Embedding Formulations for Unions of Convex Sets

Small and Strong Formulations for Unions of Convex Sets
from the Cayley Embedding

Juan Pablo Vielma

Massachusetts Institute of Technology

SIAM Conference on Optimization,

Supported by NSF grant CMMI-1351619
0-1 Mixed Integer Convex (MICP) Formulations

• 0-1 MICP = Unions of Closed Convex Sets, even with different recession cones (M. Lubin, I. Zadik, V. ‘16).

\[ x \in \bigcup_{i=1}^{n} C_i \subseteq \mathbb{R}^d \]

• General Integer MICP = much more complicated!
  – \( S = \{1\} \cup 2\mathbb{N} \) is not general MILP rep., but is MICP rep.
  – Prime numbers is not MICP rep., but is non-convex MIP rep.
  – See IPCO talk by Miles and Ilias in June at Waterloo.
"Extended" / Non-Extended Formulations for $\bigcup_{i=1}^{n} C_i$

$C_i = \{ x \in \mathbb{R}^d : A^i x \leq b^i \}$

"Extended" $\equiv$ Variable Copies

Non-Extended

$$A^i x^i \leq b^i y_i \quad \forall i \in [n]$$

$$\sum_{i=1}^{n} x^i = x$$

$$\sum_{i=1}^{n} y_i = 1$$

$y \in \{0, 1\}^n$

$x, x^i \in \mathbb{R}^d \quad \forall i \in [n]$  

Small? and strong (ideal*)
Speed: worse than expected

Small, but weak?
Speed: better than expected

*Integral $y$ in extreme points of LP relaxation
Non-Polyhedral = Different Representations

e.g. Ceria and Soares ‘99

\[ C_i = \{ x \in \mathbb{R}^d : f_i(x) \leq 0 \} \]

\[ \tilde{f}(x, y) = \begin{cases} 
  yf(x/y) & \text{if } y > 0 \\
  \lim_{\alpha \downarrow 0} \alpha f(x' - x + x/\alpha) & \text{if } y = 0 \\
  +\infty & \text{if } y < 0 
\end{cases} \]

\[ \tilde{f}_i(x^i, y_i) \leq 0 \quad \forall i \in [n] \]

\[ \sum_{i=1}^{n} x^i = x \]

\[ \sum_{i=1}^{n} y_i = 1 \]

\[ y \in \{0, 1\}^n \]

\[ x, x^i \in \mathbb{R}^d \quad \forall i \in [n] \]

e.g. Ben-tal and Nemirovski ’01

\[ C_i = \left\{ x \in \mathbb{R}^d : \exists u \in \mathbb{R}^{p_i} \text{ s.t.} \ A^i x + D^i u - b \in K^i \right\} \]

\[ K^i \text{ closed convex cone} \]

\[ A^i x^i + D^i u^i - b y_i \in K^i \quad \forall i \in [n] \]

\[ \sum_{i=1}^{n} x^i = x \]

\[ \sum_{i=1}^{n} y_i = 1 \]

\[ y \in \{0, 1\}^n \]

\[ x, x^i \in \mathbb{R}^d \quad \forall i \in [n] \]

\[ u^i \in \mathbb{R}^{p_i} \quad \forall i \in [n] \]
Generic Formulation Through Gauge Functions

• For \( C \) such that \( 0 \in \text{int} (C) \) let:
  \[
  \gamma_C (x) := \inf \{ \lambda > 0 : x \in \lambda C \}
  \]
  \[
  \text{epi} (\gamma_C) = \]

• If \( b^i \in C_i \) then ideal formulation:

  \[
  \gamma_{C^i - \{b^i\}} \left( x^i - y^i b^i \right) \leq y_i \quad \forall i \in [n]
  \]
  \[
  \sum_{i=1}^{n} x^i = x
  \]
  \[
  \sum_{i=1}^{n} y_i = 1
  \]
  \[
  y \in \{0, 1\}^n
  \]
  \[
  x, x^i \in \mathbb{R}^d \quad \forall i \in [n]
  \]
Simple Non-Extended Ideal Formulation

- Unions of (nearly) Homothetic Closed Convex Sets:

\[ C_i = \lambda_i C + b^i + C_\infty \]

\[ \gamma_C \left( x - \sum_{i=1}^{n} y_i b^i \right) \leq \sum_{i=1}^{n} \lambda_i y_i \]

\[ \sum_{i=1}^{n} y_i = 1, \ y \in \{0, 1\}^n \]
Sticking Homothetic Formulations Together

Combine 4 homothetic formulations

Valid, but not ideal!

Right relaxations yield ideal formulation
Sufficient Conditions For Ideal Formulation

\[ \sigma_S(u) := \sup \{ u \cdot x : x \in S \} \]

\[ \forall u \in \mathbb{R}^n \ \exists j \]

\[ s.t. \]

\[ \sigma_{C_i}(u) = \sigma_{C_i^j}(u) \]

\[ \forall i \in \{1, 2\} \]

Similar to “lifting” of e.g. Tawarmalani et al. ‘10
May Need to “Find” Homothetic Constraints

\[ C_1 \quad x_1^2 \leq x_2 \leq 1 \]

\[ C_2 \quad [-1, 1] \times 0 \]

\[ C_1 + (\mathbb{R}_+ \times \{0\}) : \]

\[ (\max\{x_1, 0\})^2 \leq x_2 \leq 1 \]

Similar to Bestuzheva et al. ‘16 who divide sets in two.
Existing Small Ideal Formulations (Isotone Sets)

- Studied by Hijazi et al. ’12 and Bonami et al. ’15 (n=1, 2):
  \[ C_i = \{ x \in \mathbb{R}^d : l^i \leq x \leq u^i, \ f_i(x) \leq 0 \} \]

- \( f_i(x) \) component-wise monotonous (i=1,2 opposite).

- Ideal Formulation

\[
y_1 l^1 + y_2 l^2 \leq x \leq y_1 u^1 + y_2 u^2
\]

\[
f^i_J (x, y) \leq 0 \quad \forall J \subseteq [d], i \in [2]
\]

\[
y_1 + y_2 = 1
\]

\[
y_i \in \{0, 1\} \quad \forall i \in [2]
\]
Generalization and Simplification

- More than 2 sets (with general “opposite condition”).
- Generalization of the monotone/isotone condition (beyond affine transformation)
- Significantly smaller formulation: One non-linear constraint per set.

\[ y_1 l^1 + y_2 l^2 \leq x \leq y_1 u^1 + y_2 u^2 \]
\[ f^i (x, y) \leq 0 \quad \forall J \subseteq [d], i \in [2] \]
\[ y_1 + y_2 = 1 \]
\[ y_i \in \{0, 1\} \quad i \in [2] \]
\[ \hat{f}^i (x, y) \leq 0 \quad \forall i \in [2] \]
Details of Size Reduction

\[ C_i = \{ x \in \mathbb{R}^d : l_i \leq x \leq u^i, \quad f_i(x) \leq 0 \} \]

\[ G_i = \{ x \in \mathbb{R}^d : f_i(x) \leq 0 \} \]

- Original formulation*:

\[ \gamma G_i (\lfloor x \rfloor_j) \leq y_i, \quad \forall J \subseteq [d] \quad (\lfloor x \rfloor_j)_j := \begin{cases} x_j & j \in J \\ 0 & \text{o.w.} \end{cases} \]

- Smaller formulation*:

\[ \gamma G_i (\lfloor x \rfloor^+) \leq y_i \quad (\lfloor x \rfloor^+)_j := \max\{x_j, 0\} \]

- max can cause representability issues.

*assuming some simplifying conditions on bounds
A Case in Which Both Formulations Are Small

\[ y_1 l^1 + y_2 l^2 \leq x \leq y_1 u^1 + y_2 u^2 \]

\[ f^i (x, y) \leq 0 \quad \forall i \in [2] \]

\[ y_1 + y_2 = 1 \]

\[ y_i \geq \{0, 1\} \quad \forall i \in [2] \]
**Algebraic Representation Issues**

\[ C_1 \quad \begin{align*} x_1^2 &\leq x_2 \leq 1 \end{align*} \]

\[ C_2 \quad \begin{align*} [-1, 1] \times 0 \end{align*} \]

\[ C_1 + (\mathbb{R}_+ \times \{0\}) : (\max\{x_1, 0\})^2 \leq x_2 \leq 1 \]

- Non-basic semi-algebraic set contained in formulation.
- Finite polynomial inequalities requires max or auxiliary vars.
Summary

• **Small ideal formulations without “variable copies”**.
  – Piecewise representation by (nearly) homothetic sets.
  – Representation of gauge formulation = gauge calculus.
• More on the paper (arXiv:1704.03954):
  – More examples and generalizations:
    • **Orthogonal sets**, polyhedral formulations by Balas, Blair and Jeroslow, and “truly” non-polyhedral sets.
  – More construction techniques, gauge calculus, etc.
  – **Necessary and sufficient** conditions for piecewise formulation being ideal (more geometric conditions).
• Support function matching / “Lifting” for more general non-convex sets: Tawarmalani et al. ‘10