Ordered Fields

1. Using the Field Axioms (Axiom 1-5 from lecture, or from Apostol §1.2) and the order axioms (axiom 6-9 from lecture, or from Apostol §1.3), prove that $1 > 0$. Here 0 and 1 are regarded as real numbers.

2. In this problem we will study a set that satisfies the field axioms but does not satisfy the order axioms. Consider the field $\mathbb{F}_3$. This field has three elements, which we will call 0, 1, 2 (do not confuse these elements with real numbers. We’re just using the labels 0, 1, and 2 for convenience. We could just as easily call the three elements $a, b, c$). Addition and multiplication are defined by the following addition and multiplication tables:

\[
\begin{array}{c|ccc}
+ & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
\times & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 \\
2 & 0 & 2 & 1 \\
\end{array}
\]

Using a proof by contradiction, show that it is impossible to define an operation “$<$” that satisfies the order axioms.

Remark. $\mathbb{F}_3$ is an example of a finite field. Finite fields play an important role in algebra, number theory, and computer science. We will mainly be interested in them because they behave very differently from $\mathbb{R}$, so they help illustrate some of the things that make $\mathbb{R}$ special.

3. Consider the set $S = \{x \in \mathbb{Q} : x^2 < 2\}$. Prove that if $a \in \mathbb{Q}$ is an upper bound for $S$, then there exists a number $b \in \mathbb{Q}$ with $b < a$ so that $b$ is also an upper bound for $S$ (this shows that the rational numbers do not satisfy the least upper bound property).

Sets and cardinality

4. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the natural numbers. Write down a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

5. A binary string is a string of 0s and 1s. Prove that the set of all finite binary strings is countable, but the set of all infinite binary strings is uncountable.