Problem Set 2. Solutions

Problem 1

If \( f(x) = f(y) \) for some \( x \neq y \in A \) then \( g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y) \), contradicting the injectivity of \( g \circ f \).

In order for \( g \) to be surjective, for any \( z \in C \) there must be \( y \in B \) such that \( g(y) = z \). If \( z \in C \), then, by surjectivity of \( g \circ f \), \( z = g \circ f(x) = g(f(x)) \) for some \( x \in A \), so we can take \( f(x) \) for the required \( y \in B \).

Problem 2

The set of algebraic numbers contains \( \mathbb{Z} \), so it is infinite. We will now prove that it is contained in a countable set, so that it is itself countable.

First note that the set of rational numbers \( \mathbb{Q} \) is countable: it can be expressed as a subset of \( \mathbb{Z} \times \mathbb{N} \) of pairs \((m,n)\) with \((m,n)\) having no common divisors greater than 1 (in particular, if \( m = 0 \), then \( n = 1 \)).

Let \( \mathbb{A} \) be the set of algebraic numbers. For each \( \alpha \in \mathbb{A} \) choose a polynomial \( p_\alpha \) for which \( \alpha \) is a root. Choose a numbering of roots of each \( p_\alpha \); so that \( \alpha \) is encoded by a string \((a_1, \ldots, a_k, r)\), where \( a_1, \ldots, a_k \) are integer coefficients of \( p_\alpha \), \( r \) is a number of among the roots of \( p_\alpha \).

Choose a numbering \((p_0, p_1, \ldots)\) of the set of prime numbers. Consider the following function \( f: \mathbb{A} \to \mathbb{Q} : f(a_1, \ldots, a_k, r) = p_0^r p_1^{a_1} \ldots p_k^{a_k} \). Expression on the right is a fraction with relatively prime numerator and denominator, so \( a_k \) and \( r \) are uniquely recovered, by the uniqueness of prime decomposition. So \( f \) is injective, and \( \mathbb{A} \) is a subset of a countable set.

Problem 3

We will show that any interior point of \( S \) is an interior point of \( \text{int}(S) \), that is that \( \text{int}(S) \subseteq \text{int}(\text{int}(S)) \). Let \( x \in S \) be an interior point. By definition, it is contained in \( S \) with some ball \( B_x(r) \) of radius \( r \) centered in \( x \). Let \( d(x, y) \) denote a distance between points \( x \) and \( y \), and let \( x' \) in \( B_x(r) \) be some other point of this ball. Assume that \( d(x, x') = c \). Then \( B_{x'}(r-c) \subseteq B_x(r) \); if \( z \in B_{x'}(r-c) \), then \( d(x, z) \leq d(x, x') + d(x, z) < c + r - c = r \). So any such \( x' \) is also an interior point of \( S \), and \( B_x(r) \subseteq \text{int}(S) \), so that \( x \in \text{int}(\text{int}(S)) \).

On the other hand, because \( \text{int}(S) \subseteq S \), any interior point of \( \text{int}(S) \) is also an interior point of \( S \), so \( \text{int}(\text{int}(S)) \subseteq \text{int}(S) \). We get that \( \text{int}(\text{int}(S)) = \text{int}(S) \).
Problem 4

Let $x \in S$. Then $B(x, r) \subset S$ for some $r > 0$. Since $S \subset T$, this implies $B(x, r) \subset T$, so $x$ is an interior point of $T$, i.e. every point of $S$ is an interior point of $T$.

Problem 5

Any union of open sets is open, so is the union of open balls.

On the other hand, any point $x$ in an open set $X$ is contained in a ball $B_x \subset X$, so $X = \bigcup_{x \in X} B_x$.

Problem 6

Part A

Let $X = \bigcup_{I \in \mathcal{I}} I$. $X$ is open as a union of open sets, hence $X$ is a union of disjoint intervals. Any interval $I \in \mathcal{I}$ is contained in one of these intervals. This reduces our problem to the case when $X$ is an interval itself. Also note that if $X$ is an infinite interval, then the desired inequality holds automatically for $I'$ containing one of the infinite intervals in $\mathcal{I}$.

If some point of $X$ is contained in three different intervals from $\mathcal{I}$, two of these intervals cover the third one: choose the (not necessarily unique) interval that extends furthest left, and the (not necessarily unique) interval that extends furthest right. Number the intervals of $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$. Let $\mathcal{I}_0 = \mathcal{I}$ and define $\mathcal{I}^{r+1}$ to be $\mathcal{I}^r$ if there is a point in $X$ contained only in $I_r$, or $\mathcal{I}^r \setminus \{I_r\}$ otherwise. From the discussion above we see that $X = \bigcup_{I \in \mathcal{I}^{k+1}} I$, and any point of $X$ is contained in $\leq 2$ of intervals from $\mathcal{I}^{k+1}$. Order $\mathcal{I}^{k+1} = \{I'_1, \ldots, I'_r\}$ by the coordinates of the left ends of the intervals. Let $\mathcal{I}_{\text{odd}} = \{I'_j, j \text{ is odd}\}$, $\mathcal{I}_{\text{even}} = \{I'_j, j \text{ is even}\}$. $\mathcal{I}_{\text{odd}}, \mathcal{I}_{\text{even}}$ are two sets of disjoint intervals. Let $X_1 = \bigcup_{I \in \mathcal{I}_{\text{odd}}} I$ and $X_2 = \bigcup_{I \in \mathcal{I}_{\text{even}}} I$. We have $X = X_1 \cup X_2$, so that $\text{mes}(X) \leq \text{mes}(X_1) + \text{mes}(X_2)$. This gives $\text{mes}(X) \leq 2\text{mes}(X_1)$ or $\text{mes}(X) \leq 2\text{mes}(X_2)$ and we are done.

Part B

Proof of part A shows that the constant of 3 can be improved to 2. Constant 2 in the inequality can’t be improved. Consider the set $\mathcal{I} = \{(0, 1), (1 - \epsilon, 2 - \epsilon)\}$. We must throw away one of the intervals, in both cases we get $\text{mes}(\bigcup \mathcal{I}) = (2 - \epsilon)\text{mes}(\bigcup \mathcal{I})$. This shows that any constant less than 2 will not suffice.