18.100B Problem Set 1. Solutions.

Problem 1
First note that, by Axioms 3 and 4, \( x0 = x(0 + 0) = x0 + x0 \), and hence \( x0 = 0x = 0 \). It follows \(-1x + x = x(-1 + 1) = x0 = 0\), so that \(-1x = -x\).

By Axiom 5 there is an element \( x \) of a field with \( x \neq 0 \) and \( x/x = 1 \). It follows that \( 1 \neq 0 \): otherwise we would have, by Axiom 5, \( x = x0 = 0 \). By Axiom 6 we have that either \( 1 > 0 \) or \( 1 < 0 \). If the latter holds, by Axiom 7 \( 0 = 1 + (-1) < 0 + (-1) = -1 \) and by Axiom 8 \((-1)(-1) = -(-1) = 1 > 0\). This contradicts our assumption that \( 1 < 0 \).

Problem 2
Assume that \( \mathbb{F}_3 \) is ordered. By problem 1 we have that \( 0 < 1 \). It follows, by Axiom 7 (twice), that \( 0 = 0 + 0 < 1 + 1 = 2 \), \( 0 < 1 \), \( 0 < 2 + 1 = 2 = 0 \). This contradicts our assumption that \( 1 < 0 \).

Problem 3
Let \( a = m/n, \gcd(m, n) = 1 \) be an upper bound for \( S \). Note that \( a^2 > 2 \): if \( m^2/n^2 = 2 \) then \( 2n^2 = m^2 \), which is impossible because exponents of 2 in LHS and RHS must have different parity; and if \( a^2 < 2 \), \( a \) is not an upper bound: since \( 5/4 \in S \), we can assume that \( 2 - a^2 < 1, a < 2 \). Take \( \epsilon = (2 - a^2)/8 < 1/8 \). We have \( (a + \epsilon)^2 < a^2 + 2ae + \epsilon^2 < a^2 + 4\epsilon + \epsilon/8 < a^2 + 8\epsilon = 2 \) so \( a < a + \epsilon \in S \).

We then have \( a^2 > 2 \). Note again that since \( 9/4 > 2 \) we can assume that \( a < 2 \). Take \( \epsilon = (a^2 - 2)/4 \) and \( b = a - \epsilon \). Similarly, \( b^2 = (a - \epsilon)^2 = a^2 - 2ae + \epsilon^2 > a^2 - 4\epsilon = 2 \), so \( b < a \) is also an upper bound for \( S \).

Problem 4
Take \( f : \mathbb{N} \to \mathbb{Z} \) to be

\[
f(n) = \begin{cases} 
n/2, & \text{n is even;} 
-(n-1)/2, & \text{n is odd.}
\end{cases}
\]

\( f \) is obviously injective when restricted to even or odd numbers. It takes positive values on even numbers and non-positive values on odd, so is injective on \( \mathbb{N} \). \( f \) is surjective: \( 0 < k = f(2k), 0 \geq k = f(2k + 1) \).
Problem 5

First consider a bijection: \{finite binary strings\} \to \{finite binary strings starting in 1\} taking string \(x\) to string 1\(x\). Then note that the latter set is in bijection with \(\mathbb{N}\) given by binary decomposition.

Now assume that there is a bijection \(f: \mathbb{N} \to \{\text{infinite binary strings}\}\). For an infinite binary string \(\alpha\) let \(\alpha_i \in \{0, 1\}\) be its \(i\)'th symbol. Consider the string \(\beta, \beta_i = NOT(f(i))\), where \(NOT(0) = 1, NOT(1) = 0\). \(\beta\) does not lie in the image of \(f\): if \(\beta = f(k)\), we have that \(\beta_k = NOT(f(k)_k) = NOT(\beta_k) \neq \beta_k\), contradicting our assumption that \(f\) is surjective.