This midterm is 85 minutes, closed book. Solve each problem using the paper provided, and put your full name at the top of each sheet of paper. No clarification will be given for any problems; if you believe a problem is ambiguous, interpret it as best you can and write down any assumptions you feel are necessary.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:</td>
<td>/10</td>
</tr>
<tr>
<td>2:</td>
<td>/10</td>
</tr>
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<td>3:</td>
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<tr>
<td>4:</td>
<td>/10</td>
</tr>
<tr>
<td>Total:</td>
<td>/40</td>
</tr>
</tbody>
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1. Let $S \subset \mathbb{R}^n$. Define

$$T = \{x \in S: \text{ for every } r > 0, \ B(x, r) \cap S \text{ is uncountable}\}.$$ 

Prove that $S \setminus T$ is countable.

Proof. For each $x \in S \setminus T$, there exists $r > 0$ so that $B(x, r) \cap S$ is countable. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we can select a rational number $r_x \in (r/2, r) \cap \mathbb{Q}$, and a rational point $y_x \in B(x, r/2) \cap \mathbb{Q}^n$.

Note that $x \in B(y_x, r_x)$, and $B(y_x, r_x) \subset B(x, r)$, so $B(y_x, r_x) \cap S$ is countable. Define $A = \{(r_x, y_x): x \in S\}$ (note that there could be several distinct $x \in S$ which get mapped to the same pair $(r_x, y_x)$, but $A$ is a set, so it contains each element at most once). Since $A \subset \mathbb{Q}^{n+1}$, $A$ is countable.

Now, since $x \in B(y_x, r_x)$, $x \in \bigcup_{(y_x, r_x) \in A} B(y_x, r_x)$, so $S \setminus T \subset \bigcup_{(y_x, r_x) \in A} B(y_x, r_x) \cap S$. The latter is a countable union of countable sets, so it is countable. We conclude that $S \setminus T$ is countable.
2. Let $f : (a, b) \to \mathbb{R}$ be continuous on $(a, b)$, and suppose that $f'$ exists and is finite at every point of $(a, b)$. Let $c \in (a, b)$ and suppose that $\lim_{x \to c} f'(x)$ exists and is finite. Prove that $\lim_{x \to c} f'(x) = f'(c)$.

**Proof.** Let $L = \lim_{x \to c} f'(x)$. We need to show that $L = f'(c)$. Suppose not, and let $\epsilon = |L - f'(c)|$; we will arrive at a contradiction.

Since $f'(c)$ exists, there exists $\delta_1 > 0$ so that for all $x \in (c - \delta_1, c + \delta_1), x \neq c$, we have

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon/2. \quad (1)$$

By the definition of $L$, there exists $\delta_2 > 0$ so that for all $y \in (c - \delta_2, c + \delta_2), y \neq c$, we have

$$|f'(y) - L| < \epsilon/2. \quad (2)$$

Let $\delta = \min(\delta_1, \delta_2)$; note that $\delta > 0$. Select $x \in (c - \delta, c + \delta), x \neq c$. Since $f'$ is finite on $(a, b)$, $f'(x)$ and $f'(c)$ are finite. Thus we can apply the mean value theorem: there exists a point $y$ between $x$ and $c$ so that $\frac{f(x) - f(c)}{x - c} = f'(y)$. Since $y$ lies between $x$ and $c$, we have $|c - y| < \delta$ so by (2),

$$|f'(y) - L| < \epsilon/2. \quad (3)$$

Thus

$$|f'(c) - L| = \left| f'(c) - \frac{f(x) - f(c)}{x - c} + \frac{f(x) - f(c)}{x - c} - L \right|$$

$$\leq \left| f'(c) - \frac{f(x) - f(c)}{x - c} \right| + \left| \frac{f(x) - f(c)}{x - c} - L \right|$$

$$< \epsilon/2 + |f'(y) - L|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon. \quad (4)$$

On the second line we used the triangle inequality, on the third line we used (1), and on the fourth line we used (3). (4) contradicts the definition $\epsilon = |f'(c) - L|$. We conclude that $f'(c) = L$. 

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Name: ____________
3. Let $M$ be the set of continuous functions $f: [0, 1] \to [0, 1]$, and define

$$d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|.$$  

$(M, d)$ is a complete metric space (you don’t need to prove this). Is this metric space compact? Prove that your answer is correct.

**Solution.** This metric space is not compact. We will construct a sequence in $M$ that has no converging subsequences. Consider the following sequence of functions $\{f_n\}_{n=1}^\infty : f_n(x) = |\sin(2^n \pi x)|$.

Consider $f_i, f_j, i < j$. Then $f_i(\frac{1}{2i+1}) = |\sin(\frac{\pi}{2})| = 1$ and $f_j(\frac{1}{2j+1}) = |\sin(2^{j-i-1}\pi)| = 0$. We get $d(f_i, f_j) \geq 1$ for all $i \neq j$, and so $\{f_n\}$ has no converging subsequences, because it has no Cauchy subsequences.
4. Let \( f : \mathbb{R} \to \mathbb{R} \) and suppose that there exists at least one point at which \( f \) is continuous. Suppose that \( f(x+y) = f(x)+f(y) \) for all \( x, y \in \mathbb{R} \). Prove that there exists a real number \( b \) so that \( f(x) = bx \).

**Solution.** Assume that \( f \) is continuous at a point \( x \). Fix any other point \( y \) and a sequence \( \{y_n\} \) converging to \( y \). Then the sequence \( \{y_n - y + x\} \) converges to \( x \) and we get

\[
\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} f(y_n - y + x - x + y) = \lim_{n \to \infty} f(y_n - y + x) - f(x) + f(y) = f(y),
\]

so \( f \) is continuous at \( y \). Since \( y \) was arbitrary, we get that \( f \) is everywhere continuous.

Now \( f(x) = f(x+0) = f(x) + f(0) \), so \( f(0) = 0 \). We have \( 0 = f(x-x) = f(x) + f(-x) \), so \( f(-x) = -f(x) \). For a positive integer \( n \), \( f(nx) = f(x+...+x) = nf(x) \), and \( nf(x/n) = f(x) \), so for any rational number \( \alpha = p/q, q \neq 0 \), we have \( f(\alpha) = \alpha f(1) \). Fix \( y \in \mathbb{R} \) and let \( \{\alpha_n\} \subset \mathbb{Q} \) be a sequence of rational numbers converging to \( y \). Such sequence exists since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), that is any open interval contains a rational number. Since \( f \) is continuous at \( y \), we get \( f(y) = \lim_{n \to \infty} f(\alpha_n) = \lim_{n \to \infty} \alpha_n f(1) = yf(1) \), so our function is indeed linear: \( f(y) = f(1)y \).

**Bonus (2 points).** If \( f(x+y) = f(x)+f(y) \) for all \( x,y \in \mathbb{R} \), but if we do not assume that there exists at least one point at which \( f \) is continuous, is it still true that there exists a real number \( b \) so that \( f(x) = bx \)? Prove that the answer is yes or provide a counter-example. I recommend you attempt this bonus problem only after you’ve completed the rest of the exam.

**Solution.** This problem requires some linear algebra.

**Linear algebra fact:** any vector space has a linear basis. For infinite-dimensional vector spaces, its construction involves Axiom of Choice.

Consider \( \mathbb{R} \) as a vector space over \( \mathbb{Q} \), and let \( \{e_\alpha\}_{\alpha \in I} \) be its basis. We may require 1 to be an element of this basis. Consider the following \( \mathbb{Q} \)-linear operator \( f : \mathbb{R} \to \mathbb{R} \), given by its matrix coefficients: \( f(1) = 1, f(e_\alpha) = 0 \) for \( e_\alpha \neq 1 \). \( f(x+y) = f(x)+f(y) \), since \( f \) is a linear operator, but \( f \neq bx \) for any \( b \), because \( bx \) is either 0 everywhere or only at one point, namely \( f(0) = 0 \).