The Geometry of Nonlinear Mixed Integer Programming Formulations

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Mixed Integer Convex Optimization (MICP)

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad x \in C \\
& \quad x_i \in \mathbb{Z} \quad i \in I
\end{align*}
\]

Convex \( f \) and \( C \).

• Examples:
  – MI-Second Order Cone Programming (MISOCP)
  – MI-Semidefinite Programming (MISDP)
Standing On The Shoulders of Gigants

Jonathan M. Borwein • Adrian S. Lewis

Convex Analysis and Nonlinear Optimization
Theory and Examples
Second Edition
MI-SOCP Solvers and Applications

• Effective and improving solvers:

     (V., Dunning, Huchette, Lubin, 2017)

• Applications:

   – Portfolio optimization, pricing, regression, experimental design, etc.

http://www.gurobi.com/company/example-customers
MI-SDP (& other cones) Solvers and Applications

• Emerging Solvers:
  
  ![SCIP](image1)
  ![PajaroTc](image2)
  ![JuMP ↔ Julia](image3)

  Coey, Bent, Lubin, V. and Yamangil ‘16

• Applications:
  
  – Collision avoidance with mixed integer sum-of-squares for optimal control of polynomial trajectories
Outline

• General Mixed Integer Convex Representability
  – What can be modeled with MICP?
  – Joint work with Miles Lubin and Ilias Zadik

• 0-1 Mixed Integer Convex Representability
  – Unions of Convex Sets
  – Small and Strong Formulations
A set $S \subseteq \mathbb{R}^n$ is MICP representable (MICPR) if it has an MICP formulation:

- A closed convex set $M \subseteq \mathbb{R}^{n+p+d}$
- auxiliary continuous variables $y \in \mathbb{R}^p$
- auxiliary integer variables $z \in \mathbb{Z}^d$

$x \in S \iff \exists (y, z) \in \mathbb{R}^p \times \mathbb{Z}^d$ s.t.

$$(x, y, z) \in M$$

or equivalently

$$S = \text{proj}_x \left( M \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d) \right)$$
What Sets are MICP Representable (MICPR) ?

Two sheet hyperbola?

\[ \{ x \in \mathbb{R}^2 : 1 + x_1^2 \leq x_2^2 \} \]

Spherical shell?

\[ \{ x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2 \} \]

- Discrete subsets of the real line or natural numbers:
  - Dense discrete set? \( \left\{ \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor : x \in \mathbb{N} \right\} \subseteq [0, 1] \)
  - Set of prime numbers?
A Simple Lemma for non-MICP Representability

• Obstruction for MICP representability of $S$:

infinite $R \subseteq S$ s.t. \[ \frac{u + v}{2} \notin S \quad \forall u, v \in R, u \neq v \]

Proof: Assume for contradiction there exists $M$ such that:

\[ S = \text{proj}_x \left( M \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d) \right) \]

\[ (u, y_u, z_u) \in M \quad \Rightarrow \quad \frac{z_u + z_v}{2} \notin \mathbb{Z}^d \]

\[ (v, y_v, z_v) \in M \]

\[ z_u \equiv z_v \pmod{2} \text{ component-wise} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \in \mathbb{Z}^d \]

component-wise parity classes = $2^d < |R| = \infty \quad \Rightarrow \]
Examples of non-MICPR Sets From Lemma

✗ Spherical shell  \( \{ x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2 \} \)

✗ Set of prime numbers
  • However prime numbers has a non-convex polynomial integer programming formulation

✗ Set of Matrices of rank at most \( k \)

✗ Piecewise linear interpolation of \( x^2 \) at all integers
MICPR = Convex Sets Indexed by Integers in Convex

closed convex $M$

$S = \text{proj}_x (M \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d))$

$I$ convex and $B_z : \mathbb{R}^d \Rightarrow \mathbb{R}^{n+p}$ closed and convex:

- $(w_m \in B_{z_m}, (w_m, z_m) \xrightarrow{m} (w, z)) \Rightarrow w \in B_z$
- $\lambda B_z + (1 - \lambda) B_{z'} \subseteq B_{\lambda z + (1 - \lambda) z'}$
MI-Linear Programming: Rational Polyhedral $M$

\[ S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x (B_z) = \bigcup_{z \in I \cap \mathbb{Z}^d} P_z \]

- $P_z = \text{rational polyhedra with the same recession cone}$
- Representation simplifies to (Jeroslow and Lowe ’84):
  - \[ S = \bigcup_{i=1}^k P_i + \left\{ \sum_{i=1}^t \lambda_i r^i : \lambda \in \mathbb{Z}^t_+ \right\} \]
    - $P_i = \text{rational polytopes}$
    - Very regular infinite union
- Bounded or 0-1 MILPR / MICPR = Bounded $I = \text{Finite union}$
  - MILPR: of polyhedra with the same recession cone
  - MICPR: of non-polyhedral convex sets ...
Extra from MICP 1: Non-Polyhedral Unions

\[ S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x (B_z) \]

1. Unions of Non-Polyhedral sets

Plus Projection:

2. Unions of non-closed sets

3. Unions of convex sets with different recession cones

\[ B_z = \left\{ (x, y) \in \mathbb{R}^{n+1} : x \in C_z, \|x\|_2^2 \leq y \right\} \]

Two sheet hyperbola?

\[ \{ x \in \mathbb{R}^2 : 1 + x_1^2 \leq x_2^2 \} \]
Extra from MICP 2: Non-Polyhedral Index Set

\[ S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x (B_z) \]

"God made the integers, all else is the work of man"
- Leopold Kronecker

• Integers + non-rational unbounded ray = Trouble!

✓ Dense discrete set \[ \left\{ \sqrt{2}x - \left[ \sqrt{2}x \right] : x \in \mathbb{N} \right\} \subseteq [0, 1] \]

\[ \|(z_1, z_1)\|_2 \leq z_2 + 1, \quad \|(z_2, z_2)\|_2 \leq 2z_1, \quad x_1 = y_2 - z_2, \]
\[ \|(z_1, z_1)\|_2 \leq y_1, \quad \|(y_1, y_1)\|_2 \leq 2z_1, \quad z \in \mathbb{Z}^2 \]
One (Somewhat Extreme) Way to Add Regularity

\[ S = \text{proj}_x \left( M \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d) \right), \quad I = \text{proj}_z (M) \]

- \( M = B + K \):
  - \( B \) compact convex set
  - \( K \) rational polyhedral cone

Then

- \( S = \bigcup_{i=1}^{k} C_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i : \lambda \in \mathbb{Z}_+^t \right\} \)
- \( C_i = \) compact convex sets

- Less extreme, but still well behaved “Rational MICPR”:
  - Any rational affine mapping of index set \( I \) is bounded, or has a rational recession direction
Bounded or 0-1 MICP Formulations for Unions of Convex Sets
A Classical Strong Formulation for $\bigcup_{i=1}^{k} C_i$

$C_i = \{ x \in \mathbb{R}^n : A^i x \leq_i b^i \}$, $C_i^\infty = C_j^\infty$

$A^i x^i \leq_i b^i z_i$, $\forall i \in [k]$  

$\sum_{i=1}^{k} x^i = x$, $\sum_{i=1}^{k} z_i = 1$, $z \in \{0, 1\}^k$

$x, x^i \in \mathbb{R}^n$, $\forall i \in [k]$  

- Auxiliary continuous variables are copies of original variables
- $y = (x^i)_{i=1}^{k}$
- “Ideal” Formulation Strength:
  - Extreme points of continuous relaxation satisfy integrality constraints on $z$
  - Variable copies crucial here, but slow down computations (usually worse than Big-M)
- Balas, Jeroslow and Lowe (Polyhedral), Ben-tal, Nemirovski, Helton, Nie (Conic)
Generic Geometric Formulation = Gauge Functions

• For $\mathbf{C}$ such that $0 \in \text{int} (\mathbf{C})$ let:

\[ \gamma_\mathbf{C} (x) := \inf \{ \lambda > 0 : x \in \lambda \mathbf{C} \} \]

\[ \text{epi} (\gamma_\mathbf{C}) = \text{cone} (\mathbf{C} \times \{ 1 \}) \]

• If $b^i \in C_i$ then ideal formulation:

\[ \gamma_{\mathbf{C}^i-\{b^i\}} \left( x^i - z_i b^i \right) \leq z_i \quad \forall i \in [k] \]

\[ \sum_{i=1}^{k} x^i = x \]

\[ \sum_{i=1}^{k} z_i = 1 \]

\[ z \in \{0, 1\}^k \]

\[ x, x^i \in \mathbb{R}^n \quad \forall i \in [k] \]
Simple Ideal Formulation without Variable Copies

- Unions of (nearly) Homothetic Closed Convex Sets (V. 17):

\[ C_i = \lambda_i C + b^i + C^\infty \]

\[
\gamma_C \left( x - \sum_{i=1}^{n} z_i b^i \right) \leq \sum_{i=1}^{n} \lambda_i z_i \\
\sum_{i=1}^{n} z_i = 1, \quad z \in \{0, 1\}^n
\]

\( \approx \) to polyhedral results from Balas ‘85, Jeroslow ‘88 and Blair ‘90
Embedding Formulation Construction

\[ S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x(B_z) \]

\[ M = \text{conv} \left( \bigcup_{z \in I \cap \mathbb{Z}^d} B_z \times \{z\} \right) \]

\[ C_z \]

\[ \begin{aligned}
I \text{ convex and } B_z : \mathbb{R}^d &\Rightarrow \mathbb{R}^{n+p} \text{ closed and convex:} \\
C_z^\infty &= C_{z'}^\infty \\
&\cdot (w_m \in B_{z_m}, (w_m, z_m) \xrightarrow{m} (w, z)) \Rightarrow w \in B_z \\
I \cap \mathbb{Z}^d &\subseteq \{0, 1\}^d \\
&\cdot \lambda B_z + (1 - \lambda) B_{z'} \subseteq B_{\lambda z + (1 - \lambda) z'}
\end{aligned} \]
Embedding Formulation = Automatically Ideal

\[ \text{conv} \left( \bigcup_{i=1}^{k} P^i \times \{h^i\} \right) \]

- Originally for Polyhedra (V. ’17)
  - Small size with careful choice of encoding \( \{h^i\}_{i=1}^{k} \subseteq \{0, 1\}^d \)

- Extensions to general integers, practical construction techniques, computations, applications and software tools:
  - Huchette and V. ‘17a,b,c; Huchette, Dey and V. ‘17
Focus for Non-Polyhedral Embedding Formulations

• **Unary encoding:** \( \{ h^i \}_{i=1}^k = \{ e^i \}_{i=1}^k \subseteq \{0, 1\}^k \)
  
  – Related to Cayley Embedding for Minkoski sums
  
  – Homothetic formulation
  
  \[
  \gamma_C \left( x - \sum_{i=1}^n z_i b^i \right) \leq \sum_{i=1}^n \lambda_i z_i \\
  \sum_{i=1}^n z_i = 1, \; z \in \{0, 1\}^n
  \]

• **How to write convex hull:**

\[ C_1 \quad \rightarrow \quad C_2 \quad \rightarrow \quad Z_1 \]
Sticking Homothetic Formulations Together

$C_1$

$C_2$

Valid, but not ideal!

Combine 4 homothetic formulations

Right relaxations yield ideal formulation
Sufficient Conditions For Ideal Formulation

\[ \sigma_S(u) := \sup\{u \cdot x : x \in S\} \]

\[ \forall u \in \mathbb{R}^n \quad \exists j \quad s.t. \]

\[ \sigma_{C_i}(u) = \sigma_{C_j^i}(u) \quad \forall i \in \{1, 2\} \]
May Need to “Find” Homothetic Constraints

\[ C_1 \quad x_1^2 \leq x_2 \leq 1 \]

\[ C_2 \quad [-1, 1] \times 0 \]

\[ C_1 + (\mathbb{R}_+ \times \{0\}) : \\
(\max\{x_1, 0\})^2 \leq x_2 \leq 1 \]

Similar to Bestuzheva et al. ‘16 who divide sets in two.
Algebraic Representation Issues

\[ C_1 \]

\[ x_1^2 \leq x_2 \leq 1 \]

\[ C_2 \]

\[ [-1, 1] \times 0 \]

\[ C_1 + (\mathbb{R}_+ \times \{0\}) : (\max\{x_1, 0\})^2 \leq x_2 \leq 1 \]

- Non-basic semi-algebraic set contained in formulation.
- Finite polynomial inequalities requires max or auxiliary vars.
Summary

• General mixed integer convex representability (MICPR):
  – Infinite union of convex sets with special structure
  – More results/questions on regularity (arXiv:1706.05135)
• Bounded MICPR = Finite unions of Convex Sets
  – Variable copies = strong (ideal), but slow computation
  – Copies can be removed, but possibly at a prize
    • MIP-solver compatible formulations = gauge calculus.
    • More examples: generalizations and size reductions
    • Conditions for piecewise formulations to be ideal