Solutions 2

*Hard Lagrangian Bashing.* For each of the following systems, write the number of degrees of freedom, a good choice of generalized coordinates, the kinetic and potential energies, the Lagrangian, and the Euler-Lagrange equations of motion:

a) A bead of mass \( m \) on a hoop of radius \( r \) rotating with angular velocity \( \omega \) about a diameter

First we draw a picture: A bead moves freely along a hoop of radius \( r \) rotating about the dashed line with angular velocity \( \omega \). Choose as generalized coordinate the angle between the diameter the bead is on and the diameter the hoop is spinning around (call it \( \phi \)). Then the velocity of the bead consists of two orthogonal components: one due to the rotation of the hoop (in blue) and one due to motion of the bead (in orange). If the bead is at \( \phi \), the magnitude of the former will be \( r \sin \phi \omega \) and that of the latter will be \( r \dot{\phi} \), so the total kinetic energy (since the components are orthogonal) is \( T = \frac{1}{2}mr^2(\omega^2 \sin^2 \phi + r^2 \dot{\phi}^2) \). The particle moves freely so \( V = 0 \). We thus have \( \mathcal{L} = \frac{1}{2}mr^2(\omega^2 \sin^2 \phi + \dot{\phi}^2) \). The Euler-Lagrange equation is then

\[
\frac{\partial \mathcal{L}}{\partial \phi} = m r^2 \omega^2 \cos \phi \sin \phi = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2 \ddot{\phi},
\]

or \( \ddot{\phi} = \omega^2 \cos \phi \sin \phi \).
b) A particle of mass $m$ moving in two dimensions under an arbitrary angularly symmetric potential $U(r)$

We have two degrees of freedom and in order to take advantage of the symmetry of the potential we use polar coordinates. Thus the particle’s velocity consists of its radial rate of change, $\dot{r}$, and its angular rate of change, $r\dot{\theta}$, and these components are orthogonal, so $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. Thus $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$. Our Euler-Lagrange equation in $\theta$ is $\frac{d}{dt}(mr^2\dot{\theta}) = 0$: the conservation of angular momentum. The Euler-Lagrange equation in $r$ is $m\ddot{r} = mr\dot{\theta}^2 - U'(r)$, which has the outward “centrifugal force” term $mr\dot{\theta}^2$ in addition to the usual radially directed force $-U'(r)$.

c) A pendulum of length $l$ whose bob has mass $m$ whose pivot is accelerating parallel to the ground with acceleration $a$

Let $\theta$ be the angle the bob makes with the vertical. The position of the pendulum in a fixed Cartesian coordinate system will be $(\frac{1}{2}at^2 + l\sin\theta, -l\cos\theta)$, so its velocity in that fixed coordinate system will be $(at + \dot{\theta}l\cos\theta, \dot{\theta}l\sin\theta)$. Thus the kinetic energy is $T = \frac{1}{2}m(a^2t^2 + 2at\dot{\theta}l\cos\theta + \dot{\theta}^2l^2)$ and since $V = -mgl\cos\theta$, we have $L = \frac{1}{2}m(a^2t^2 + 2at\dot{\theta}l\cos\theta + \dot{\theta}^2l^2) + mgl\cos\theta$. We write

$$\frac{\partial L}{\partial \theta} = -mat\dot{l}\sin\theta - mgl\sin\theta = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mal\cos\theta - mat\dot{l}\sin\theta + m\ddot{l},$$

so $g\sin\theta + a\cos\theta = -\ddot{l}$.

d) A bead constrained to move on a bent wire whose height $h(x)$ depends on horizontal position $x$

Choose as generalized coordinate the horizontal position of the particle. Then the velocity can be broken into orthogonal components of vertical and horizontal motion: We have $\frac{dx}{dt} = \dot{x}$ and $\frac{du}{dt} = \frac{dx}{dt}\frac{dx}{dt} = h'(x)\dot{x}$, so $T = \frac{1}{2}m\dot{x}^2(1 + h'(x)^2)$. The potential is of course $V = mgh(x)$. Thus $L = \frac{1}{2}m\dot{x}^2(1 + h'(x)^2) - mgh(x)$. We first calculate

$$\frac{\partial L}{\partial x} = m\dot{x}h'(x)h''(x) - mgh'(x),$$
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}(1 + h'(x)^2),$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 2m\dot{x}h'(x)h''(x) + m\ddot{x}(1 + h'(x)^2),$$

so our Euler-Lagrange equation is

$$m\dot{x}h'(x)h''(x) + mgh'(x) + m\ddot{x}(1 + h'(x)^2) = 0.$$