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LINEAR ESTIMATION OF STATIONARY STOCHASTIC PROCESSES, VIBRATING STRINGS, AND INVERSE SCATTERING*

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ABSTRACT

Some vibrating string equations are derived for estimating a stationary stochastic process given some observations of this process over a finite interval. These equations are the time-domain counterpart of equations introduced by Krein, and Dym and McKean in the frequency domain. They are obtained by decomposing the observation and signal processes into even and odd processes, and by solving some associated filtering problems. The resulting linear estimation procedure is very efficient and is well adapted to the estimation of random fields. We also show that it is identical to the Gelfand-Levitan inverse scattering method of quantum mechanics.

I. Introduction

The problem of finding the linear least-squares estimate of a stationary stochastic process given some observations of this process over a finite interval has received a large amount of attention over the years [1]. The best known solution of this problem was proposed by Levinson [2] in 1947 for discrete-time scalar processes. In this approach, a set of recursions is used to compute the one-step ahead predictor of the observed process as the order N of this predictor increases. This algorithm is very efficient, since it requires only $O(N^2)$ operations to compute the predictor of order N. Another feature of these recursions is that they compute implicitly the inverse covariance matrix of the observed process, and therefore they can be used to solve general estimation and detection problems [3]. The Levinson recursions were subsequently extended to the multivariable case by Whittle [4] and Wiggins and Robinson [5], and to the continuous case by Krein [6]. Because of its efficiency and of its conceptual simplicity, this estimation technique is now widely used in areas such as speech processing, geophysical signal processing, and spectral estimation.

Another method of solving the linear estimation problem over a finite interval was proposed by Krein [7] and Dym and McKean [8], [9]. This method identifies the spectral density function of the observed process with that of a vibrating string. Then, by using inverse spectral techniques [10], [11], the vibrating string is reconstructed from its spectral function and is used to solve our original estimation problem. However, since the relation existing between stochastic processes and vibrating strings is not obvious, this estimation technique has received only a limited amount of attention thus far.

In this paper, for the simple case when the observed process contains a white noise component, it will be shown that some vibrating string equations similar to those of Krein and Dym and McKean can be derived entirely in the time-domain by using elementary principles only. These string equations are closely related to the Levinson recursions, and they have the same numerical efficiency. After transformation to the spectral domain, these equations take the form of Schrödinger equations, and we use this observation to show that the estimation method proposed here is identical to the Gelfand-Levitan procedure for solving the inverse scattering problem of quantum mechanics.

Since the Levinson recursions provide already an efficient solution of the estimation problem over a finite interval, the results described in this paper may appear unncessary. However, the Levinson recursions suffer from an important limitation, which is that they cannot be extended to several dimensions for the estimation of random fields. By comparison the Gelfand-Levitan approach that we have used here can be extended easily, as will be shown in [12]. Furthermore, even in the one-dimensional case, the string equations that we have obtained may be useful if we consider random fields (noncausal processes) instead of causal processes. This is demonstrated below on a simple example.

The connection appearing between inverse scattering techniques and linear estimation problems should not also come as a complete surprise, since such a relation was already observed in [13] - [15].

This paper is organized as follows. In Section II the observation process is decomposed into some even and odd processes obtained by symmetrizing the observations with respect to the time origin. These processes are not stationary and have covariances which are sums of Toeplitz and Hankel kernels.

By exploiting the structure of these kernels, we obtain some vibrating string equations which solve the even and odd filtering problems associated to our decomposition. These string equations are used in Section III to estimate arbitrary random variables, and the resulting estimation procedure is illustrated by a random field example. In Section IV, we compare the vibrating string equations to the Levinson recursions and discuss their numerical implementation. A spectral domain interpretation of our results is given in Section V for the case when the observed process has an absolutely continuous spectral function. In this context, the string equations are transformed into Schrödinger equations whose spectral function is identical to the spectral function of the observed process. This observation is then used in Section VI to show that our estimation method is identical to the Gelfand-Levitan procedure for reconstructing a Schrödinger operator from its spectral function. Section VII describes some resolvent identities which can be used to compute efficiently the linear least-squares smoothed estimates. Section VIII contains some conclusions, and the Appendix discusses the case when the spectral function of the observed process contains a singular part.

II. Vibrating String Equations

Let

$$y(t) = z(t) + v(t), \quad -T \leq t \leq T \tag{2.1}$$

be some observations of a stationary, zero-mean Gaussian signal $z(\cdot)$ with covariance

$$E(z(t)z(s)] = k(t-s), \tag{2.2}$$

where $v(\cdot)$ is a white Gaussian noise process with unit intensity i.e.

$$E[v(t)v(s)] = \delta(t-s) \quad . \tag{2.3}$$

For convenience, assume that $z(\cdot)$ and $v(\cdot)$ are uncorrelated, so that

$$E[z(t)v(s)] \equiv 0. \tag{2.4}$$

Then, given an arbitrary random variable $\underline{a}$ belonging to the Hilbert space Z spanned by the signal $z(t)$ for $-\infty < t < \infty$, the linear estimation problem that will be considered in this paper is the one of computing the conditional mean of $\underline{a}$ given Y_T, where Y_T denotes the Hilbert space spanned by the observations $y(t)$ for $-T \leq t \leq T$. The space Z is obtained by taking the mean-square limit of linear combinations of $z(t)$ for $-\infty < t < \infty$, and elements of Y_T are of the form

$$\underline{b} = \int_{-T}^{T} b(t)\ y(t)\ dt$$

where $b(\cdot)$ belongs to $L_2\ [-T, T]$.

A. Even and odd processes

Our approach to solve this problem is to decompose the processes $y(\cdot)$, $z(\cdot)$, and $v(\cdot)$ into even and odd processes. The even observation, signal and noise processes are defined as

$$y_+(t) = \frac{1}{2}\ (y(t) + y(-t))$$

$$z_+(t) = \frac{1}{2}\ (z(t) + z(-t)) \tag{2.6}$$

$$v_+(t) = \frac{1}{2}\ (v(t) + v(-t))$$

so that we have

$$y_+(t) = z_+(t) + v_+(t), \quad 0 \leq t \leq T. \tag{2.7}$$

The equation (2.7) defines an even estimation problem where we want to estimate $z_+(\cdot)$ given the observations $y_+(t)$, $0 \leq t \leq T$. Similarly, the odd processes are given by

$$\begin{aligned} y_-(t) &= \frac{1}{2}(y(t) - y(-t)) \\ z_-(t) &= \frac{1}{2}(z(t) - z(-t)) \\ v_-(t) &= \frac{1}{2}(v(t) - v(-t)) \end{aligned} \tag{2.8}$$

and

$$y_-(t) = z_-(t) + v_-(t), \quad 0 \leq t \leq T \tag{2.9}$$

defines an odd estimation problem.

The main feature of this decomposition is that even and odd processes are uncorrelated, i.e.

$$E[z_+(t)z_-(s)] = E[v_+(t)v_-(s)] \equiv 0 . \tag{2.10}$$

Consequently, if we denote by $Z^{\pm} = H(z_{\pm}(t), -\infty < t < \infty)$ and $Y_T^{\pm} = H(y_{\pm}(t), -T \leq t \leq T)$ the Hilbert spaces spanned by the even and odd signals and observations, we have the orthogonal decomposition

$$Z = Z^+ \oplus Z^- \tag{2.11a}$$

$$Y_T = Y_T^+ \oplus Y_T^- \tag{2.11b}$$

where $Z^+ \perp Y_T^-$ and $Z^- \perp Y_T^+$. This shows that the signal estimate can be expressed as

$$E[z(t)|Y_T] = E[z_+(t)|Y_T^+] + E[z_-(t)|Y_T^-] \tag{2.12}$$

and our original estimation problem over the interval [-T,T] is now decomposed into even and odd estimation problems over [0,T].

An apparent disadvantage of this approach is that the signal processes $z_{\pm}(\cdot)$ are <u>not stationary</u>. However, these processes have some structure since their covariance

$$k_{\pm}(t,s) = E[z_{\pm}(t)z_{\pm}(s)] = \frac{1}{2}(k(t-s) \pm k(t+s))$$

is the sum or difference of Toeplitz and Hankel kernels. A way to characterize this structure is to note that the kernels $k_{\pm}(\cdot,\cdot)$ have the properties

(i) $$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right)k_{\pm}(t,s) \equiv 0 \tag{2.13}$$

(ii) $$\frac{\partial}{\partial s} k'_{+}(t,0) = 0 \text{ and } k_{-}(t,0) = 0 \tag{2.14}$$

The property (2.13) is called the <u>displacement property</u> of $k_{\pm}(.,.)$, and the wave operator

$$\underline{\delta} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$$

appearing in this identity can be viewed as obtained by composing the operators

$$\underline{\tau} = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \quad \text{and } \underline{\theta} = \frac{\partial}{\partial t} - \frac{\partial}{\partial s}$$

that were used in [16], [17] to characterize respectively the displacement properties of Toeplitz and Hankel kernels.

To specify completely the even and odd estimation problems we note finally that the processes $v_{\pm}(\cdot)$ are white Gaussian noise processes with intensity 1/2, i.e.

$$E[v_{\pm}(t)v_{\pm}(s)] = \frac{1}{2}\delta(t-s),$$

and $v_{\pm}(\cdot)$ are uncorrelated with $z_{\pm}(\cdot)$.

B. Vibrating String Equations

To solve the even and odd estimation problems, the first step is to construct the filtering estimates

$$\hat{z}_{\pm}(T|T) = E[z_{\pm}(T)|Y_T^{\pm}] = \int_0^T g_{\pm}(T,t)y_{\pm}(t)dt \quad . \tag{2.15}$$

By denoting the filtering errors as

$$\tilde{z}_{\pm}(T|T) = z_{\pm}(T) - \hat{z}_{\pm}(T|T)$$

and by using the orthogonality property $\tilde{z}_{\pm}(T|T) \perp Y_T^{\pm}$ of linear least-squares estimates, we find that $g_{\pm}(T,\cdot)$ satisfies the integral equation

$$k_{\pm}(t,T) = \int_0^T k_{\pm}(t,s)g_{\pm}(T,s)ds + \frac{1}{2}g_{\pm}(T,t) \tag{2.16}$$

with $0\leq t\leq T$.

To guarantee the existence and unicity of a solution to (2.16), we assume that $k_{\pm}(.,.)$ is square-integrable over $[0,T]^2$, or equivalently that $k(\cdot)$ is square-integrable over $[0,T]$. Then, the operator

$$\underline{K}_{\pm}: a(t) \to b_{\pm}(t) = \int_0^T k_{\pm}(t,s)a(s)ds$$

is defined over $L_2[0,T]$. Since $k_{\pm}(.,.)$ is a covariance kernel, the operator $\underline{K}_{\pm}$ is self-adjoint and nonnegative definite, so that $\underline{K}_{\pm} + \underline{I}/2$ is invertible. This guarantees the existence and unicity of a solution in $L_2[0,T]$ to the integral equation (2.16).

To compute $g_{\pm}(T,\cdot)$ one method would be to discretize the interval $[0,T]$ into N subintervals of length $\Delta = T/N$, and to solve the resulting system of equations. However, this method requires $O(N^3)$ operations. A more efficient procedure is to exploit the displacement property of $k_{\pm}(.,.)$ as is shown now.

<u>Theorem 1</u>: If $k(\cdot)$ is twice differentiable, the functions $g_{\pm}(\cdot,\cdot)$ satisfy the differential equations

$$\left(\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial t^2}\right) g_{\pm}(T,t) = V_{\pm}(T)\, g_{\pm}(T,t) \tag{2.17}$$

with the boundary conditions

$$V_{\pm}(T) = -2\frac{d}{dT}\, g_{\pm}(T,T) \tag{2.18}$$

$$\frac{\partial}{\partial t}\, g_{+}(T,0) = 0 \text{ and } g_{-}(T,0) = 0 \quad . \tag{2.19}$$

<u>Proof</u>: By operating with

$$\underline{\delta} = \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial t^2}$$

on both sides of (2.16) and integrating by parts, we obtain

$$V_{\pm}(T)k_{\pm}(t,T) = \int_0^T k_{\pm}(t,s)\ \underline{\delta} g_{\pm}(T,s)\,ds + \frac{1}{2}\,\underline{\delta}\ g_{\pm}(T,t)$$

where $V_{\pm}(\cdot)$ satisfies (2.18). Then, by using the unicity of the solution to (2.16), we get (2.17).

<u>Remark</u>: These differential equations are those satisfied by <u>elastically braced vibrating strings</u>. In this context, T and t denote respectively the space and time variables, and $V_{\pm}(T)$ are the elasticity constants of the strings per unit length at point T. The mass density of these strings is uniform, and since $0 \leq T < \infty$, their length is infinite. The vibrating string equations (2.17) show that the filters $g_{\pm}(.,.)$ which are functions of <u>two variables</u> are parametrized entirely by $V_{\pm}(.)$, which depend only

on <u>one variable</u>. From this point of view, the functions $V_{\pm}(\cdot)$ play exactly the same role for the string equations (2.17) as the reflection coefficient function for the Levinson recursions. The relation between $V_{\pm}(\cdot)$ and the reflection coefficient function will be examined in Section IV.

Now, let

$$e_{\pm}^2(T) = E[\tilde{z}_{\pm}^2(T|T)]$$

be the even and odd mean-square filtering errors. By using the orthogonality property of linear least-squares estimates, we find that

$$e_{\pm}^2(T) = E[\tilde{z}_{\pm}(T|T)z_{\pm}(T)] = \frac{1}{2}\, g_{\pm}(T,T) \quad , \tag{2.20}$$

so that the functions $V_{\pm}(\cdot)$ given by (2.18) can be interpreted as

$$V_{\pm}(T) = -4\,\frac{d}{dT}\,(e_{\pm}^2(T)) \quad .$$

C. <u>Even and odd innovations</u>

An important feature of the vibrating string equations (2.17) is that in the process of finding $g_{\pm}(T,\cdot)$, they compute all the filters $g_{\pm}(t,\cdot)$ for $0 \leq t \leq T$. Thus, by using these equations, we can construct the filtering estimates $\hat{z}_{\pm}(t|t)$ for $0 \leq t \leq T$. These estimates generate some even and odd innovations processes

$$\nu_{\pm}(t) \triangleq y_{\pm}(t) - \hat{z}_{\pm}(t|t)$$

which, by construction, are white Gaussian noise processes of intensity 1/2, i.e.

$$E[\nu_{\pm}(t)\nu_{\pm}(s)] = \frac{1}{2}\,\delta(t-s) \quad . \tag{2.22}$$

Furthermore, if $\nu_T^{\pm} = H(\nu_{\pm}(t),\ 0 \leq t \leq T)$ denote the Hilbert spaces spanned by $\nu_{\pm}(\cdot)$, we have (cf. [18])

$$\nu_T^{\pm} = Y_T^{\pm} \quad .$$

III. General Estimation Procedure

Let $\underline{a}$ be an arbitrary zero-mean random variable, whose joint statistics with $y(\cdot)$ are Gaussian, and such that

$$E[y_{\pm}(t)\underline{a}] = a_{\pm}(t).$$

A special case is of course $\underline{a} = z(s)$. In this case $a_{\pm}(t) = k_{\pm}(t,s)$. Then, by using the orthogonal decomposition (2.11b), the conditional mean of $\underline{a}$ given Y_T can be expressed as

$$E[\underline{a}|Y_T] = \int_0^T c_+(T,t)y_+(t)\,dt + \int_0^T c_-(T,t)y_-(t)\,dt \quad . \tag{3.1}$$

If $\tilde{\underline{a}} = \underline{a} - E[\underline{a}|Y_T]$ denotes the estimation error, by noting that $\tilde{\underline{a}} \perp Y_T^{\pm}$, we find that the filters $c_{\pm}(T,.)$ satisfy the integral equations

$$a_{\pm}(t) = \int_0^T k_{\pm}(t,s)c_{\pm}(T,s)\,ds + \frac{1}{2}\,c_{\pm}(T,t) \tag{3.2}$$

for $0 \leq t \leq T$. These equations can also be written in operator notation as

$$a_{\pm} = (\underline{K}_{\pm} + \underline{I}/2)c_{\pm} \quad . \tag{3.3}$$

A method of solving (3.2) would be to discretize the interval [0,T] into N subintervals and to solve the resulting system of linear equations. However, this procedure requires $O(N^3)$ operations. A simpler method is to note that

<u>Theorem 2</u>: If $a_{\pm}(\cdot)$ and $k(\cdot) \in L_2[0,T]$, the filters $c_{\pm}(\cdot,\cdot)$ satisfy the recursions

$$\frac{\partial}{\partial T}\, c_{\pm}(T,t) = -c_{\pm}(T,T)\,g_{\pm}(T,t) \tag{3.4}$$

where $0 \leq t \leq T$.

Proof: Take the partial derivative of (3.2) with respect to T. This gives

$$-c_{\pm}(T,T)k_{\pm}(t,T) = \int_0^T k_{\pm}(t,s)\,\frac{\partial c_{\pm}}{\partial T}(T,s)\,ds + \frac{1}{2}\,c_{\pm}(T,t) \quad . \tag{3.5}$$

Then, by comparing (3.5) with (2.16) and using the unicity of the solution to (2.16) we obtain (3.4).

The recursions (3.4) for $c_{\pm}(.,.)$ can be propagated simultaneously with the string equations (2.17) for increasing values of T, starting from T=0. By discretizing these equations with a step size $\Delta = T/N$, we obtain a numerical scheme which requires only $O(N^2)$ operations to compute $c_{\pm}(T,.)$, as is shown in Section IV. This procedure is therefore very efficient if we compare it to direct discretization methods for solving the integral equation (3.2). Note however that an exact comparison would need to take into account differences in step sizes for approximating (3.2) and (3.4).

The recursions (3.4) can be interpreted as follows. Let $H_{\pm}(.,.;T)$ be the Fredholm resolvent associated to the kernel $k_{\pm}(.,.)$, i.e.

$$k_{\pm}(t,s) = \int_0^T k_{\pm}(t,u)H_{\pm}(u,s;T)\,du + \frac{1}{2}\,H_{\pm}(t,s;T) \tag{3.6}$$

for $0 \leq t,\ s \leq T$, or equivalently in operator notation

$$(\underline{I}/2 + \underline{K}_{\pm})(\underline{I} - \underline{H}_{\pm}) = \underline{I}/2 \quad . \tag{3.7}$$

Then, the solution $c_{\pm}(T,.)$ of (3.2) is given by

$$c_{\pm}(T,t) = 2\left(a_{\pm}(t) - \int_0^T H_{\pm}(t,s;T)\,a_{\pm}(s)\,ds\right) \tag{3.8}$$

and if we take the partial derivative of (3.8) with respect to T, and use the Bellman-Krein resolvent identity [19]

$$\frac{\partial H_{\pm}}{\partial T}(t,s;T) = -g_{\pm}(T,t)\,g_{\pm}(T,s) \tag{3.9}$$

we obtain the recursions (3.4). This shows that the recursions (3.4) are a direct consequence of the resolvent identity (3.9). Conversely, since (3.6) is obtained by setting $a_{\pm}(t) = k_{\pm}(t,s)$ and $c_{\pm}(T,t) = H_{\pm}(t,s;T)$ in (3.2) (s is viewed here as a parameter) the resolvent identity (3.9) can be viewed as a special case of (3.4).

Random Field Example: Consider the case where we want to estimate $\underline{a} = z(0)$ given observations over $[-T,T]$. The motivation for considering this symmetric estimation configuration is that random fields are not causally generated. It is therefore natural to base our estimate of the field at a given point on a symmetric set of observations around this point. In this case, we have $\underline{a} = z(0) = z_+(0)$ and $z_-(0) = 0$, so that

$$a_+(t) = k_+(t,0), \quad c_+(T,t) = H_+(t,0;T)$$

and

$$a_-(t) = c_-(T,t) \equiv 0.$$

This shows that the estimate (3.1) depends on one filter only; and the recursions (3.4) can be used to compute $c_+(T,.)$ for increasing values of T.

IV. Numerical Considerations

A. Relation with the Levinson Recursions

The vibrating string equations (2.17) can be related to the Levinson recursions for the estimation problem over a finite interval. To see this, denote by $H_T = H(y(t),\ 0 \le t \le T)$ the Hilbert space spanned by the observations over [0,T], and let

$$\hat{z}(T|T) = E[z(T)|H_T] = \int_0^T A(T,s)y(s)ds \qquad (4.1)$$

by the linear least-squares estimate of z(T) given H_T. Then, the <u>Levinson recursions</u> [3], [16] for the filter A(.,.) are given by

$$\left(\frac{\partial}{\partial T} + \frac{\partial}{\partial t}\right) A(T,t) = -\rho(T)A(T,T-t), \qquad (4.2)$$

where the reflection coefficient function $\rho(.)$ can be computed by using the relation

$$\rho(T) = A(T,0) = k(T) - \int_0^T A(T,s)k(s)ds \quad . \qquad (4.3)$$

The function $\rho(\cdot)$ depends on one variable only, and therefore it plays the same role for the Levinson recursions as $V_{\pm}(\cdot)$ for the vibrating string equations (2.17).

By comparing the definitions (4.1) and (2.15) for A(.,.) and $g_{\pm}(.,.)$, we find that

$$g_{\pm}(T,t) = A(2T,T+t) \pm A(2T,T-t), \quad 0 \le t \le T \qquad (4.4)$$

so that it is <u>equivalent</u> to compute A(2T,.) and the functions $g_{\pm}(T,.)$.

This indicates a strong connection between the Levinson recursions and the string equations (2.17). We can go one step further and show that the string equations (2.17) can be derived from the Levinson recursions. To do so, we rewrite (4.2) as

$$\left(\frac{\partial}{\partial T} + \frac{\partial}{\partial t}\right) A(2T,T\pm t) = -2\rho(2T)A(2T,T\mp t)$$

and take (4.4) into account, so that we get the system of <u>first-order</u> partial differential equations

$$\frac{\partial}{\partial T} g_+(T,t) + \frac{\partial}{\partial t} g_-(T,t) = -2\rho(2T)g_+(T,t) \tag{4.5a}$$

$$\frac{\partial}{\partial T} g_-(T,t) + \frac{\partial}{\partial t} g_+(T,t) = 2\rho(2T)g_-(T,t) \tag{4.5b}$$

where

$$2\rho(2T) = g_+(T,T) - g_-(T,T) \quad . \tag{4.6}$$

Then, by taking respectively the partial derivatives of (4.5a) and (4.5b) with respect to T and t and subtracting the resulting equations, we obtain (2.17) with

$$V_{\pm}(T) = 4(\rho^2(2T) \mp \dot{\rho}(2T)) \ . \tag{4.7}$$

The relation (4.7) shows that $V_{\pm}(\cdot)$ can be expressed in terms of $\rho(\cdot)$. Conversely, if either $V_+(\cdot)$ or $V_-(\cdot)$ is given, to compute $\rho(\cdot)$ we need only to solve the Riccati equation (4.7) with the initial condition $\rho(0) = k(0)$. Thus, $\rho(\cdot)$ and $V_{\pm}(\cdot)$ provide two equivalent parametrizations of the process $y(\cdot)$.

B. Discretization Scheme

To compute $g_+(.,.)$ and $g_-(.,.)$ we can use either the vibrating strings (2.17) or the equivalent system of first-order equations (4.5). It turns out it is preferable to discretize (4.5). Also, since the functions $c_{\pm}(.,.)$ appear in our estimation procedure for the random variable $\underline{a}$, we will include them in our analysis.

If $\Delta > 0$ is the discretization step, denote $G_{\pm}(m,n) = g_{\pm}(m\Delta, n\Delta)$ and $C_{\pm}(m,n) = c_{\pm}(m\Delta, n\Delta)$. Suppose now that at stage N, $G_{\pm}(N,n)$ and $C_{\pm}(N,n)$ have been computed for $0 \le n \le N$. By discretizing (4.5) and (3.4), we get

$$\begin{aligned} G_+(N+1,n) &= G_+(N,n) + G_-(N,n-1) - G_-(N,n) \\ &\quad - R(N)G_+(N,n)\Delta \end{aligned} \tag{4.8a}$$

$$G_-(N+1,n) = G_-(N,n) + G_+(N,n-1) - G_+(N,n) + R(N)G_-(N,n) \tag{4.8b}$$

where $R(N) = G_+(N,N) - G_-(N,N)$ and $1 \le n \le N$, and

$$C_\pm(N+1,n) = C_\pm(N,n) - C_\pm(N,N)G_\pm(N,n)\Delta \tag{4.9}$$

with $0 \le n \le N$. The boundary conditions (2.19) give

$$G_+(N+1,0) = G_+(N+1,1) \text{ and } G_-(N+1,0) = 0, \tag{4.10}$$

and to compute $G_\pm(N+1, N+1)$ and $C_\pm(N+1, N+1)$, we can discretize (2.16) and (3.2), so that

$$\frac{1}{2} G_\pm(N+1, N+1) = k_\pm((N+1)\Delta, (N+1)\Delta) - \left(\sum_{i=0}^{N} k_\pm((N+1)\Delta, i\Delta) G_\pm(N+1,i)\right)\Delta \tag{4.11a}$$

$$\frac{1}{2} C_\pm(N+1, N+1) = a_\pm((N+1)\Delta) - \left(\sum_{i=0}^{N} k_\pm((N+1)\Delta, i\Delta) C_\pm(N+1,i)\right)\Delta . \tag{4.11b}$$

This shows that the function $G_\pm(N,.)$ and $C_\pm(N,.)$ can be computed recursively, starting from N=0. The number of operations required for each step of the recursions (4.8) - (4.11) is 8N so that the overall number of operations required to compute $G_\pm(N,.)$ and $C_\pm(N,.)$ is $4N^2$. The amount of storage required is 4N.

It turns out that this is exactly the number of operations and the storage that would be required by the Levinson recursions, a fact which is not surprising in light of the close relation existing between these recursions and the string equations.

V. Spectral Domain Viewpoint

To express the previous results in the spectral domain, we will use the Kolmogorov isometry [9] between Y and $L_2(\hat{r}(\lambda)d\lambda)$, where $\hat{r}(\lambda)$ is the spectral density function of $y(\cdot)$, i.e.

$$\begin{aligned}\hat{r}(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} r(t)\exp{-j\lambda t}\,dt \\ &= \frac{1}{2\pi}(1 + \int_{-\infty}^{\infty} k(t)\exp{-j\lambda t}\,dt) \qquad (5.1)\end{aligned}$$

with

$$r(t) = E[y(t)y(0)] = \delta(t) + k(t) \quad . \qquad (5.2)$$

We assume that $k(\cdot)$ is summable, so that its Fourier transform

$$\hat{k}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(t)\ \exp{-j\lambda t}\,dt$$

exists. Then, the isometry between Y and $L_2(\hat{r}(\lambda)d\lambda)$ is defined by the correspondence

$$y(t) \leftrightarrow \exp j\lambda t \qquad (5.3)$$

where we have

$$\begin{aligned}E[y(t)y(s)] &= \int_{-\infty}^{\infty} \exp j\lambda t\ \exp{-j\,\lambda s}\ \hat{r}(\lambda)d\lambda \\ &= \delta(t-s) + k(t-s) \quad . \qquad (5.4)\end{aligned}$$

Since neither $y(t)$ nor $\exp j\lambda t$ belong respectively to Y and $L_2(\hat{r}(\lambda)d\lambda)$ ((5.4) contains a delta function), the isometry (5.3) should be interpreted as

$$\int_{-\infty}^{\infty} a(t)y(t)dt \leftrightarrow \int_{-\infty}^{\infty} a(t)\exp j\,\lambda t\ dt \qquad (5.5)$$

where a(.) is an arbitrary square-integrable function. However, for convenience we will continue to use y(t) and exp $j\lambda t$ as the defining elements of the isometry between Y and $L_2(\hat{r}(\lambda)d\lambda)$.

In this framework, the mapping

$$y_+(t) \leftrightarrow \cos \lambda t, \quad y_-(t) \leftrightarrow j \sin \lambda t \tag{5.6}$$

defines an isometry between $Y^{\pm}$ and the spaces of even and odd functions of $L_2(\hat{r}(\lambda)d\lambda)$. Consequently, the even and odd filtering problems of Section II are equivalent to the problems of orthogonalizing the functions $\{\cos \lambda t,\ 0 \le t < \infty\}$ and $\{\sin \lambda t,\ 0 \le t < \infty\}$ with respect to the spectral density $\hat{r}(\lambda)$. The solution of these problems is obtained by noting that we have the correspondence

$$\nu_+(t) \leftrightarrow \gamma_+(t,\lambda) = \cos \lambda t - \int_0^t g_+(t,s) \cos \lambda s \, ds \tag{5.7a}$$

$$\nu_-(t) \leftrightarrow \gamma_-(t,\lambda) = j(\sin \lambda t - \int_0^t g_-(t,s) \sin \lambda s \, ds) \tag{5.7b}$$

where $\gamma_+(t,.)$ and $\gamma_-(t,.)$ can be viewed as the cosine and sine transforms of $g_+(t,.)$ and $g_-(t,.)$.

Then, since the innovations $\nu_{\pm}(\cdot)$ are orthogonal, the functions $\gamma_{\pm}(.,.)$ are also orthogonal, so that

$$E[\nu_{\pm}(t)\nu_{\pm}(s)] = \frac{1}{2}\,\delta(t-s) = \int_{-\infty}^{\infty} \gamma_{\pm}(t,\lambda)\gamma_{\pm}^*(s,\lambda)\hat{r}(\lambda)d\lambda \ . \tag{5.8}$$

To characterize $\gamma_{\pm}(.,.)$, we note that

Theorem 3: The functions $\gamma_{\pm}(.,.)$ satisfy the Schrödinger equations

$$\frac{d^2}{dt^2}\,\gamma_{\pm}(t,\lambda) + (\lambda^2 - V_{\pm}(t))\gamma_{\pm}(t,\lambda) = 0 \tag{5.9}$$

with the boundary conditions

$$\gamma_+(0,\lambda) = 1 \qquad \frac{d}{dt}\gamma_+(0,\lambda) = -2k(0) \tag{5.10}$$

$$\gamma_-(0,\lambda) = 0 \qquad \frac{d}{dt}\gamma_-(0,\lambda) = j\lambda \quad . \tag{5.11}$$

Proof: Differentiate (5.7) twice with respect to t, and integrate by parts. This gives (5.9).

Thus, $V_+(\cdot)$ and $V_-(\cdot)$ can be viewed as potential functions. Another property of $\gamma_+(.,.)$ and $\gamma_-(.,.)$ which is not apparent from (5.9) is that they can be expressed in function of each other. To prove this, transform the system of equations (4.5) into

$$j\lambda\gamma_-(t,\lambda) = \frac{d}{dt}\gamma_+(t,\lambda) + 2\rho(2t)\gamma_+(t,\lambda) \tag{5.12a}$$

$$j\lambda\gamma_+(t,\lambda) = \frac{d}{dt}\gamma_-(t,\lambda) - 2\rho(2t)\gamma_-(t,\lambda). \tag{5.12b}$$

Now, substitute

$$2\rho(2t) = -\dot{w}(t)/w(t) \tag{5.13}$$

with $w(0) = 1$ inside the Riccati equation (4.7). This gives

$$\frac{d^2}{dt^2}w(t) - V_+(t)w(t) = 0$$

so that we can identify $w(t) = \gamma_+(t,0)$. Then, by combining (5.13) and (5.12a) we get

$$j\lambda\gamma_-(t,\lambda) = W[\gamma_+(t,\lambda)\,,\ \gamma_+(t,0)]/\gamma_+(t,0), \tag{5.14}$$

where

$$W[u(t)\,,\ v(t)] = \dot{u}(t)\ v(t) - u(t)\ \dot{v}(t)$$

denotes the Wronskian of $u(\cdot)$ and $v(\cdot)$. This expression was first obtained by Krein [7], and it shows that only one of the Schrödinger equations (5.9) or of the string equations (2.17) needs to be solved.

Now, if the random variable $\underline{a} \in Y$ that we want to estimate is mapped into $a(\lambda) \in L_2(\hat{r}(\lambda)d\lambda)$, the problem of finding the conditional mean of $\underline{a}$ given Y_T is equivalent to the one of projecting $a(\lambda)$ on the subspace S_T of $L_2(\hat{r}(\lambda)d\lambda)$ which is spanned by $\{\exp j\lambda t, -T\leq t\leq T\}$. We have therefore the correspondence

$$E[\underline{a}|Y_T] \leftrightarrow P_T a(\lambda) \tag{5.15}$$

where P_T denotes the projection operator on S_T, and the expression (3.1) for the conditional mean of $\underline{a}$ can be written in the spectral domain as

$$P_T a(\lambda) = \hat{c}_+(T,\lambda) + \hat{c}_-(T,\lambda) \tag{5.16}$$

where

$$\hat{c}_+(T,\lambda) = \int_0^T c_+(T,t)\cos\lambda t\, dt \tag{5.17a}$$

$$\hat{c}_-(T,\lambda) = j\int_0^T c_-(T,t)\sin\lambda t\, dt\ . \tag{5.17b}$$

To compute $\hat{c}_{\pm}(.,.)$, we can transform the differential equations (3.4), so that

$$\frac{d}{dT}\hat{h}_{\pm}(T,\lambda) = c_{\pm}(T,T)\gamma_{\pm}(T,\lambda)\ , \qquad 0\leq T<\infty \tag{5.18}$$

with

$$\hat{c}_{\pm}(0,\lambda) = 0\ .$$

Thus, the functions $\gamma_{\pm}(.,.)$ play the same role for the solution of the linear estimation problem in the spectral domain as $g_{\pm}(.,.)$ in the time-domain.

At this point, it is worth pointing out that several of the expressions described in this section (e.g., the Schrödinger equations (5.9) or the system of equations (5.12)) are not new and can be found in Krein [7] or

Dym [11]. The novelty of our approach, however, is that instead of using these expressions as our starting point, we have obtained them only as a byproduct of our time-domain solution of the estimation problem over a finite interval.

By denoting

$$\xi(t,\lambda) = \gamma_+(t,\lambda)/\gamma_+(t,0) \tag{5.19}$$

and

$$x(t) = \int_0^t \frac{ds}{\gamma_+^2(s,0)} , \tag{5.20}$$

we can also transform the Schrödinger equations (5.9) into the string equations used by Dym and McKean [8], [9] to solve the estimation problem over a finite interval. Indeed, if $\xi(t,\lambda) = y(x(t),\lambda)$, the function $y(x,\lambda)$ satisfies the string equation

$$\frac{d^2}{dx^2} y(x,\lambda) = -\lambda^2 \mu(x) y(x,\lambda) , \qquad 0 \leq x < \ell \tag{5.21}$$

$$y(0,\lambda) = 1, \ \frac{d}{dx} y(0,\lambda) = 0 \tag{5.22}$$

where $\ell = x(\infty)$ and where the mass density $\mu(x)$ of the string is given by

$$\mu(x(t)) = \gamma_+^4(t,0) . \tag{5.23}$$

This transformation is due to Liouville [20], and its main feature is that it is isospectral, i.e. if $S(\lambda)$ and $S'(\lambda)$ are respectively the spectral functions of the Schrödinger operator

$$\underline{\Delta}_{\pm} = -\frac{d^2}{dt^2} + V_{\pm}(t)$$

and of the string operator

$$\underline{M} = -\frac{1}{\mu(x)}\frac{d^2}{dx^2}$$

we have

$$\frac{dS(\lambda)}{d\lambda} = \frac{dS'(\lambda)}{d\lambda} = 2\hat{r}(\lambda) \quad . \tag{5.24}$$

The spectral functions of $\underline{\Delta}_{\pm}$ and $\underline{M}$ are defined as follows. Let $a(\cdot)$ and $a'(\cdot)$ be some functions of $L_2[o,\infty)$ and of $L_2(\mu dx, [o,\ell))$, and let $\gamma_{\pm}(\cdot,\lambda)$ and $y(\cdot,\lambda)$ be the solutions of (5.9)-(5.11) and (5.21)-(5.22). Then, consider the mappings

$$T_{\pm} : \quad a(x) \to A_{\pm}(\lambda) = \int_0^{\infty} a(x)\ \gamma_{\pm}(x,\lambda)\,dx \tag{5.25}$$

and

$$T_{+}' : \quad a'(x) \to A'_{+}(\lambda) = \int_0^{\ell} a'(x)y(x,\lambda)\mu(x)\,dx. \tag{5.26}$$

The functions $S(\lambda)$ and $S'(\lambda)$ are the spectral functions of $\underline{\Delta}_{\pm}$ and $\underline{M}$ if we have

$$\int_0^{\infty} a(x)b(x)dx = \int_{-\infty}^{\infty} A_{\pm}(\lambda)B^*_{\pm}(\lambda)dS(\lambda) \tag{5.27}$$

and

$$\int_0^{\ell} a'(x)b'(x)\mu(x)dx = \int_{-\infty}^{\infty} A'_{+}(\lambda)\ B'_{+}(\lambda)dS'(\lambda) \tag{5.28}$$

where $b(\cdot)$ and $b'(\cdot)$ are some arbitrary functions of $L_2[0,\infty)$ and $L_2(\mu dx,\ [0,\ell))$ which are mapped respectively into $B_{\pm}(\lambda)$ and $B'_{+}(\lambda)$. Thus, T_+ and T'_+ are some isometries between $L_2[0,\infty)$ and $L_2(\mu dx,\ [0,\ell))$ respectively and the subspaces of even functions of $L_2(dS(\lambda))$ and $L_2(dS'(\lambda))$. Similarly, T_- is an isometry between $L_2[0,\infty)$ and the subspace of odd functions of $L_2(dS(\lambda))$. The mapping

$$T'_- \ : a'(x) \to A'_-(\lambda) = -\frac{1}{\lambda}\int_0^{\ell} a'(x)dy(x,\lambda)$$

where $a'(\cdot)$ is an arbitrary function of $L_2[0,\ell]$ defines also an isometry between $L_2[0,\ell]$ and the subspace of odd functions of $L_2(dS'(\lambda))$, i.e.

$$\int_0^{\ell} a'(x)b'(x)dx = \int_{-\infty}^{\infty} A'_-(\lambda)\ B'_-(\lambda)dS'(\lambda)$$

(see [8], [9]).

By using the orthogonality relation (5.8) for the eigenfunctions $\gamma_{\pm}(t,\lambda)$, we find that

$$\frac{dS(\lambda)}{d\lambda} = 2\hat{r}(\lambda) \tag{5.29}$$

and by substituting the transformation (5.19)-(5.20) inside (5.28), it is easy to check that

$$dS(\lambda) = dS'(\lambda) \ . \tag{5.30}$$

This shows that the spectral functions $S(\cdot)$ and $S'(\cdot)$ are absolutely continuous and are specified by the spectral density function $\hat{r}(\lambda)$ of the observations process $y(\cdot)$. Consequently, if we are given $\hat{r}(\lambda)$, it does not matter whether we use $V_{\pm}(\cdot)$ or $\mu(\cdot)$ to parametrize $\hat{r}(\cdot)$, since one can go from one description to the other by using the transformation (5.19)-(5.20).

It is important to note that our results depend on the fact that

$$\hat{r}(\lambda) = \frac{1}{2\pi} (1 + \hat{k}(\lambda)), \tag{5.31}$$

i.e. on the assumption that

(i) $y(\cdot)$ contains a white noise component

(ii) $z(\cdot)$ has an absolutely continuous spectral function. It will be shown in the Appendix that the second assumption can be removed by adding a singular part to the spectral function of the operators $\underline{\Delta}_{\pm}$ and $\underline{M}$. However, the first assumption is essential, and if it does not hold our estimation technique cannot be used. By comparison, the method of Dym and McKean [8], [9] is more general, since it starts from the string equation

$$\frac{d^2}{dmdx} y(x,\lambda) = - \lambda^2 y(x,\lambda), \quad o \leq x < \ell \tag{5.32}$$

with initial conditions (5.22), and where the mass distribution $m(\cdot)$ is <u>arbitrary</u>. The string (5.32) can then be used to construct an isometry between $L_2(dm, [o,\ell))$ and the space of even functions of $L_2(dR(\lambda))$, where $dR(\lambda)/d\lambda$ cannot necessarily be represented as in (5.31). However, from a physical point of view, it is reasonable to assume that $y(\cdot)$ contains white noise. An additional advantage of making this assumption is that the algorithms that we obtain are simpler to implement than the procedure described in [9], [10].

VI. The Gelfand-Levitan Inverse Scattering Method

Our estimation procedure can now be related to the Gelfand-Levitan method [21]-[23] for solving the inverse scattering problem of quantum mechanics. Let $\Psi_{\pm}(\cdot,\cdot)$ be the solutions of

$$\left(\frac{d^2}{dr^2} + \lambda^2 - V_{\pm}(r)\right) \psi_{\pm}(r,\lambda) = 0, \qquad 0<r<\infty \tag{6.1}$$

with initial conditions

$$\psi_{+}(0,\lambda) = 1 \qquad \frac{d}{dr}\psi_{+}(0,\lambda) = h \tag{6.2}$$

$$\psi_{-}(0,\lambda) = 0 \qquad \frac{d}{dr}\psi_{-}(0,\lambda) = 1 \tag{6.3}$$

where h is a constant to be determined from the spectral data. Then, if $V_{+}(r)$ and $V_{-}(r)$ decay sufficiently rapidly as r becomes large, when $r \to \infty$ we have

$$\psi_{+}(r,\lambda) \cong |F(\lambda)| \cos(\lambda r-\eta(\lambda)) \tag{6.4}$$

$$\psi_{-}(r,\lambda) \cong \frac{|F(\lambda)|}{\lambda} \sin(\lambda r-\eta(\lambda)) \tag{6.5}$$

where $\eta(\lambda) = \text{Arg } F(\lambda)$, and where $F(\lambda)$ is the Jost function associated to the Schrödinger equation (6.1). If $f_{\pm}(r,\lambda)$ are the solutions of (6.1) such that

$$\lim_{r\to\infty} \exp -j\lambda r \; f_{\pm}(r,\lambda) = 1, \tag{6.6}$$

$F(\lambda)$ is given by the Wronskians

$$F(\lambda) = \frac{1}{j\lambda} \; W[f_{+}(r,\lambda), \psi_{+}(r,\lambda)] \tag{6.7}$$

$$= -W[f_{-}(r,\lambda), \psi_{-}(r,\lambda)] . \tag{6.8}$$

The function $F(\lambda)$ is analytic in the upper half-plane [22], [23]. Furthermore, since the spectral function of

$$\underline{\Delta}_{\pm} = -\frac{d^2}{dr^2} + V_{\pm}(r)$$

is absolutely continuous with density $2\hat{r}(\lambda)$, the operators $\underline{\Delta}_{+}$ and $\underline{\Delta}_{-}$ have no bound states, so that $F^{-1}(\lambda)$ is analytic in the upper half-plane [22].

The inverse scattering problem that we study is specified by the knowledge of the magnitude of $\Psi_{\pm}(r,\lambda)$ for large values of r and for λ real. This means that we are given $|F(\lambda)|$ for λ real. The objective is to reconstruct the potentials $V_{\pm}(\cdot)$. The Gelfand-Levitan procedure for solving this problem is as follows.

Step 1: Let

$$\hat{r}(\lambda) = \frac{1}{2\pi} |F(\lambda)|^{-2} \tag{6.9}$$

and compute

$$k(t) = \int_{-\infty}^{\infty} \exp j\lambda t(\hat{r}(\lambda) - \frac{1}{2\pi})d\lambda. \tag{6.10}$$

Then, in (6.2) we have

$$h = -2k(0) \ . \tag{6.11}$$

Step 2: Define

$$k_{\pm}(t,s) = \frac{1}{2}(k(t-s) \pm k(t+s))$$

and solve the Gelfand-Levitan equations (2.16) for $g_{\pm}(.,.)$.

Step 3: The potentials $V_{\pm}(\cdot)$ are given by

$$V_{\pm}(t) = -2 \frac{d}{dt} g_{\pm}(t,t). \tag{6.12}$$

It is obvious that the estimation method of Section II is identical to the Gelfand-Levitan procedure. Note that we have used here a single function $|F(\lambda)|$ to reconstruct two potentials $V_+(\cdot)$ and $V_-(\cdot)$. Usually in quantum mechanics, we are faced with the reverse situation: there is only one potential $V(\cdot)$, and two Jost functions $F_+(\cdot)$ and $F_-(\cdot)$ which are associated to the partial waves $\psi_+(.,.)$ and $\psi_-(.,.)$. Then, to reconstruct $V(\cdot)$ one can use either $|F_+(\cdot)|$ or $|F_-(\cdot)|$ [22].

Note that there is no difference between the inverse scattering problem that we have considered above and the inverse spectral problem where $\hat{r}(\lambda)$ is given and where the objective is to reconstruct $V_{\pm}(\cdot)$. One benefit of using the scattering point of view is that the Jost function $F(\cdot)$ can be interpreted as follows: let

$$A(s) = 1 - \int_0^\infty a(u) \exp{-su} du \tag{6.13}$$

be the whitening filter generating the innovations

$$\nu(t) = y(t) - E\,[y(t)|y(\tau),\ -\infty < \tau < t]$$

$$= y(t) - \int_0^\infty a(u)\ y(t-u) du$$

from the observations $y(\cdot)$. Then, $A(s)$ is an outer function [1] in the Hardy space $H_2\ (d\lambda/1+\lambda^2)$ of functions $f(\cdot)$ which are analytic in the right half-plane and such that

$$\sup_{\sigma>0} \int_{-\infty}^{\infty} |f(\sigma+j\lambda)|^2 \frac{d\lambda}{1+\lambda^2} < \infty.$$

Furthermore

$$\hat{r}(\lambda) = \frac{1}{2\pi} |A(j\lambda)|^{-2}, \tag{6.14}$$

and by noting that $F(\cdot)$ is outer in the upper half-plane [22] and satisfies the identity (6.9), we find that

$$A(s) = F(js). \tag{6.15}$$

VII. Resolvent Identities

The results of Section III show that for $|t| \le T$ the smoothed estimate of z(t) given Y_T is

$$E[z(t)|Y_T] = \hat{z}_+(t|T) + \hat{z}_-(t|T) \tag{7.1}$$

where

$$\hat{z}_\pm(t|T) = \int_0^T H_\pm(t,s;T)y_\pm(s)ds \quad . \tag{7.2}$$

To compute this estimate, it is therefore important to find efficient methods of implementing the even and odd resolvents $H_\pm(.,.;T)$.

A. Even/Odd Resolvent Factorization

By integrating the Bellman-Krein identity (3.9), we obtain the operator factorization

$$\underline{I} - \underline{H}_+ = (\underline{I} - \underline{g}_\pm^*)(\underline{I} - \underline{g}_\pm) \tag{7.3}$$

where the operator $\underline{g}_+$ associated to the kernel $g_\pm(.,.)$ is causal, i.e.

$$g_\pm(t,s) = 0 \quad \text{for } s>t$$

and where $\underline{g}_\pm^*$ denotes the adjoint operator of $\underline{g}_\pm$, so that

$$g_\pm^*(t,s) = g_\pm(s,t). \tag{7.4}$$

This factorization provides an implementation of the smoothed estimates (7.2) in terms of causal operations. If we substitute (7.3) inside (7.2), and note that the even and odd innovations are given by

$$\nu_{\pm}(t) = y_{\pm}(t) - \int_0^t g_{\pm}(t,s)y_{\pm}(s)ds,$$

we obtain

$$\hat{z}_{\pm}(t|T) = \hat{z}_{\pm}(t|t) + \int_t^T g_{\pm}(t,s)\nu_{\pm}(s)ds \ . \tag{7.5}$$

This expression shows that for $0<t<T$, $\hat{z}_{\pm}(t|T)$ can be computed by performing first a forwards pass on $y_{\pm}(\cdot)$ to generate the filtered estimates $\hat{z}_{\pm}(t|t)$ and innovations $\nu_{\pm}(t)$, and then by performing a backwards pass to generate $\hat{z}_{\pm}(t|T)$ for all t (see [19] for more details).

Since the factorization (7.3) requires only the knowledge of the kernels $g_{\pm}(\cdot,\cdot)$, the Gelfand-Levitan procedure can be viewed as a factorization technique [10], [11].

B. Sobolev Identities

The factorization (7.3) has the disadvantage of not exploiting the structure of $H_{\pm}(\cdot,\cdot;T)$. The kernels $k_{\pm}(\cdot,\cdot)$ have a Toeplitz plus Hankel structure. Consequently, we would expect the resolvents $H_{\pm}(\cdot,\cdot;T)$ to have a similar structure, in the same way as resolvents of Toeplitz kernels are close to Toeplitz [16], [17]. However, this property is not displayed by (7.3).

To display the Toeplitz plus Hankel structure of $H_{\pm}(\cdot,;T)$, we observe that

$$\frac{\partial}{\partial t} k_{+}(t,s) + \frac{\partial}{\partial s} k_{-}(t,s) \equiv 0 \tag{7.6a}$$

and

$$\frac{\partial}{\partial t} k_{-}(t,s) + \frac{\partial}{\partial s} k_{+}(t,s) \equiv 0 \ . \tag{7.6b}$$

This structure is partially inherited by $H_{\pm}(\cdot,\cdot;T)$, since

Lemma 1: If $k(\cdot)$ is differentiable, the resolvents $H(\cdot,\cdot;T)$ satisfy

$$\frac{\partial}{\partial t} H_+(t,s;T) + \frac{\partial}{\partial s} H_-(t,s;T) = g_-(T,t)g_+(T,s) \tag{7.7a}$$

$$\frac{\partial}{\partial t} H_-(t,s;T) + \frac{\partial}{\partial s} H_+(t,s;T) = g_+(T,t)g_-(T,s) \tag{7.7b}$$

for $0\le t,\ s\le T$.

Proof: Take the partial derivatives of the equations satisfied by H_+ and H_- with respect to t and s respectively, and add the resulting equations. Then, use (7.6a) and integrate by parts. This gives

$$k_-(t,T)g_+(T,s) = \int_C^T k_-(t,u)c(u,s)\,du + \frac{1}{2}c(t,s)$$

where

$$c(t,s) = \frac{\partial}{\partial t} H_+(t,s;T) + \frac{\partial}{\partial s} H_-(t,s;T) \quad .$$

Then, by linearity, and using the unicity of the solution to (2.16), we obtain (7.7a) The identity (7.7b) can be derived similarly. □

Now, we note that a kernel $k(\cdot,\cdot)$ has a Toeplitz plus Hankel structure if and only if

$$\delta K(t,s) \equiv 0 \tag{7.8}$$

where

$$\underline{\delta} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} \quad .$$

Consequently, to check whether a kernel $L(t,s)$ has a structure which is close to Toeplitz plus Hankel, we can apply the displacement operator $\underline{\delta}$ to $L(t,s)$, and if the resulting displacement kernel $\underline{\delta}L(t,s)$ has finite rank, i.e. if

$$\underline{\delta}L(t,s) = \sum_{i=1}^{N} c_i(t)d_i(s) \quad , \tag{7.9}$$

$L(\cdot,\cdot)$ can be considered as close to Toeplitz plus Hankel. In the case of $H(\cdot,\cdot;T)$, this gives

Theorem 4: If $k(\cdot)$ is twice differentiable, the resolvents $H_{\pm}(\cdot,\cdot;T)$ are such that

$$\delta H_{\pm}(t,s;T) = g_{\pm}(T,t)\frac{\partial}{\partial T}g_{\pm}(T,s) - \frac{\partial}{\partial T}g_{\pm}(T,t)g_{\pm}(T,s) \tag{7.10}$$

with the boundary conditions

$$H_{\pm}(t,s;T) = g_{\pm}(T,s) \tag{7.11}$$

and

$$\frac{\partial}{\partial t}H_{+}(0,s;T) = H_{-}(0,s;T) \equiv 0. \tag{7.12}$$

Proof: Differentiate (7.7a) and (7.7b) with respect to t and s respectively, and subtract the resulting equations. This gives

$$\underline{\delta}H_{+}(t,s;T) = \frac{\partial}{\partial t}g_{-}(T,t)g_{+}(T,s) = g_{+}(T,t)\frac{\partial}{\partial s}g_{-}(T,s)$$

and by using (4.5a) to replace the derivatives of g_{-} with respect to t and s, we get the identity (7.10) for $\underline{\delta}H_{+}$. By symmetry, we get a similar identity for δH_-. The boundary condition (7.11) is obtained by comparing the integral equations satisifed by $H_\pm(\cdot,\cdot;T)$ and $g\ (T,\cdot)$, and (7.12) is a consequence of the property (2.14) for the kernels $k_\pm(\cdot,\cdot)$.

Remark: Thus, the displacement kernels $\delta H_\pm(t,s;T)$ have rank two, and the resolvents $H_\pm(\cdot,\cdot;T)$ have a structure close to Toeplitz plus Hankel. The identity (7.10) is an exact generalization of the so-called Sobolev identity [16], [17] for resolvents of Toeplitz kernels.

Another way of expressing (7.10) is to consider the transformed resolvents

$$\hat{H}_+(\lambda,\mu;T) = \int_0^T \int_0^T (\delta(t-s) - H_+(t,s;T)) \cos \lambda t \cos \mu s \, dt \, ds$$

and

$$\hat{H}_-(\lambda,\mu;T) = \int_0^T \int_0^T (\delta(t-s) - H_-(t,s;T)) \sin \lambda t \sin \mu s \, dt \, ds .$$

The kernels $2\hat{H}_\pm(\lambda,\mu;T)$ can be interpreted as reproducing kernels for the subspaces $S_T^\pm$ of $L_2(\hat{r}(\lambda)d\lambda)$ which are spanned respectively by the functions $\{\cos \lambda t,\ 0 \le t \le T\}$ and $\{\sin \lambda t,\ 0 \le t \le T\}$. To see this, observe that by transforming the factorization (7.3) one gets

$$\hat{H}_\pm(\lambda,\mu;T) = \int_0^T \gamma_\pm(t,\lambda)\ \gamma_\pm(t,\mu) dt . \tag{7.13}$$

Then, let $\hat{f}_\pm(\lambda)$ be an arbitrary function of $S_T^\pm$. Since the functions $\{\gamma_\pm(t,\lambda),\ 0 \le t \le T\}$ span $S_T^\pm$, $\hat{f}_\pm(\lambda)$ can be expressed as

$$\hat{f}_{\pm}(\lambda) = \int_0^T f_{\pm}(s)\gamma_{\pm}(s,\lambda)ds. \tag{7.14}$$

By using (7.13) and (7.14), and the orthogonality relation (5.8) for the eigenfunctions $\gamma_{\pm}(\cdot,\lambda)$, we obtain

$$\int_{-\infty}^{\infty} 2H_{\pm}(\lambda,\mu;T)f_{\pm}(\lambda)\hat{f}(\lambda)d\lambda$$

$$= \int_0^T f_{\pm}(s)\ \gamma_{\pm}(s,\mu)ds = \hat{f}_{\pm}(\mu) \tag{7.15}$$

so that $2\hat{H}_{\pm}\ (\lambda,\mu;T)$ is a reproducing kernel for $S_T^{\pm}$. The Sobolev identity (7.10) can then be transformed into

$$\hat{H}_{\pm}(\lambda,\mu;T) = \frac{1}{\lambda^2 - \mu^2} W[\gamma_{\pm}(T,\lambda),\ \gamma_{\pm}(T,\mu)], \tag{7.16}$$

and (7.16) can be viewed as generalization of the Christoffel-Darboux formula for reproducing kernels associated to orthogonal polynomials on the unit circle or on the real line [3].

C. Even/Odd Resolvent Representation

The Sobolev identity (7.10) shows that the resolvents H $(\cdot,\cdot;T)$ have a structure which is close to Toeplitz plus Hankel. A consequence of this structure is that $\underline{I} - \underline{H}_+$ and $\underline{I} - \underline{H}_-$ can be represented as sums of products of triangular Toeplitz and Hankel operators (see [16], [17] for a discussion of representations of this type in the Toeplitz case). The first step is to add and substract (7.7a) and (7.7b), so that

$$\underline{\tau}(H_+ + H_-) = g_-(T,t)g_+(T,s) + g_+(T,t)g_-(T,s) \tag{7.17}$$

$$\underline{\theta}(H_+ - H_-) = g_-(T,t)g_+(T,s) - g_+(T,t)g_-(T,s) \tag{7.18}$$

where

$$\underline{\tau} = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \quad \text{and} \quad \underline{\theta} = \frac{\partial}{\partial t} - \frac{\partial}{\partial s}$$

are the displacement operators associated respectively to Toeplitz and Hankel kernels. By integrating (7.16) with the boundary condition (7.11), we find that

$$(\underline{I} - \underline{H}_+) + (\underline{I} - \underline{H}_-) = (\underline{I} - \underline{a}_+^*)(\underline{I} - \underline{a}_+) + (\underline{I} - \underline{a}_-^*)(\underline{I} - \underline{a}_-) - (\underline{a}_+ + \underline{a}_-)^*(\underline{a}_+ + \underline{a}_-) \tag{7.19}$$

where $\underline{a}_\pm$ is the causal Toeplitz operator associated to the kernel

$$a_\pm(t,s) = \begin{cases} g_\pm(T,T-t-s)) & t>s \\ 0 & t<s \end{cases} \tag{7.20}$$

and where $\underline{a}_\pm^*$ is the corresponding adjoint operator. Similarly, by integrating (7.18), one gets

$$(\underline{I} - \underline{H}_+) - (\underline{I} - \underline{H}_-) = \underline{b}_-(\underline{I} - \underline{a}_+) - \underline{b}_+(\underline{I} - \underline{a}_-) + \underline{d}_- - \underline{d}_+ \tag{7,21}$$

where $\underline{b}_{\pm}$ and $\underline{d}_{\pm}$ are the triangular Hankel operators given by

$$b_{\pm}(t,s) = \begin{cases} g_{\pm}(T,t+s-T) & t+s \geq T \\ 0 & t+s \leq T \end{cases} \tag{7.22}$$

and

$$d_{\pm}(t,s) = \begin{cases} 0 & t+s > T \\ H(0,T-(t+s);T) & t+s \leq T \; . \end{cases} \tag{7.23}$$

Note that from (7.12), we can conclude that $d_{-}(T,s) \equiv 0$ for all t and s.

The expressions (7.19) and (7.21) can now be combined to express $\underline{I} - \underline{H}_{+}$ and $\underline{I} - \underline{H}_{-}$ as sums of products of triangular Toeplitz and Hankel operators. Since triangular Toeplitz and Hankel operators can be implemented by causal and time-invariant operations, this implies that the even and odd smoothed estimates (7.2) can be computed with causal and time-invariant filters. Furthermore, by using the same procedure as in [26], we can also use the identities (7.19) and (7.21) to obtain time-invariant causal implementations of Gaussian signal detectors.

VIII. Conclusions

In this paper a set of string equations has been obtained for estimating a stationary stochastic process over a finite interval. These string equations can be related to the Levinson recursions, and from a numerical point of view, they have the same efficiency. In the spectral domain, these equations take the form of Schrödinger equations which have the same spectral function as the observed process. By using this observation, we have shown that our estimation procedure is identical to the Gelfand-Levitan inverse scattering method of quantum mechanics.

The results discussed here can be extended in several directions. First, it should be clear that although we have considered the case where we are given some observations of a stationary signal over $|t| < T$, the case where $|t| \geq T$ can be treated similarly. In fact, the solution of this problem is given in [9]. The case where we want to estimate a two-dimensional isotropic random field is more difficult since there are no Levinson recursions in this case. However, it will be shown in [12] that the approach developed here can be generalized directly. This will give some vibrating membrane equations which can be viewed as arising from the Gelfand-Levitan procedure for reconstructing a circularly symmetric potential from its spectral function.

Another important subject of future research is the relation existing between linear estimation and inverse scattering theory. It turns out for example that the Marchenko inverse scattering method of quantum mechanics [22], [23] can also be interpreted from an estimation point of view [24]. In the case when the spectral function $\hat{r}(\lambda)$ is rational, an efficient method of solving linear estimation problems is to use a Kalman filter [1]. It would be interesting to see whether this can be used for inverse scattering problems.

Appendix. Spectral Functions With a Singular Part.

In Section V and VI, it was assumed that the covariance $k(\cdot)$ is summable, i.e. that the signal process $z(\cdot)$ has an absolutely continuous spectral function. When $k(\cdot)$ is only continuous, by Bochner's theorem [9], there exists an odd non-decreasing function $M(\lambda)$ such that

$$k(t) = \int_{-\infty}^{\infty} \exp j\lambda t \, dM(\lambda), \tag{A.1}$$

and $M(\cdot)$ can be decomposed as

$$dM(\lambda) = m(\lambda)\, d\lambda + dM_s(\lambda) \tag{A.2}$$

where $M_s(\cdot)$ denotes the singular part of $M(\cdot)$. Consequently, we have

$$r(t) = k(t) + \delta(t) = \int_{-\infty}^{\infty} \exp j\lambda t \, dR(\lambda), \tag{A.3}$$

where

$$dR(\lambda) = dM(\lambda) + \frac{d\lambda}{2\pi} \tag{A.4}$$

is the spectral function of the observations process $y(\cdot)$. The function $R(\cdot)$ can also be decomposed as

$$dR(\lambda) = w(\lambda)d\lambda + dM_s(\lambda) \tag{A.5}$$

with

$$w(\lambda) = m(\lambda) + \frac{1}{2\pi} \ . \tag{A.6}$$

By using the correspondence (5.3), we can construct an isometry between Y and $L_2(dR(\lambda))$, and the results of Section V remain valid provided that we replace $\hat{r}(\lambda)d\lambda$ by $dR(\lambda)$. This implies in particular that the spectral function of the Schrödinger operators $\underline{\Delta}_{\pm}$ and of the string operator $\underline{M}$ is $R(\cdot)$. Then, the orthogonality relation (5.8) for the eigenfunctions $\gamma_{\pm}(t,\lambda)$ of $\Delta_{\pm}$ becomes

$$\frac{1}{2}\ \delta(t-s) = \int_{-\infty}^{\infty} \gamma_{\pm}(t,\lambda)\ \gamma_{\pm}^{*}(s,\lambda)\ dR(\lambda)\ . \tag{A.7}$$

The results of Section VI are harder to extend since unlike in the absolutely continuous case, the magnitude $|F(\lambda)|$ of the Jost function is not sufficient to specify the spectral function $R(\cdot)$. More precisely, if $F(\cdot)$ is defined as in (6.7) and (6.8), we have

$$w(\lambda) = \frac{1}{2\pi}\ |F(\lambda)|^{-2} \tag{A.8}$$

almost everywhere (see [27]). Consequently, to reconstruct $V_{\pm}(\cdot)$, we need to assume that we are given both $|F(\lambda)|$ and $M_s(\lambda)$, or equivalently that we are given the spectral function $R(\cdot)$. In this case, if we replace (6.10) by

$$k(t) = \int_{-\infty}^{\infty} \exp j\lambda t\ (dR(\lambda) - \frac{d\lambda}{2\pi}), \tag{A.9}$$

we can use the Gelfand-Levitan procedure described in Section VI to reconstruct $V_{\pm}(\cdot)$ (see also [21]).

<u>Special Case</u>: Let

$$z(t) = z_1(t) + \sum_{i=1}^{N} a_i \cos \lambda_i t + \sum_{i=1}^{N} b_i \sin \lambda_i t \tag{A.10}$$

where $z_1(\cdot)$ is a zero-mean Gaussian stationary process with summable covariance $k_1(\cdot)$, and where the amplitudes a_i, b_i are some zero-mean Gaussian random variables uncorrelated with $z_1(\cdot)$ and such that

$$E[a_i a_j] = E[b_i b_j] = A_i \delta_{ij} \tag{A.11a}$$

$$E[a_i b_j] \equiv 0$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. The frequencies λ_i, $1 \leq i \leq N$ are fixed. Then, the covariance of $z(\cdot)$ is given by

$$\begin{aligned} k(t-s) &= E[z(t)z(s)] \\ &= k_1(t-s) + \sum_{i=1}^{N} A_i \cos \lambda_i (t-s) \ , \end{aligned} \tag{A.12}$$

so that in the decomposition (A.2) we have $m(\lambda) = \hat{k}_1(\lambda)$, where $\hat{k}_1(\cdot)$ denotes the Fourier transform of $k_1(\cdot)$, and

$$M_s(\lambda) = \sum_{i=1}^{M} \frac{A_i}{2} (u(\lambda - \lambda_i) + u(\lambda + \lambda_i)) \tag{A.13}$$

where

$$u(\lambda) = \begin{cases} 1 & \text{for } \lambda \geq 0 \\ 0 & \text{for } \lambda < 0 \end{cases}$$

is the unit step function. Thus, the singular continuous part of $M(\cdot)$ is identically zero, so that the measure $M_s(\cdot)$ is <u>purely discrete</u>. In this case, the completeness relation (A.7) becomes

$$\frac{1}{2}\,\delta(t-s) = \int_{-\infty}^{\infty} \gamma_{\pm}(t,\lambda)\gamma_{\pm}^{*}(s,\lambda)w(\lambda)d\lambda \qquad (A.14)$$

$$+ \sum_{i=1}^{N} A_i \gamma_{\pm}(t,\lambda_i)\gamma_{\pm}^{*}(s,\lambda_i)$$

and the spectrum of $\underline{\Delta}_{\pm}$ can be decomposed in two parts: a continuous spectrum of energy levels $E = \lambda^2$ corresponding to the eigenfunctions $\gamma_{\pm}(\cdot,\lambda)$ with $0 \leq E < \infty$, and a discrete spectrum embedded in the continuous spectrum which is associated to the eigenvalues $E_i = \lambda_i^2$ and eigenfunctions $\gamma_{\pm}(\cdot,\lambda_i)$ with $1 \leq i \leq N$. The eigenfunctions $\gamma_{\pm}(\cdot,\lambda_i)$ can be viewed as positive energy bound states. Indeed, it is shown in [27] that $\gamma_{\pm}(\cdot,\lambda_i)$ is square-integrable, and that the Jost function $F(\cdot)$ is such that

$$F(\pm\lambda_i) = 0, \quad 1 \leq i \leq N\,. \qquad (A.15)$$

These bound states differ from negative energy bound states by the fact that the Jost functions has zeros on the real axis, whereas for a bound state at the negative energy level $E = -\kappa^2$ where κ is real and positive, one has

$$F(j\kappa) = 0\,, \qquad (A.16)$$

i.e. $F(\cdot)$ has a zero on the positive imaginary axis [22], [23].

Remark: Negative energy bound states do not appear in the estimation problem that we have considered here. However, it will be shown in [24] that if the signal $z(t)$ contains a deterministic component of the type

$$z_D(t) = \sum_{i=1}^{N} a_i \cosh \kappa_i t + \sum_{i=1}^{N} b_i \sinh \kappa_i t \qquad (A.17)$$

where the amplitudes a_i, b_i satisfy (A.11), the operators $\underline{\Delta}_{\pm}$ have some bound states with energy $E_i = -\kappa_i^2 \leq 0$ and with normalizing constants $A_i = E[a_i^2] = E[b_i^2]$. Signals such as (A.17) have been excluded from consideration in this paper because they are not stationary.

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