $x < x_0$, $e^x < 2^x < \varepsilon/2$. Note that this implies $|e^x - e^y| \leq |e^x| + |e^y| < \varepsilon$ for all $x, y < x_0$.

Now, since $\cos(x)$ is a periodic continuous function, it is uniformly continuous on $\mathbb{R}$: since every continuous function on a compact set is uniformly continuous (this is called the Heine-Cantor theorem), for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in [-2\pi, 2\pi], |x - y| < \delta, |\cos(x) - \cos(y)| < \varepsilon$, because $[-2\pi, 2\pi]$ is a compact set. Now if $x, y, |x - y| < \delta$ are not necessarily in $[-2\pi, 2\pi]$ we have some $x', y' \in [-2\pi, 2\pi], |x' - y'| < \delta, \cos(x) = \cos(x'), \cos(y) = \cos(y')$: without loss of generality, we can assume $x \leq y$. Suppose $x$ lies in the interval $[n(2\pi), (n + 1)(2\pi)]$ for some integer $n$. Then if $\delta < 2\pi$, then $y \in [n(2\pi), (n + 2)(2\pi)]$. Let $x' = x - (n + 1)(2\pi)$ and let $y' = y - (n + 1)(2\pi)$. Then $\cos(x) = \cos(x'), \cos(y) = \cos(y')$, and $x', y' \in [-2\pi, 2\pi].$

Recall that we fixed $\varepsilon > 0$. Let $S_0 = (-\infty, x_0], S_1 = [x_0 - 1, 1], S_2 = [0, \infty)$. Let $1 > \delta > 0$ be such that $f$ satisfies the condition for uniform continuity for $\varepsilon$ on $S_1$: such $\delta_1$ exist for $S_0, S_2$ as we have shown above, and it exists for $S_1$ by Heine-Cantor theorem since $S_1$ is compact. Let $\delta = \min\{\delta_1\}$. Since any interval of length $\delta$ lies inside one of the sets $S_0, S_1, S_2$, we have that $f$ satisfies the condition of uniform continuity for $\varepsilon, \delta$.

Problem 2

We first prove that $d(f, g)$ is a metric.

\[ d(f, g) = d(g, f) = |f(x) - g(x)| = |g(x) - f(x)|, \]
\[ d(f, g) = 0 \text{ if } f = g: \text{ if there is an } x \in [0, 1] \text{ such that } |f(x) - g(x)| > 0, \sup_{x \in [0, 1]}|f(x) - g(x)| > 0. \]
\[ d(f, g) \leq d(f, h) + d(h, g): \text{ as we have seen in previous problem sets, triangular inequality for a metric of such a form follows from the inequality} \]
sup_{x \in [0, 1]} |f(x) + g(x)| \leq \sup_{x \in [0, 1]} |f(x)| + \sup_{x \in [0, 1]} |g(x)| \quad \text{(via the substitution } f \to f - h, \ g \to h - g). \ \text{But sup}_{x \in \mathbb{R}} |f(x) + g(x)| \leq \sup_{x \in \mathbb{R}} (|f(x)| + |g(x)|) \leq \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \in \mathbb{R}} |g(x)|, \text{ and we are done.}

We now prove that $C([0, 1], d)$ is complete. Let $\{f_n\}$ be a Cauchy sequence. Then for any $x \in [0, 1]$, since $|f_n(x) - f_m(x)| \leq \sup_{x \in [0, 1]} |f_n(x) - f_m(x)|$, we get that $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we can define a function $f(x) = \lim_{n \to \infty} f_n(x)$. We will now prove that $f \in C([0, 1])$ and $\lim_{n \to \infty} f_n = f$. Let $x \in [0, 1]$, $\varepsilon > 0$. We want to prove that there is a $\delta > 0$ such that for any $y \in [0, 1]$, $|x - y| < \delta, \ |f(x) - f(y)| < \varepsilon$. Fix $\varepsilon > 0$. Let $N, \delta$ be such that the following conditions hold:

1. for all $n, m \geq N, d(f_n, f_m) < \varepsilon/4$ (can do, because $\{f_n\}$ is Cauchy);
2. for all $n \geq N, |f_n(x) - f(x)| < \varepsilon/4$ (can do, because $\lim_{n \to \infty} f_n(x) = f(x)$);
3. for all $y \in [0, 1], |x - y| < \delta, |f_N(x) - f_N(y)| < \varepsilon/4$ (can do, because $f_N$ is continuous).

Let $y$ be such that $|x - y| < \delta$. Fix $M_y > N$ such that $|f_{M_y}(y) - f(y)| < \varepsilon/4$ (can do that, because $\lim_{n \to \infty} f_n(y) = f(y)$). We finally get

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f(y)| \leq \\
\varepsilon/4 + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leq \\
\varepsilon/4 + \varepsilon/4 + |f_N(y) - f_M(y)| + |f_M(y) - f(y)| \leq 4\varepsilon/4 = \varepsilon. \quad (1)$$

We just proved that $f \in C([0, 1])$. Let $x \in [0, 1]$. There exists $M_x > N$ such that $|f_{M_x}(x) - f(x)| < \varepsilon/2$ (can do, since $\lim_{n \to \infty} f_n(x) = f(x)$). Then for any $n > N$ we have

$$|f_{n}(x) - f(x)| \leq |f_{n}(x) - f_{m_x}(x)| + |f_{m_x}(x) - f(x)| < 2\varepsilon/2 = \varepsilon. \quad (2)$$

Since this holds for all $x \in [0, 1]$ we get that $d(f_n, f) < \varepsilon$ for $n > N$.

**Problem 3**

Consider the function $f(x) = e^{i\pi(x-1)} := \cos(\pi(x - 1)) + i \sin(\pi(x - 1))$. Since both cos and sin are continuous, $f$ is also a continuous function: if $x_n = a_n + ib_n$ converges to $x = a + ib$, since $\sqrt{x^2 + y^2} \geq \max(|x|, |y|)$ and $d(x_n, x) \geq \max(|a_n - a|, |b_n - b|)$ we get that $a_n$ converges to $a$, $b_n$ converges to $b$, so $f(x_n) = \cos(\pi(x_n - 1)) + i \sin(\pi(x_n - 1))$ converges to $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \cos(\pi(x_n - 1)) + i \lim_{n \to \infty} \sin(\pi(x_n - 1)) = f(x)$. We have $f(0) = \cos(-\pi) = -1, f(1) = i \sin 0 = 1$. Now $f(\epsilon) = 0$ iff $\cos(\pi(\epsilon - 1)) = 0$ and $\sin(\pi(\epsilon - 1)) = 0$, which never happens since cos and sin don’t have common zeroes.
Problem 4

We know from the previous problem set that any subset of a discrete metric space is both open and closed. Now any finite subset of $M$ is compact: from any open cover we can choose a finite subcover picking for each element of a finite set one open set containing it. On the other hand, if $U \subset M$ is infinite, an open cover $\bigcup_{x \in U} \{x\}$ contains no finite subcover, since all the sets in this cover are disjoint and every point of $U$ is contained in exactly one of them. We conclude that a subset of $M$ is compact if and only if it is finite.

Problem 5

Fix $x \in \mathbb{R}$. Several times in the past homework, we have shown that the rational numbers are dense in the reals, i.e. that any open interval in $\mathbb{R}$ contains a rational point. So we can pick a sequence $\{x_n\}, x_n \in \mathbb{Q}$ converging to $x$ (e.g. take $x_n$ to be any rational number with $|x_n - x| < 1/n$). Since $f$ is continuous, we get $f(x) = \lim_{n \to \infty} f(x_n) = 0$.

Problem 6

First we prove that $f$ is continuous at irrational points. Let $x \notin \mathbb{Q}$, and take any $\varepsilon > 0$. There is an integer $n$ such that $\varepsilon > 1/n$. Consider some bounded interval $I$ around $x$. Since $I$ is bounded, for each positive integer $m$ there is only a finite number of rationals of the form $k/m$ in $I$. So there is only a finite number of rationals of the form $k/m$ with $0 < m < n$ in $I$. Since $x$ is not rational, there is an open interval $B(x, \delta) \subset I$ such that none of these finite number of points are in $B(x, \delta)$. Now for any $y \in B(x, \delta)$, $f(y) = 0$ or $f(y) \leq 1/n$, and in both cases $|f(x) - f(y)| = |f(y)| < \varepsilon$, so $f$ is continuous at $x$.

If $x \in \mathbb{Q}$, then $f(x) > 0$. Since any interval in reals contains an irrational point, for $\varepsilon < f(x)$ for any $\delta > 0$ there is $y \notin \mathbb{Q}, |y-x| > 0$, so that $|f(x) - f(y)| = f(x) > \varepsilon$, that is $f$ is not continuous at $x$.

Problem 7

Let $S$ be a convex subset of $\mathbb{R}^n$. Assume that $S = U_1 \cup U_2$ is a union of two disjoint open sets. Note that, by definition, $U_i = U_i' \cap S$, for $U_i'$ open in $\mathbb{R}^n$.

Let $x \in U_1$, $y \in U_2$ and let $I$ be a line segment between $x$ and $y$. Since $S$ is convex, $I \subset S$. We get that $I = (U_1 \cap I) \cup (U_2 \cap I) = (U_1' \cap I) \cup (U_2' \cap I)$, with $x \in (U_1' \cap I), y \in (U_2' \cap I)$, so $I$ is also a union of two non-empty disjoint open sets. But $I$ is a continuous image of a line segment in $\mathbb{R}$, so $I$ is connected, and we get a contradiction.