Coordinating a Constrained Channel with Linear Wholesale Price Contracts.

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We show that when a supply channel is capacity-constrained and the constraint is tight, there is a set of linear wholesale price contracts that coordinates the channel while allowing the supplier to make a profit. We prove this for the one-supplier/one-newsvendor supply channel as well as the many-supplier/one-newsvendor channel configuration (with each supplier selling a unique product). We analyze how this set of wholesale prices changes as we change the channel's capacity constraint. We also explore conditions under which these channel-efficient linear wholesale price contracts result from the equilibrium behavior of a newsvendor procurement game. Our newsvendor procurement game generalizes the Stackelberg game introduced in Lariviere and Porteus (2001) to allow for multiple suppliers as well as a capacity constraint at the newsvendor. Finally, we find the set of risk-sharing contracts (such as buy-back and revenue-sharing contracts) that coordinate a constrained supply channel and contrast that set with the set of risk-sharing contracts that coordinate an unconstrained channel.

Key words: Wholesale Price Contract; Supply Contracts; Channel Coordination; Capacity Constraint; Newsvendor.

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1. Introduction

There is a wealth of supply contracts available that coordinate a newsvendor's decision for unconstrained supplier-retailer channels: buy-back contracts, revenue-sharing contracts, etc. (Cachon 2003) A contract *coordinates* the actions of a newsvendor *for* a supply channel if the contract causes the newsvendor to take actions when solving his *own* decision problem that are also optimal for the *channel*.[1] Our paper shows that simpler contracts, namely linear wholesale price contracts, (which are thought to be unable to coordinate a newsvendor's decision for unconstrained channels) can, in fact, coordinate a newsvendor's procurement decision for resource-constrained channels. This is relevant for supply channels in which capacity of some resource is limited. For example, shelf space at retail stores, seats on airlines, warehouse space, procurement budgets, time available for manufacturing, raw materials, etc. (Corsten 2006)

[1] Sometimes we also say a contract *channel-coordinates* a newsvendor's decision. Therefore, *achieving coordination* for the channel equates to attaining channel optimality (and thus efficiency) when the newsvendor is allowed to decide for himself.

Towards the end of our paper we show how risk-sharing contracts such as buy-back and revenue-sharing coordinate the procurement decision of a resource-constrained newsvendor thereby generalizing the treatment of these contracts. But the primary insight we show in this paper, is that if newsvendor capacity is a binding constraint, then a *set* of linear wholesale-price contracts can coordinate the procurement decision of a capacity-constrained newsvendor.[2] Furthermore, this set includes wholesale prices that allow both the supplier and the newsvendor to profit.

Wholesale price contracts are commonplace since they are straightforward and easy to implement. While risk-sharing contracts such as revenue-sharing agreements can coordinate a retailer's decision in a newsvendor setting, Cachon and Lariviere (2005) note that these alternative contracts impose a heavier administrative burden. For example, these alternative contracts may require an investment in information technology or a higher level of trust between the trading partners due to the additional processes involved. Our stylized capacity-constrained newsvendor setting provides a laboratory for understanding the set of wholesale price contracts that lead the retailer to take coordinating actions under various channel configurations: one-supplier/one-retailer and multiple-suppliers/one-retailer.

In this paper, we are concerned with the coordination capability of wholesale price contracts for a supply channel in both a negotiation setting and an equilibrium setting. In our negotiation setting, we are concerned with the entire set of coordinating wholesale-price contracts. The wholesale prices in this set are Pareto-optimal, a useful property for getting 'win/win' results in negotiation settings. This is in contrast to an equilibrium setting, where choosing the wholesale price(s) is an initial stage of a game for the supplier(s). In the equilibrium setting we explore conditions for the game's equilibria wholesale-price vector to coordinate the newsvendor's procurement decision for the channel (i.e., necessary and sufficient conditions so that the game's equilibria are included in the set of coordinating wholesale price contracts), and characterize the extent of the efficiency loss when these conditions are violated.

1.1. Organization of this paper

In Section 2, we provide an overview of the supply contracts literature and emphasize the point that the literature has underestimated and not considered the coordination capability of wholesale

[2] In addition to capacity being a binding constraint, the relative power of the parties and their competitive environments are also important for the wholesale-price contract to coordinate the actions of the newsvendor in practice. For example, even if the set of wholesale prices $\mathcal{W}(k)$ that coordinate the retailer's actions is enlarged beyond the supplier's marginal cost (due to the retailer's capacity constraint k), the retailer and supplier still need to agree upon some wholesale price in that set. Their outside-alternatives and the power in the supply channel could determine if some wholesale price in the set $\mathcal{W}(k)$ is acceptable for the parties involved.

price contracts for a constrained supply channel. In Section 3, we provide a stylized 1-supplier/1-retailer model and formally define what it means for a wholesale price contract to coordinate the retailer's ordering decision for a supply channel. This model is analyzed in Section 4. In Section 5, we extend the model to include more suppliers and generalize our earlier analysis. Section 6 shows that the set of revenue-sharing and buyback contracts that coordinates a newsvendor's decision for a constrained channel is a superset of the set of coordinating contracts in the unconstrained setting. Finally, we summarize our findings and provide managerial insights in Section 7.

2. Literature Review

The supply contracts literature has been based on the observation, pointed out, for example, by Lariviere and Porteus (2001), that wholesale price contracts are simple but *do not* coordinate the retailer's order quantity decision for a supplier-retailer supply chain in a newsvendor setting. This observation has led to the study of an assortment of alternative contracts. For example, buy back contracts (Pasternack 1985), quantity flexibility contracts (Tsay 1999), and many others. Cachon (2003) provides an excellent survey of the many contracts and models that have been studied in the supply contracts literature. The mindset surrounding wholesale price contract's inability to channel-coordinate is true under appropriate assumptions— which the supply contracts literature has been implicitly assuming: that there are no capacity constraints (e.g., shelf space, budget, etc.).

As mentioned before, we also consider the case of multiple suppliers serving a single retailer. This exploration is motivated, in part, by Cachon (2003) and Cachon and Lariviere (2005), who emphasize that coordination for channel configurations with multiple suppliers has yet to be explored. The relevant literature on multi-product newsvendors with side constraints (which has developed independently from the coordination literature) includes Lau and Lau (1995), Abdel-Malek and Montanari (2005a,b).

Considering capacity constraints in a supply channel is not new to the supply contracts literature. However, most other papers in the literature consider choosing capacity as one stage of a game (before downstream demand is realized) that also involves a production decision after demand is finally realized (Cachon and Lariviere 2001, Gerchak and Wang 2004, Wang and Gerchak 2003, Tomlin 2003). Our paper, although complementary to this stream of literature, does not involve an endogenous capacity choice for any party but rather analyzes how an exogenous capacity constraint determines the set of wholesale prices that can coordinate the retailer's decision for the channel. Pasternack (2001) considers an exogenous budget constraint, but not for the purposes of studying coordination. Rather, he analyzes a retailer's optimal procurement decision when the retailer has two available strategies: buying on consignment and outright purchase.

Also our paper is not the first to reconsider wholesale price contracts and their benefits beyond simplicity. Cachon (2004) looks at how inventory risk is allocated according to wholesale price contracts and the resulting impact on supply chain efficiency. As far as we are aware, our paper is the first to consider the coordination-capability of linear wholesale price contracts under a simple capacity-constrained production/procurement newsvendor model.

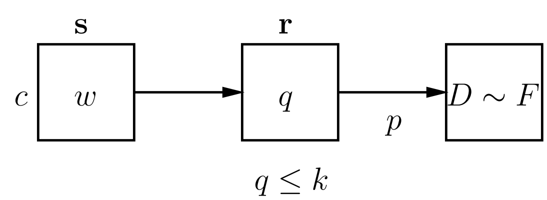
3. Model

A risk-neutral retailer r faces a newsvendor problem in ordering from a risk-neutral supplier for a single good: there is a single sales season, the retailer decides on an order quantity q and orders well in advance of the season, the entire order arrives before the start of the season, and finally demand is realized, resulting in sales for the retailer (without an opportunity for replenishment). Without loss of generality, we assume that units remaining at the end of the season have no salvage value and that there is no cost for stocking out.

The model's parameters are summarized in Figure 1 with the arrows denoting the direction of product flow. In particular, the supplier has a fixed marginal cost of c per unit supplied and charges the retailer a wholesale price $w \geq c$ per unit ordered. The retailer's price p per unit to the market is fixed, and we assume that $p > w$. For that price, the demand D is random with probability density function (p.d.f.) f and cumulative distribution function (c.d.f.) F. We also define $\bar{F}(x) \overset{\text{def}}{=} 1 - F(x) = P(D > x)$. We say that a c.d.f. F has the *IGFR property* (increasing generalized failure rate), if $g(x) \overset{\text{def}}{=} \frac{x \cdot f(x)}{\bar{F}(x)}$ is weakly increasing on the set of all x for which $\bar{F}(x) > 0$ (Lariviere and Porteus 2001). Most distributions used in practice (such as the Normal, the Uniform, the Gamma, and the Weibull distribution) have the IGFR property.

We assume that the retailer's capacity is constrained by some $k > 0$; for example, the retailer can only hold k units of inventory, or accept a shipment not larger than k. For a different interpretation, k could represent a constraint on the capacity of the channel or a budget constraint.

Figure 1 **"single supplier & single capacity constrained retailer" model.**



Note. Supplier s with marginal cost c (per unit) offers a product at wholesale price w (per unit) to a capacity-constrained retailer r that faces uncertain demand D downstream, when the price for the product is fixed at p (per unit). The retailer must decide on a quantity q to order from the supplier.

Assumption 1. *The p.d.f. f for the demand D has support $[0,l]$, with $l > k$, on which it is positive and continuous.*

As a consequence, $\bar{F}(0) = 1$ and $\bar{F}$ is continuously differentiable, strictly decreasing, and invertible on $(0,l)$.

3.1. Retailer's problem

Faced with uncertain sales $S(q) \stackrel{\text{def}}{=} \min\{q, D\}$ (when ordering q units) and a wholesale price w (from the supplier), the retailer decides on a quantity to order from the supplier in order to maximize expected profit $\pi_r(q) \stackrel{\text{def}}{=} E[pS(q)] - wq$ while keeping in mind the capacity constraint k. Namely, it solves the following convex program with linear constraints in the decision variable, q:

RETAILER(k,w)

$$\begin{aligned} \text{maximize} \quad & pE[S(q)] - wq \\ \text{subject to} \quad & k - q \geq 0 \\ & q \geq 0. \end{aligned} \tag{1}$$

Because of our assumptions on the c.d.f. F, it can be shown that *RETAILER(k,w)* has a unique solution which we denote by $q^r(w)$.

3.2. Channel's problem

Denote the channel's expected profit by $\pi_s(q) \stackrel{\text{def}}{=} E[pS(q) - cq]$. Under capacity constraint k, the optimal order quantity q^s for the system/channel is the solution to convex program (2), *CHANNEL(k)*. Note that *CHANNEL(k)* has identical linear constraints but a slightly altered objective function when compared to *RETAILER(k,w)*:

CHANNEL(k)

$$\begin{aligned} \text{maximize} \quad & pE[S(q)] - cq \\ \text{subject to} \quad & k - q \geq 0 \\ & q \geq 0. \end{aligned} \tag{2}$$

Again because of our assumptions on the c.d.f. F it can be shown that *CHANNEL(k)* also has a unique solution which we denote by q^s. We denote the unique solution, $\arg\max_{0 \leq q < \infty} \pi_s(q)$, for the unconstrained channel problem by q^*. It is well known that $q^* = \bar{F}^{-1}(c/p)$ (e.g., Cachon and Terwiesch (2006)). Because of convexity, it is also easily seen that $q^s = \min\{q^*, k\}$.

3.3. Definition: Coordinating the retailer's action

A wholesale price contract w *coordinates* the retailer's ordering decision *for* the supply channel when it causes the retailer to order the channel-optimal amount, i.e., $q^r(w) = q^s$. In Section 4 we are interested in the following questions: For a fixed capacity k, what is the set of wholesale prices $\mathcal{W}(k)$ for which $q^r(w) = q^s$? What does this set $\mathcal{W}(k)$ resemble geometrically?

If there is no capacity constraint (or equivalently if k is very large), recall that 'double marginalization' results in the retailer not ordering enough (i.e., $q^r(w) < q^s$) under any wholesale price contract, $w > c$. In the next section, we will show that when the capacity constraint k is small relative to demand, there exist a set of wholesale price contracts $w > c$ that can coordinate the retailer's order quantity, i.e., $q^r(w) = q^s$.

4. Analysis

Our first result describes the set of coordinating wholesale prices under a capacity constraint.

THEOREM 1. *In a 1-supplier/1-retailer configuration where the retailer faces a newsvendor problem and has a capacity constraint k, any wholesale price*

$$w \in \mathcal{W}(k) \stackrel{def}{=} \left[c, p\bar{F}(\min\{q^*, k\})\right]$$

will coordinate the retailer's ordering decision for the supply channel, i.e., $q^r(w) = q^s$. Furthermore, if $q^r(w) = q^s$ and $c \le w \le p$, then $w \in \mathcal{W}(k)$.

Proof. See Appendix A.

Notice that if the capacity constraint k is larger than or equal to the unconstrained channel's optimal order quantity, q^*, then $p\bar{F}(\min\{q^*, k\}) = p\bar{F}(q^*) = c$, reducing to the 'classic' result in the supply contracts literature. However, this is true only when *the capacity constraint is not binding for the channel* (i.e., $q^* \le k$). When *the capacity constraint k is binding for the channel* (i.e., $q^* > k$), then any wholesale price $w \in [c, p\bar{F}(k)]$ will coordinate the retailer's action *and* only wholesale prices in the range $[c, p\bar{F}(k)]$ can coordinate the retailer's action.

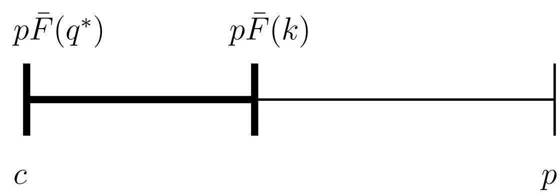
Many factors such as 'power in the channel', 'outside alternatives', 'inventory risk exposure', and 'competitive environment' ultimately influence the actual wholesale price (selected from the set $[c, p]$) charged by the supplier. In the unconstrained setting, regardless of these factors, coordination is not possible with a linear wholesale price contract (because the supplier presumably would not agree to price at cost). However, when the capacity constraint is binding for the channel, coordination becomes *possible* (because the set of coordinating wholesale price contracts becomes

$[c, p\bar{F}(k)]$ (rather than $\{c\}$) and ultimately depends on these other factors. Theorem 6 in Section 4.4 considers a equilibrium setting where the retailer takes on all the inventory risk (akin to the 'Stackelberg game' in Lariviere and Porteus (2001) and 'push mode' in Cachon (2004)), and provides additional conditions that must be met so that the 'equilibrium' wholesale price contract is a member of the set of coordinating wholesale price contracts, $[c, p\bar{F}(k)]$.

4.1. Size of $\mathcal{W}(k)$.

The geometry of the set of wholesale prices $\mathcal{W}(k)$ that coordinate the retailer's decision for the supply channel is depicted in Figure 2.

Figure 2 The set of wholesale prices that coordinates the actions of a single retailer when procuring from a single supplier.



Note. Note that $p\bar{F}(q^*) = c$ and $\mathcal{W}(k) = [c, p\bar{F}(k)]$ (the interval denoted in bold) when $k \le q^*$.

Note that the size of $\mathcal{W}(k)$ is increasing as k decreases. Corollary 1 formalizes this notion and follows directly from Theorem 1 because $\bar{F}(k)$ is decreasing in k.

Corollary 1. *If* $0 \le k_1 \le k_2$*, then* $\mathcal{W}(k_2) \subseteq \mathcal{W}(k_1) \subseteq [c, p]$.

Thus, the more constrained the channel is with respect to the channel optimal order quantity, q^*, the larger the set of coordinating wholesale price contracts $\mathcal{W}(k)$.

Consider two supply channels selling the same good with the same retail price p and supplier cost c. Assume that the probability of excess demand in the first channel is larger, in the sense $\bar{F}_1(k) \ge \bar{F}_2(k)$. Let $\mathcal{W}_i(k)$ denote the set of coordinating wholesale price contracts for channel i when the channel is constrained by k units. The channel with the higher probability of excess demand has a larger set of coordinating wholesale prices. Corollary 2 to Theorem 1 makes this precise.

Corollary 2. *Given two demand distributions* F_1 *and* F_2*, if* $\bar{F}_1(k) \ge \bar{F}_2(k) > 0$*, then*

$$\mathcal{W}_2(k) \subseteq \mathcal{W}_1(k) \subseteq [c, p].$$

Proof. See Appendix B.

4.2. Revenue requirement implicit in $\mathcal{W}(k)$.

By agreeing to focus on the set $\mathcal{W}(k)$ in negotiating over a wholesale price for coordination purposes, the supplier and retailer are implicitly agreeing to a '*minimum* share of expected revenue' requirement for the retailer and thus a '*maximum* share of expected revenue' restriction for the supplier. This notion is formalized in Theorem 2.

THEOREM 2. *If the capacity constraint k is binding for the channel (i.e., $q^* > k$), then any coordinating linear wholesale price contract $w \in \mathcal{W}(k)$ guarantees that the retailer receive at least a fraction $\frac{\int_0^k \bar{F}(x)\,dx - k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}$ of the channel's expected revenue, and that the supplier receive at most a fraction $\frac{k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}$ of the channel's expected revenue. Furthermore, if F has the IGFR property, then the supplier's maximum revenue share is weakly decreasing as k increases.*

Proof. See Appendix C.

An important distinction regarding the supplier and retailer 'share of expected revenue' guarantees formalized in Theorem 2 is that the supplier's share results in a guaranteed income (i.e., no uncertainty) whereas the retailer's share results in an uncertain income. For example, from Theorem 2 there exists some wholesale price $w \in \mathcal{W}(k)$, where the supplier receives a fraction $\frac{k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}$ of the expected channel revenue, $pE[S(k)]$. But the supplier's income is certain, wk, whereas the retailer's income is an uncertain amount, $pS(k) - wk$.

As a numerical example, if $\frac{k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx} = 1/2$, the supplier can receive *up to* fifty percent of the expected channel revenue and still keep the channel coordinated, whereas we require that the retailer receive *at least* fifty percent of the revenue in order for the wholesale price to coordinate the actions of the retailer.

Recall that the set of coordinating wholesale price contracts $\mathcal{W}(k)$ increases with the probability $\bar{F}(k)$ of excess demand, when k is held fixed (Corollary 2). Theorem 3 formalizes a related idea: the larger the expected excess demand, the greater the maximum possible share of revenue at the supplier without sacrificing channel-coordination.

THEOREM 3. *Consider two different demands D_1 and D_2, with each D_i associated with a c.d.f. F_i, that have the same mean and such that $\bar{F}_1(k) \geq \bar{F}_2(k)$. Suppose that (a) the capacity constraint k is binding for the channel under both distributions (i.e., $\min\{q_1^*, q_2^*\} > k$), and (b) $E[(D_1 - k)^+] \geq E[(D_2 - k)^+]$ (i.e., the expected excess demand under D_1 is higher than that under D_2). Then,*

$$\frac{k\cdot\bar{F}_1(k)}{\int_0^k \bar{F}_1(x)\,dx} \geq \frac{k\cdot\bar{F}_2(k)}{\int_0^k \bar{F}_2(x)\,dx}.$$

Proof. See Appendix D.

4.3. Wholesale price contracts and flexibility in allocating channel-optimal profit

The benefits of risk sharing contracts in the unconstrained setting include the ability to channel-coordinate the retailer's decision *as well as* flexibility (due to the extra contract parameters) that allows for any allocation of the optimal channel profit between the supplier and retailer. Cachon (2003) provides excellent examples of the 'channel-profit allocation flexibility' inherent in these more complex contracts.

Theorem 4 demonstrates that in a resource constrained setting, wholesale price contracts also have flexibility in allocating the channel-optimal profit. Namely, these simpler contracts allow for a range of divisions of the optimal channel profit among the firms. The divisions allowed (without losing coordination) depend on the channel's capacity, k. Similar to our observations in Section 4.2 for the implicit revenue requirements, the supplier's share results in a guaranteed income (i.e., no uncertainty) whereas the retailer's share results in an uncertain income.

THEOREM 4. *If the capacity constraint is binding for the channel (i.e., $q^* > k$), there exists a wholesale price contract $w \in \mathcal{W}(k)$ that can allocate a fraction t_s of the channel-optimal profit to the supplier and a fraction $1 - t_s$ to the retailer, if and only if $t_s \in [0, t_s^{max}(k; \bar{F})]$, where*

$$t_s^{max}(k; \bar{F}) \stackrel{def}{=} \frac{k \cdot \left(\bar{F}(k) - c/p\right)}{\int_0^k \left(\bar{F}(x) - c/p\right) dx}.$$

Furthermore, if F has the IGFR property, then $t_s^{max}(k; \bar{F})$ is weakly decreasing as k increases in the range $[0, q^*)$.

Proof. See Appendix E.

Let us interpret Theorem 4 at two extremes values for the capacity k. As k approaches q^*, $t_s^{\max}(k; \bar{F})$ approaches zero. Thus the supplier can not get any fraction of the channel-optimal profit with any wholesale price contract from $\mathcal{W}(k)$ (this was to be expected because $\mathcal{W}(k) = \{c\}$ when $k \geq q^*$). At the other extreme, as k tends to zero, $t_s^{\max}(k; \bar{F})$ tends to one. Thus any allocation of the channel-optimal profit becomes possible with some wholesale price contract from $\mathcal{W}(k)$ (this is natural, because as k tends to zero, the interval $\mathcal{W}(k)$ becomes $[c, p]$). See Figure 3.

Theorem 5 parallels Theorem 3 and makes precise the idea that when we serve a larger market the 'flexibility' in allocating the channel-optimal profit 'increases'.

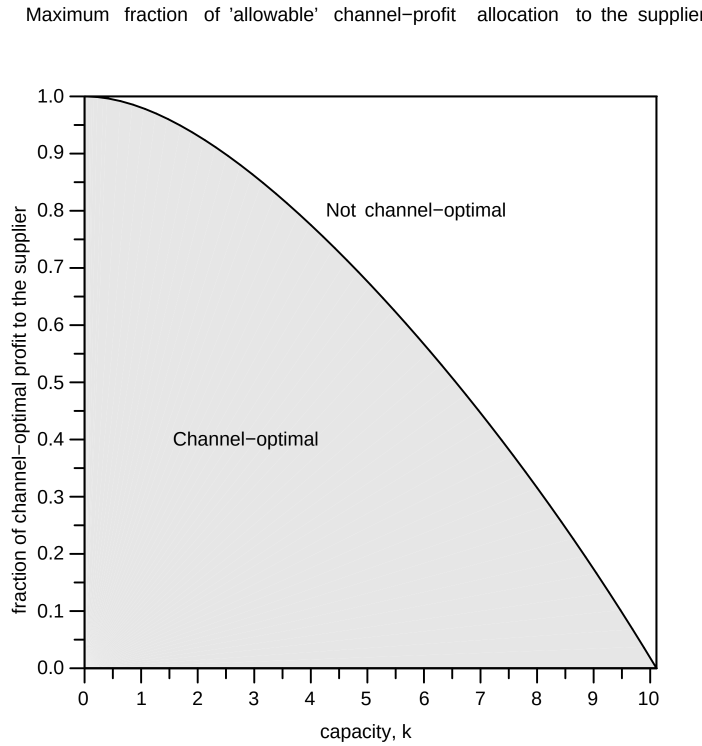
THEOREM 5. *Under the same assumptions as in Theorem 3, we have*

$$t_s^{max}(k; \bar{F}_1) \geq t_s^{max}(k; \bar{F}_2).$$

Proof. See Appendix F.

Theorem 5 suggests that a supplier (and retailer) can find flexibility in profit allocation by joining a supply channel that serves a larger market.

Figure 3 Flexibility in allocating channel-optimal profit as a function of the capacity constraint.



Note. Demand is distributed according to a Gamma distribution with mean 10 and coefficient of variation $2^{-1/2} \approx .707$. The retail price is $p = 10$, and the cost is $c = 4$. (these are similar to parameters used in Cachon (2004)). Thus, $q^* \approx 10.112$. The shaded region denotes the fractions of profit to the supplier consistent with a channel-optimal outcome (i.e., the set $[0, t_s^{\max}(k; \bar{F})]$). Or in other words, the shaded region represents the fractional allocations of channel-optimal profit to the supplier that are achievable with some wholesale price contract $w \in \mathcal{W}(k)$.

4.4. Equilibrium setting.

The equilibrium setting we analyze is a two-stage (Stackelberg) game. In the *first* stage, the supplier (the *'leader'*) sets a wholesale price w. In the *second* stage, the retailer (the *'follower'* chooses an optimal response q, given the wholesale price w. The supplier produces and delivers q units before the sales season starts and offers no replenishments. Both the supplier and retailer aim to maximize their own profit. The supplier's payoff function is $\pi_s(w; q) = (w - c)q$ and the retailer's payoff function is $\pi_r(q; w) = E[pS(q) - wq]$. Lariviere and Porteus (2001) analyze this Stackelberg game, for an unconstrained channel with one supplier and one retailer. They find that when F has the IGFR property, the game results in a unique outcome (q^e, w^e) defined implicitly in terms of the equations

$$p\bar{F}(q^e)\,(1 - g(q^e)) - c = 0, \tag{3}$$

$$p\bar{F}(q^e) - w^e = 0, \tag{4}$$

where g is the generalized failure rate function $g(y) \stackrel{\text{def}}{=} yf(y)/\bar{F}(y)$. Furthermore, they show that the outcome is not channel optimal. In this section, and in Section 4.6, we explore the efficiency of the outcome when the channel has a capacity constraint (i.e., $q \leq k$).

Theorem 6 provides necessary and sufficient conditions on the channel's capacity constraint k for the Stackelberg game to result in a channel-optimal equilibrium.

THEOREM 6. *Assume F has the IGFR property. Consider the above described game, when the channel capacity is k units. This game has a unique equilibrium, given by $q^{eq}(k) = \min\{k, q^e\}$ and $w^{eq}(k) = \max\{p\bar{F}(k), w^e\}$, where q^e and w^e are defined by equations* (3) *and* (4)*, respectively. This equilibrium is channel optimal if and only if*

$$k \leq q^e. \tag{5}$$

Under this condition, we have $q^{eq} = k$ and $w^{eq} = p\bar{F}(k)$.

Proof. See Appendix G.

The function $p\bar{F}(y)\,(1 - g(y)) - c$ represents the supplier's marginal profit on the yth unit, when $y < k$. When F has the IGFR property, the supplier's marginal profit is decreasing in y, while the marginal profit is nonnegative. This fact and equation (3) imply that inequality (5) is equivalent to the inequality $p\bar{F}(k)\,(1 - g(k)) - c \geq 0$, which can be interpreted as a statement that the supplier's marginal profit (when relaxing the capacity constraint) on the kth unit is greater than zero. Therefore, inequality (5) suggests that when the capacity constraint is binding for the supplier's problem (the 'leader' in the Stackelberg game), then the outcome of the game is channel optimal and vice-versa.

If the channel capacity k is 'large enough', so that inequality (5) is not satisfied, how inefficient is the channel? In Section 4.6, we provide a distribution-free 'measuring stick' for the efficiency loss in channels with a capacity constraint.

4.5. When can both parties be better off?

The set of coordinating wholesale price contracts $\mathcal{W}(k)$ introduced in Theorem 1 has many merits in a negotiation setting. For example, such contracts are Pareto optimal. In contrast, Theorem 7 examines the set of wholesale price contracts $\mathcal{D}(k)$ that have little merit in that they are Pareto-dominated by some other wholesale price contract in $[c, p]$. A contract is *Pareto-dominated* if there exists an alternative linear wholesale price contract that makes one party better off without making any other party worse off. Having a complete picture of the contracts that are channel-optimal and the contracts that are Pareto-dominated is helpful in a negotiation setting.

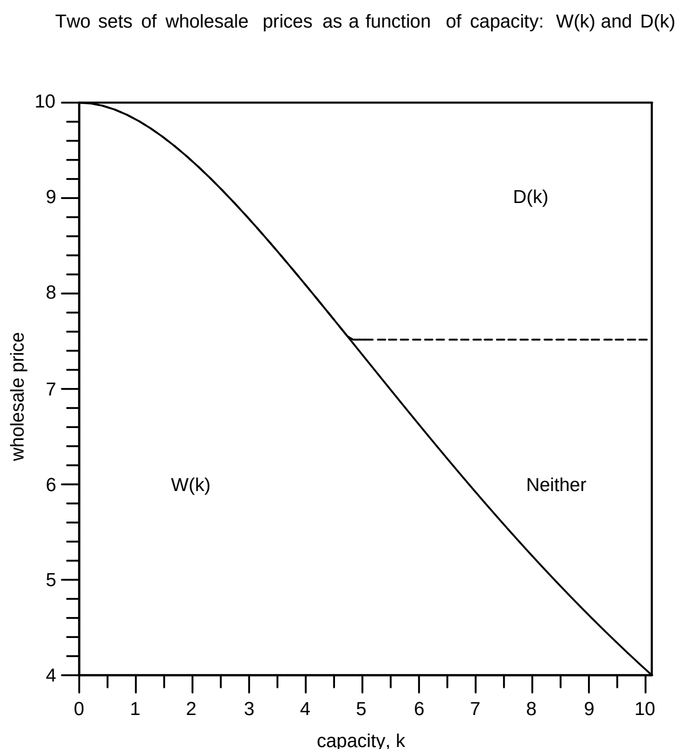
Theorem 7. *Assume F has the IGFR property and that the quantity q^e and wholesale price w^e are defined implicitly in terms of equations* (3) *and* (4). *If $k \le q^*$, then the set of Pareto-dominated wholesale price contracts $\mathcal{D}(k)$ is*

$$\mathcal{D}(k) \stackrel{def}{=} (\max\{w^e, p\bar{F}(k)\}, p] = (p\bar{F}(\min\{q^e, k\}), p].$$

Proof. See Appendix H.

Note that $\mathcal{W}(k)$ and $\mathcal{D}(k)$ are disjoint. Corollary 3 to Theorem 7 formalizes the idea that when k is 'small enough', $\mathcal{W}(k)$ and $\mathcal{D}(k)$ partition the set $[c, p]$. Figure 4 illustrates these ideas when demand has a Gamma distribution.

Figure 4 An example illustrating $\mathcal{W}(k)$ and $\mathcal{D}(k)$.



Note. We use the same parameters as in Figure 3, resulting in $q^* \approx 10.112$, $q^e \approx 4.784$, and $w^e \approx 7.516$. The set of coordinating wholesale price contracts $\mathcal{W}(k)$ lies under the solid curve. The set of Pareto-dominated wholesale price contracts $\mathcal{D}(k)$ lies above both the solid and dashed curves. The set of contracts that lie *between* the solid and dashed curves are neither in $\mathcal{W}(k)$ nor in $\mathcal{D}(k)$. Such contracts do not coordinate the channel, but nevertheless, are not Pareto dominated by coordinating wholesale contracts.

Corollary 3. *Assume F has the IGFR property. If $k \le q^e$, then*

$$\mathcal{W}(k) \cup \mathcal{D}(k) = [c, p], \tag{6}$$

$$\mathcal{W}(k) \cap \mathcal{D}(k) = \emptyset. \tag{7}$$

Corollary 3 is especially interesting: it asserts that when capacity is small enough there are only two types of contracts: 'good contracts', $\mathcal{W}(k)$, and 'bad contracts', $\mathcal{D}(k)$. Furthermore, both parties will always have a reason to avoid the 'bad contracts' because they are Pareto-dominated by some channel-optimal contract in the set $\mathcal{W}(k)$.

4.6. Efficiency Loss.

When the outcome of the Stackelberg game we described in Section 4.4 results in a wholesale price contract that is not channel optimal, how much does the channel 'lose' as a result? What is the 'price' paid for the 'gaming' between the supplier and retailer? To quantify the answer we analyze the worst-case efficiency. Our definition of efficiency is related to the concept of *Price of Anarchy*, "PoA", as used by Koutsoupias and Papadimitriou (1999), and Papadimitriou (2001). PoA has been used as a 'measuring stick' in an assortment of gaming contexts: facility location (Vetta 2002), traffic networks (Schulz and Moses 2003), resource allocation (Johari and Tsitsiklis 2004). More recently Perakis and Roels (2006) analyze the PoA for an assortment of supply channel configurations with the IGFR restriction, but not for resource constrained channels. Theorem 8 complements their results, by providing an efficiency result for the Stackelberg game of Section 4.4, in the presence of a capacity constraint k.

For a channel with a capacity constraint k and probability $\bar{F}(k)$ of excess demand, we define the parameter $\beta \stackrel{\text{def}}{=} \frac{\max\{\bar{F}(k), c/p\}}{c/p}$. The parameter β depends on the probability $\bar{F}(k)$ of excess demand and takes values from the set $[1, p/c]$. It quantifies how constrained the channel is with respect to the channel optimal order quantity q^*, because $\beta \stackrel{\text{def}}{=} \frac{\max\{\bar{F}(k), c/p\}}{c/p} = \frac{\max\{\bar{F}(k), \bar{F}(q^*)\}}{\bar{F}(q^*)}$. In the Stackelberg game with a capacity constraint k and parameter β, the efficiency, $\text{Eff}(k, \beta)$, is defined according to equation (8) below.

$$\text{Eff}(k,\beta) = \inf_{F \in \mathcal{F}(k,\beta)} \frac{\text{Channel profit under 'gaming'}}{\text{Optimal channel profit}} = \inf_{F \in \mathcal{F}(k,\beta)} \frac{E[pS(q^{eq}(k)) - cq^{eq}(k)]}{E[pS(q^s(k)) - cq^s(k)]} \tag{8}$$

The set $\mathcal{F}(k,\beta)$ represents the set of probability distributions that satisfy Assumption 1, have the IGFR property, and such that the probability $\bar{F}(k)$ of excess demand satisfies $\frac{\max\{\bar{F}(k), c/p\}}{c/p} = \beta$. Note that $\text{Eff}(k,\beta)$ is a distribution-free method of quantifying the *worst-case* efficiency. When $\text{Eff}(k,\beta)$ is low (much smaller than one), there is significant efficiency loss due to 'gaming'.

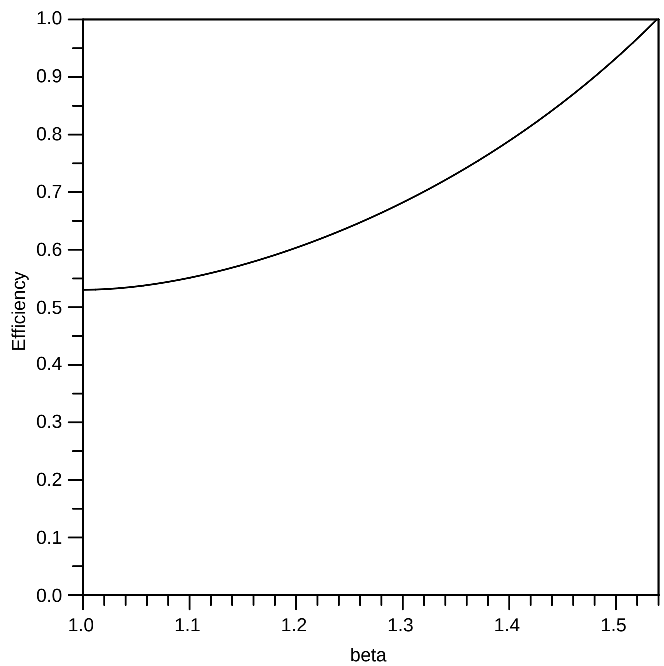
THEOREM 8. *Define* $m \stackrel{def}{=} (p-c)/p$ *(the channel's gross profit margin). For the Stackelberg game described in Section 4.4, we have*

$$\mathit{Eff}(k,\beta) = \left(\left(\frac{\beta - 1 + m}{m} \right) \left(\frac{1}{\beta} \cdot \frac{1}{1-m} \right)^{1/m} - \left(\frac{1}{1-m} \right) \right)^{-1}. \tag{9}$$

Proof. See Appendix I.

Note that $\text{Eff}(k,\beta)$ is decreasing in the channel's gross profit margin m and increasing in β. When $\beta = 1$, the channel is not constrained and $\text{Eff}(k,\beta)$ equals $\left(\left(\frac{1}{1-m}\right)^{1/m} - \frac{1}{1-m}\right)^{-1}$ which, after some algebraic manipulation, matches the result in Perakis and Roels (2006). On the other hand, when the channel is most constrained (i.e., $k \approx 0$, $\bar{F}(k) \approx 1$, and $\beta \approx p/c$), then $\text{Eff}(k,\beta)$ simplifies to 1. In other words there is no efficiency loss because the equilibrium outcome involves the retailer ordering exactly k. Our result is thus a more general version of the 'two-stage push-mode PoA' result in Perakis and Roels (2006) in that we account for a capacity constraint. Also our proof technique differs from and complements Perakis and Roels (2006), in that we indirectly optimize over the space of probability distributions by optimizing over the space of generalized failure rates.

Figure 5 An example illustrating Eff(k,β) when $m = 0.35$.



Note. We fix the margin $(p-c)/p = 0.35$ and see how $\text{Eff}(k,\beta)$ changes as a function of β.

Figure 5 provides an example of the $\text{Eff}(k,\beta)$ when the channel's gross profit margin is 35 percent. Figure 5 illustrates that for channels with smaller capacity (i.e., higher β), the worst-case efficiency (as measured by $\text{Eff}(k,\beta)$) is larger.

5. Extension to the case of Multiple Suppliers

We consider a retailer who orders from multiple suppliers (where each supplier offers one differentiated product), subject to a constraint on the total amount of inventory that can be stocked. The market price for each product is fixed. The retailer faces a random demand for each one of the products (product substitution is not allowed), which is independent of the quantities stocked. In this context, the retailer must make a portfolio decision: which suppliers to order from, and how much to order from each.

For this model, we explore questions similiar to those studied for the single-product case. Do there exist nontrivial (wholesale price different than the unit cost) wholesale contracts that coordinate the retailer's portfolio decision, resulting in an order quantity vector which is optimal from the channel's point of view? How does the set of coordinating wholesale price vectors change as we change the retailer's capacity constraint? Is everyone better off or no worse off by picking a wholesale price vector in this set? We will show that our main findings for the 1-supplier/1-retailer case (Theorems 1 and 6) extend to this more general setting with many suppliers.

5.1. Many-suppliers/1-retailer model

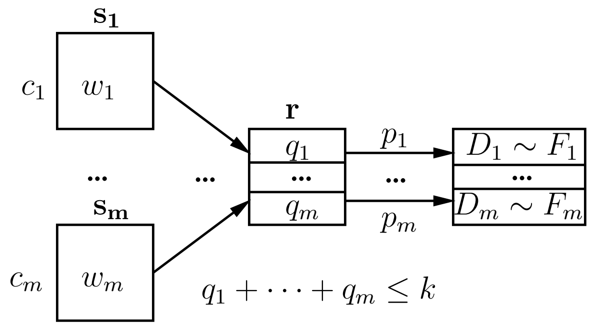
A risk-neutral retailer r orders from $m \geq 2$ risk-neutral suppliers, for m different goods, differentiated by supplier. There is a single sales season, the retailer decides on an order quantity vector/portfolio $(q_1, q_2, \ldots, q_m)$ and orders well in advance of the season, the entire order arrives before the start of the season, and finally demand is realized, resulting in sales for the retailer (without an opportunity for replenishment). Without loss of generality, units remaining at the end of the season are assumed to have no salvage value, and there is no cost for stocking out.

Supplier i has a fixed marginal cost of c_i per unit supplied and charges the retailer a wholesale price $w_i \geq c_i$ per unit ordered. The retailer's price p_i per unit to the market for good i is fixed and, at that price, the demand for good i, D_i, is random with p.d.f. f_i and c.d.f. F_i. We assume that the demands D_i are independent random variables, in the sense that their distribution does not depend on the ordered quantities $(q_1, q_2, \ldots, q_m)$.

The retailer's total capacity is again constrained by some $k > 0$. We assume that the capacity as well as the quantities of the different products are measured with a common set of units (e.g., shelf-space), so that the capacity constraint can be expressed in the form $q_1 + \cdots + q_m \leq k$. The models parameters are summarized in Figure 6, with the arrows denoting the direction of product flow.

As before, we assume that the p.d.f. f_i for the demand D_i has support $[0, l_i]$, with $l_i > k$, on which it is positive and continuous. As a consequence, $\bar{F}_i(0) = 1$ and $\bar{F}_i$ is continuously differentiable, strictly decreasing, and invertible on $(0, l_i)$.

Figure 6 **"m suppliers & 1 capacity constrained retailer" model with independent downstream demands.**



Note. There are m suppliers. Supplier s_i with marginal cost c_i (per unit) offers good i at wholesale price w_i (per unit) to a capacity-constrained retailer r who faces uncertain demand D_i downstream with c.d.f. F_i (for good i) when the price for the good is fixed at p_i (per unit). The retailer decides on a portfolio of goods to order from the suppliers.

5.2. Retailer's problem

For product $i \in \{1,\ldots,m\}$, let $S_i(q) \stackrel{\text{def}}{=} \min\{q, D_i\}$ denote the (uncertain) amount of sales for product i given that the retailer orders q_i units of product i. The retailer decides on a quantity vector $q^r(w) = (q_1^r, q_2^r, \ldots, q_m^r)$ to order (for a given wholesale price vector w) that maximizes the expected profit $\pi_r(q) \stackrel{\text{def}}{=} E[\sum_{i=1}^m p_i \min\{q_i, D_i\} - w_i q_i]$, subject to the capacity constraint k. In particular, it solves the following convex program with linear constraints in the decision vector, q:

RETAILER(k,w)

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^m \big(p_i E[S_i(q_i)] - w_i q_i\big) \qquad (10) \\ \text{subject to} \quad & k - \sum_{i=1}^m q_i \geq 0 \\ & q_i \geq 0, \quad i = 1,\ldots,m. \end{aligned}$$

Because of our assumptions on the distribution of the demand D_i for each product, it can be shown that *RETAILER(k,w)* has a unique solution (vector), which we denote by $q^r(w)$.

5.3. Channel's problem

Given the channel's expected profit $\pi_s(q) \stackrel{\text{def}}{=} E[\sum_{i=1}^m p_i \min\{q_i, D_i\} - c_i q_i]$ and capacity constraint k, the optimal order quantity vector q^s for the system/channel is the solution to the following convex program, *CHANNEL(k)*, with the same linear constraints on the decision vector, q, but a slightly altered objective function:

CHANNEL(k)

$$\text{maximize} \quad \sum_{i=1}^m \big(p_i E[S_i(q_i)] - c_i q_i\big) \qquad (11)$$

$$\text{subject to} \quad k - \sum_{i=1}^{m} q_i \geq 0$$
$$q_i \geq 0, \quad i = 1, \ldots, m.$$

Again, because of our assumptions on the demand distributions, it can be shown that *CHANNEL(k)* also has a unique solution (vector) which we denote by q^s. Finally, we denote the unique solution for the unconstrained channel problem, $\max_{0 \leq q < \infty} \pi_s(q)$, by q^*.

5.4. The set $\mathcal{W}(k)$.

In this subsection, (cf. Theorem 9 below), we derive conditions under which the vector $w = (w_1, ..., w_m)$ belongs to the set $\mathcal{W}(k)$ of wholesale price vectors that coordinate the retailer's order quantity vector, i.e., $q^r(w) = q^s$.

Throughout this subsection, we assume that the capacity constraint is binding for the channel, that is, $\sum_{i=1}^{m} q_i^* \geq k$ or equivalently

$$\sum_{i=1}^{m} \bar{F}_i^{-1}\Big(\frac{c_i}{p_i}\Big) \geq k.$$

Otherwise, the problem degenerates into m standard 1-supplier/1-retailer problems in which the only way to coordinate the retailer's action for the supply channel is with a wholesale price contract $w = (c_1, .., c_m)$.

Theorem 9. *Let $Z = \{i \mid q_i^* = 0\} \subset M = \{1, \ldots, m\}$ be the set of products that are not ordered in the channel's portfolio decision problem, and define λ_{m+1} implicitly by the equation:*

$$\sum_{j \in M \setminus Z} F_j^{-1}\Big(\frac{p_j - c_j - \lambda_{m+1}}{p_j}\Big) = k.$$

For any wholesale price vector $w = (w_1, w_2, ..., w_m)$, the following two conditions are equivalent.

(a) The vector w coordinates the retailer's portfolio decision, i.e., $q^r(w) = q^s$.

(b) There exists some α that satisfies

$$\alpha \in [0, \lambda_{m+1}], \tag{12}$$
$$w_j = c_j + \alpha, \qquad \forall\ j \in M \setminus Z, \tag{13}$$
$$w_j \geq p_j - \lambda_{m+1} + \alpha, \qquad \forall\ j \in Z. \tag{14}$$

Proof. See Appendix J.

Let $\mathcal{W}(k)$ be the set of all w for which $q^r(w) = q^s$. If $Z = \emptyset$ (so that every product is in the channel's optimal portfolio), $\mathcal{W}(k)$ can be represented geometrically by a line segment that starts at the point $(c_1, c_2, \ldots, c_m)$, has unit partial derivatives, and ends at the intersection of the line with the set of vectors w that satisfy $\sum_{i=1}^{m} \bar{F}_i^{-1}(\frac{w_i}{p_i}) = k$. More generally, if $Z \neq \emptyset$, then $W(k)$ is the set described by the conditions (12) through (14).

5.5. The Stackelberg game with multiple suppliers.

We now consider a generalization of the Stackelberg game analyzed in Section 4.4. In the first stage, all the suppliers (the 'leaders') simultaneously choose their wholesale prices w_i. In the second stage, the retailer (the 'follower') chooses an order quantity vector q. When does an equilibrium wholesale price vector of this game belong to the set $\mathcal{W}(k)$? A full exploration of this and other questions related to this interesting game is beyond the scope of this paper and is reported elsewhere (Sabbaghi et al. 2007). We only provide here one result that connects to and generalizes Theorem 6.

THEOREM 10. *Assume the game is symmetric for the suppliers, that is, $c_i = c$, $p_i = p$, and $F_i = F$, for every supplier i. Furthermore, assume that F has the IGFR property. Recall the definition of q^e given in equation (3). If $k \le m \cdot q^e$, then there exists a symmetric equilibrium that belongs to $\mathcal{W}(k)$.*

Proof. See Appendix K.

6. Risk sharing contracts

We have provided necessary and sufficient conditions so that linear wholesale price contracts coordinate a newsvendor's procurement decision and allow both the supplier(s) and the newsvendor to profit. A natural related question is whether more complicated contracts such as buy-back contracts and revenue-sharing contracts also coordinate a newsvendor's procurement decision when the newsvendor is capacity-constrained.

In this section, we prove that revenue-sharing contracts and buy-back contracts continue to coordinate a newsvendor's ordering decision even when the newsvendor has a constrained resource. Furthermore, we examine the advantages of these more complex contracts over a linear wholesale price contract for a constrained newsvendor.

6.1. Buyback and revenue-sharing contracts for unconstrained newsvendor's still coordinate

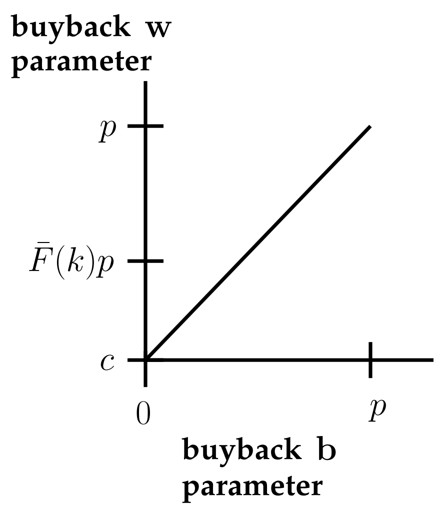
In Theorem 11, we show that buyback contracts, which are known to coordinate an *unconstrained* newsvendor's procurement decision, continue to coordinate a constrained newsvendor's procurement decision.

THEOREM 11. *Consider a 1-supplier/1-retailer configuration in the presence of a capacity constraint $k \ge 0$. Buyback and revenue sharing contracts coordinate the retailer's ordering decision for the channel, and allow for any profit allocation. In particular, the buyback and revenue sharing contracts that coordinate an unconstrained retailer (in the corresponding unconstrained channel) continue to coordinate the constrained retailer's order decision and allow for any profit allocation.*

Proof. See Appendix L.

Figure 7 illustrates the set of buyback contracts (w,b) that channel-coordinate a capacity-constrained newsvendor (as well as unconstrained retailer) as described in Theorem 11. The buyback contracts in Figure 7 are the *only* buyback contracts that can coordinate an unconstrained newsvendor. However, the buyback contracts in Figure 7 are *not* the only buyback contracts that can coordinate a constrained newsvendor. There are more. In Subsection 6.2 we find necessary and sufficient conditions for a buyback contract (w,b) to coordinate a capacity-constrained newsvendor.

Figure 7 Some buyback contracts (w,b) that channel-coordinate a constrained newsvendor.



Note. The buyback contracts (w,b) that channel-coordinate an unconstrained newsvendor's ordering decision (the ones graphed in this figure) still coordinate a capacity-constrained newsvendor. $\bar{F}(k)p$ is labelled on the y-axis purely for comparison with Figure 8.

6.2. Necessary and sufficient conditions for coordination

In Theorem 12, we show that the set of buyback contracts that coordinate an constrained newsvendor's procurement decision is a superset of the set of buyback contracts that coordinate an unconstrained newsvendor's procurement decision.

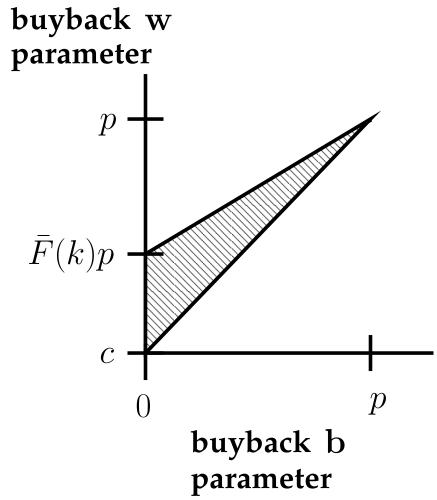
THEOREM 12. *Consider a 1-supplier/1-retailer configuration in the presence of a capacity constraint $k \geq 0$, and assume that $\bar{F}(k) > c/p$. A buy-back contract $(w,b) \in \{(u,v) \mid c \leq u \leq p,\, v \leq u\}$ coordinates a newsvendor's procurement decision for the channel if and only if*

$$(w,b) \in \{(u,v) \mid u = (1-\lambda)v + \lambda p,\ \lambda \in \left[c/p, \bar{F}(k)\right]\}.$$

Proof. See Appendix M.

Notice that if capacity becomes large enough (so that $k \geq q^*$), then the set of coordinating buyback contracts implied by Theorem 12 and Figure 8 simplifies to the 'classical' set of coordinating buyback contracts implied by Theorem 11 and Figure 7.

Figure 8 **Necessary and sufficient conditions for a buyback contract (w, b) to channel-coordinate a constrained newsvendor.**



Note. The shaded area represents all the buyback contracts (w, b) that channel-coordinate a capacity-constrained newsvendor when $k \leq q^c$. Compare with Figure 7.

7. Discussion

One of the reasons that revenue-sharing, buyback, and an assortment of other contracts are able to coordinate the retailer in an unconstrained setting is because those contracts have two or more parameters. Intuitively, the 'flexibility' of those parameters creates contracts where the retailer has an incentive to order the system-optimal amount and that allows the supplier to earn a profit. Interestingly, our model also introduces another 'parameter', capacity. But capacity is not part of the contract. Rather it is part of the system. So instead of introducing complexity into the contract (with another contract parameter) one should check if an inherent resource parameter (such as capacity) can lead to the use of simpler contracts.

Demand and capacity are both levers in the system. If demand is large relative to capacity for the *channel's* problem, then wholesale price contracts that coordinate the channel and allow both the supplier and retailer to profit *exist*. Consequently the potential to reach a channel optimal outcome in a negotiation setting exists.

Furthermore, in the Stackelberg game (Section 4.4) where the supplier acts as the 'leader', if the capacity constraint is tight, the equilibrium outcome is channel optimal. Otherwise, when the equilibrium is not efficient (because the capacity k is not small enough), we provide a distribution-free worst-case characterization of the efficiency loss, as measured by $\mathrm{Eff}(k,\beta)$ (see Section 4.6).

Another lesson for *constrained* channels is that buyback and revenue sharing contracts *still* coordinate the channel (see Section 6.1). And those contracts coordinate the constrained channel for a larger set of parameters than for the unconstrained case.

Cachon (2003) mentions that coordination in multiple supplier settings has not been explored. Theorems 9 and 10 are initial steps in that direction.

Resource constraints are a part of many supply channels. This paper shows that taking them into consideration in the analysis is important in assessing the actual efficiency of contracts for constrained channels.

Acknowledgments

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Appendix

In order to not disrupt the flow of presentation, the proofs for our results are contained here.

A. Proof: 1-supplier/1-retailer, Set of wholesale prices $\mathcal{W}(k)$

Proof of Theorem 1. We start by proving that if $w \in \mathcal{W}(k)$, then $q^r(w) = q^s$. Suppose first that $q^* \le k$. We then have $p\bar{F}(\min\{q^*, k\}) = p\bar{F}(q^*) = c$. Therefore, $\mathcal{W}(k) = \{c\}$. Thus, for any $w \in \mathcal{W}(k)$, the problems *RETAILER(k,w)* and *CHANNEL(k)* are the same and $q^r(w) = q^s$.

Suppose now that $q^* > k$. We then have $q^s = k$ and, furthermore, $p\bar{F}(\min\{q^*, k\}) = p\bar{F}(k) > p\bar{F}(q^*) = c$. (The strict inequality is obtained because $\bar{F}$ is strictly decreasing.) Therefore, $\mathcal{W}(k) = [c, p\bar{F}(k)]$. Solving $\frac{\partial}{\partial x}\left(E[pS(x)] - p\bar{F}(k)x\right) = 0$ for $x \in [0, l]$ and noting $\frac{\partial S(x)}{\partial x} = \bar{F}(x)$, we obtain $q^r(p\bar{F}(k)) = k$. Since $q^r(w)$ is nondecreasing as we decrease w, we see that for all $w \in \mathcal{W}(k)$, $q^r(w) = k = q^s$.

Suppose now that $q^r(w) = q^s$ and $c \le w \le p$. We have shown that

$$\mathcal{W}(k) = \begin{cases} \{c\}, & \text{if } q^* \le k; \\ [c, p\bar{F}(k)], & \text{if } q^* > k. \end{cases}$$

When $q^* \le k$, the first order conditions imply that $p\bar{F}(q^r(w)) - w = 0 = p\bar{F}(q^s) - c$ for any $w \ge c$, which implies w must equal c. When $q^* > k$, we know that $q^s = k$. Assume $w > p\bar{F}(k)$ when $q^r(w) = q^s$. Due to invertibility around k, $q^r(w) < k$. This is a contradiction because $q^s = q^r(w) < k$. $\square$

B. Proof: Impact of size of Market on size of $\mathcal{W}(k)$

Proof of Corollary 2. Let $q_i^* = \bar{F}_i^{-1}(c/p)$ be the order quantity (for an unconstrained channel) under the demand distribution F_i.

If $k \le q_2^*$, then $c/p \le \bar{F}_2(k) \le \bar{F}_1(k)$, which implies that $k \le q_1^*$. Thus, $\mathcal{W}_i(k) = [c, p\bar{F}_i(k)]$ for $i \in 1, 2$. Since $\bar{F}_2(k) \le \bar{F}_1(k)$, we can conclude that $\mathcal{W}_2(k) \subseteq \mathcal{W}_1(k) \subseteq [c, p]$.

Similarly, if $q_2^* < k$, then $\mathcal{W}_2(k) = \{c\}$. Thus, $\mathcal{W}_2(k) \subseteq \mathcal{W}_1(k)$. $\square$

C. Proof: Revenue requirement implicit in $\mathcal{W}(k)$

Proof of Theorem 2. If the capacity constraint k is binding for the channel (i.e., $q^* > k$), then $\mathcal{W}(k) = \left[c, p\bar{F}(k)\right]$. For any wholesale price, the supplier's fraction of expected revenue is $r_s(w) \stackrel{\text{def}}{=} wq(w)/E[pS(q(w))]$ where $q(w)$ is the retailer's order quantity for a wholesale price w. Thus for any coordinating linear wholesale price contract $w \in \mathcal{W}(k)$,

$$r_s(w) = \frac{wk}{E[pS(k)]} = \frac{wk}{p\int_0^k \bar{F}(x)\,dx}.$$

The maximum possible value for $r_s(w)$, when $w \in \mathcal{W}(k)$, is

$$r_s^{\max}(k;\bar{F}) = \frac{\left(p\bar{F}(k)\right)k}{pE[S(k)]} = \frac{k \cdot \bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}.$$

Accordingly, the expected revenue that the retailer receives with any linear wholesale price contract $w \in \mathcal{W}(k)$ is at least a fraction

$$1 - \frac{k \cdot \bar{F}(k)}{\int_0^k \bar{F}(x)\,dx} = \frac{\int_0^k \bar{F}(x)\,dx - k \cdot \bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}$$

of the total.

Next we show that if F has the IGFR property, then $r_s^{\max}(k;\bar{F})$ is weakly decreasing as k increases. We first note that

$$\frac{\partial r_s^{\max}(k;\bar{F})}{\partial k} = \frac{\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx} \cdot \left(1 - g(k) - r_s^{\max}(k;\bar{F})\right), \tag{15}$$

where $g(x) \stackrel{\text{def}}{=} \frac{xf(x)}{\bar{F}(x)}$ is the generalized failure rate function. From L'Hôpital's rule, we also have $\lim_{k\to 0} r_s^{\max}(k;\bar{F}) = 1$. Furthermore, the function $r_s^{\max}(k;\bar{F})$ is bounded above by 1 and goes to zero as $k \to \infty$. If this function is not weakly decreasing, there must exist some value t such that the derivative of $r_s^{\max}(k;\bar{F})$ at t is zero, and positive for values slightly larger than t. We then have

$$r_s^{\max}(t;\bar{F}) = 1 - g(t) \tag{16}$$

since the derivative of $r_s^{\max}(k;\bar{F})$ at t is zero. For k slightly larger than t, the function $r_s^{\max}(k;\bar{F})$ increases, and $g(k)$ is nondecreasing, by the IGFR assumption. But then, equation (15) implies that the derivative of $r_s^{\max}(k;\bar{F})$ is negative, which is a contradiction. $\square$

D. Proof: Revenue requirement as we 'vary' F

Proof of Theorem 3. Note that $\int_0^k \bar{F}_i(x)\,dx = (\int_0^\infty \bar{F}_i(x)\,dx) - (\int_k^\infty \bar{F}_i(x)\,dx) = E[D_i] - E[(D_i - k)\cdot 1_{\{D_i > k\}}]$. Thus,

$$\begin{aligned}
\int_0^k \bar{F}_1(x)\,dx &= E[D_1] - E[(D_1 - k)\cdot 1_{\{D_1 > k\}}] \\
&= E[D_2] - E[(D_1 - k)\cdot 1_{\{D_1 > k\}}] \\
&\le E[D_2] - E[(D_2 - k)\cdot 1_{\{D_2 > k\}}] \\
&= \int_0^k \bar{F}_2(x)\,dx.
\end{aligned} \tag{17}$$

The inequalities (17) and $\bar{F}_1(k) \ge \bar{F}_2(k)$ imply that $\frac{\bar{F}_1(k)}{\int_0^k \bar{F}_1(x)\,dx} \ge \frac{\bar{F}_2(k)}{\int_0^k \bar{F}_2(x)\,dx}$. $\square$

E. Proof: $\mathcal{W}(k)$'s flexibility in allocating the channel-optimal profit

Proof of Theorem 4. We first recall that given our assumption $k < q^*$, the set of coordinating wholesale price contracts is $\mathcal{W}(k) = \left[c, p\bar{F}(k)\right]$.

First we prove that $t_s \in [0, t_s^{\max}(k; \bar{F})]$, if and only if there exists a wholesale price contract $w \in \mathcal{W}(k)$ such that w allocates a fraction t_s of the channel-optimal profit to the supplier (and thus a fraction $1 - t_s$ to the retailer).

For any wholesale price w, the supplier's fraction of the channel's expected profit is $t_s(w) \stackrel{\text{def}}{=} \frac{(w-c)q(w)}{E[pS(q(w))-cq(w)]}$ where $q(w)$ is the retailer's order quantity for a wholesale price w. For any coordinating linear wholesale price contract $w \in \mathcal{W}(k)$, the retailer orders k units; thus we can simplify $t_s(w)$:

$$t_s(w) = \frac{(w-c)k}{E[pS(k)] - ck} = \frac{k(w/p - c/p)}{\int_0^k \left(\bar{F}(x) - c/p\right)\, dx}. \tag{18}$$

Observe that $t_s(c) = 0$, $t_s(p\bar{F}(k)) = t_s^{\max}(k; \bar{F})$, and $t_s(w)$ is strictly increasing and continuous in w for $w \in \left[c, p\bar{F}(k)\right]$. Thus, $t_s(w)$ is a one-to-one and onto map from the domain $\left[c, p\bar{F}(k)\right]$ to the range $[0, t_s^{\max}(k; \bar{F})]$.

Next we show that if F has the IGFR property, then $t_s^{\max}(k; \bar{F}) \stackrel{\text{def}}{=} \frac{k\cdot(\bar{F}(k)-c/p)}{\int_0^k (\bar{F}(x)-c/p)\,dx}$ is weakly decreasing as k increases. Define $\bar{H}(x) = \frac{\bar{F}(x)-c/p}{1-c/p}$. Since $\bar{F}(q^*) = c/p$, $\bar{H}(x)$ restricted to the domain $[0, q^*)$ is equal to $1 - H(x)$, where H is a c.d.f. with support $[0, q^*)$.

The generalized failure rate function $g_H(x)$ for H, defined in equation (19) below, can be rewritten in terms of the generalized failure rate function $g_F(x)$ for F, as follows:

$$\begin{aligned} g_H(x) &\stackrel{\text{def}}{=} -\frac{x\frac{\partial \bar{H}(x)}{\partial x}}{\bar{H}(x)} \qquad (19)\\ &= \frac{xf(x)}{\bar{F}(x) - c/p} \\ &= \frac{\bar{F}(x)}{\bar{F}(x) - c/p} \cdot \frac{xf(x)}{\bar{F}(x)} \\ &= \frac{\bar{F}(x)}{\bar{F}(x) - c/p} \cdot g_F(x). \qquad (20) \end{aligned}$$

Furthermore,

$$\frac{\partial}{\partial x}\left(\frac{\bar{F}(x)}{\bar{F}(x) - c/p}\right) = \frac{f(x)\cdot c/p}{(\bar{F}(x) - c/p)^2} \geq 0, \tag{21}$$

which implies that $\frac{\bar{F}(x)}{\bar{F}(x)-c/p}$ is weakly increasing (over the domain $[0, q^*)$).

Since $\frac{\bar{F}(x)}{\bar{F}(x)-c/p}$ is positive and weakly increasing and F has the IGFR property, we can deduce that H also has the IGFR property when restricted to the domain $[0, q^*)$ (because of equation (20)).

Then, Theorem 2 (applied to $\bar{H}$) implies that $\frac{k\cdot\bar{H}(k)}{\int_0^k \bar{H}(x)\,dx}$ is weakly decreasing as k increases (while k is restricted to the domain $[0, q^*)$). But $t_s^{\max}(k; \bar{F}) = \frac{k\cdot\bar{H}(k)}{\int_0^k \bar{H}(x)\,dx}$, which proves that $t_s^{\max}(k; \bar{F})$ is weakly decreasing as k increases (and $k < q^*$). □

F. Proof: Flexibility in allocating the channel-optimal profit as we 'vary' F

Proof of Theorem 5. Given the definition of $t_s^{\max}(k;\bar{F})$ (cf. Theorem 4), we need to prove that

$$\frac{\bar{F}_1(k)-c/p}{\int_0^k \left(\bar{F}_1(x)-c/p\right)\,dx} \geq \frac{\bar{F}_2(k)-c/p}{\int_0^k \left(\bar{F}_2(x)-c/p\right)\,dx}. \tag{22}$$

We know that $\bar{F}_1(k)\geq \bar{F}_2(k)$ and that the capacity constraint is binding for the channel's problem under both distributions. Thus,

$$\bar{F}_1(k)-c/p \geq \bar{F}_2(k)-c/p > 0. \tag{23}$$

From inequality (17) in the proof of Theorem 3, we also know that $\int_0^k \bar{F}_1(x)\,dx \leq \int_0^k \bar{F}_2(x)\,dx$. Thus, we can deduce that

$$0 < \int_0^k \left(\bar{F}_1(x)-c/p\right)\,dx \leq \int_0^k \left(\bar{F}_2(x)-c/p\right)\,dx. \tag{24}$$

Inequalities (23) and (24) imply that inequality (22) holds. $\square$

G. Proof: When is the equilibrium of the Stackelberg game channel optimal?

Proof of Theorem 6. The retailer's profit function $\pi_r(q;w)$ under a wholesale price contract w is defined as $\pi_r(q;w)\stackrel{\text{def}}{=}E[pS(q)-wq]$. Since $\pi_r(q;w)$ is concave, in q, we can use the first order conditions and conclude that for a wholesale price $w\in[c,p]$, the constrained retailer's order quantity $q^r(w)$ is given by

$$q^r(w)=\min\{k,\bar{F}^{-1}(w/p)\}. \tag{25}$$

The supplier's profit function $\pi_s(w;q)$ under a wholesale price contract w is defined as $\pi_s(w;q)\stackrel{\text{def}}{=}(w-c)q$. Since $q^r(w)$ is the retailer's best response in the second stage to a wholesale price w by the supplier in the first stage, equation (25) allows us to express the supplier's objective function as follows:

$$\pi_s(w)=\begin{cases}(w-c)\,k, & \text{if } c\leq w\leq \max\{c,p\bar{F}(k)\};\\ \left(p\bar{F}(q^r(w))-c\right)q^r(w), & \text{if } \max\{c,p\bar{F}(k)\}<w\leq p.\end{cases} \tag{26}$$

For $w>\max\{c,p\bar{F}(k)\}$, note that $\frac{\partial \pi_s(w)}{\partial w}=\left(p\bar{F}(q^r(w))\,(1-g(q^r(w)))-c\right)\cdot\frac{\partial q^r(w)}{\partial w}$. Since the function $p\bar{F}(y)\,(1-g(y))-c$ is strictly decreasing in y when it is nonnegative and equals zero at q^e (see equation (3)), we can deduce that $\left(p\bar{F}(q^r(w))\,(1-g(q^r(w)))-c\right)>0$ for $w>w^e$ (because $q^r(w)<q^e$). Furthermore, $\frac{\partial q^r(w)}{\partial w}<0$ for $w>p\bar{F}(k)$. Therefore, we can conclude that $\frac{\partial \pi_s(w)}{\partial w}<0$ for $w>\max\{w^e,p\bar{F}(k)\}$.

Either the inequality $p\bar{F}(k)<w^e$ holds or the inequality $w^e\leq p\bar{F}(k)$ holds. First assume the inequality $p\bar{F}(k)<w^e$ holds. Equation (26) implies that $\pi_s(w)$ is increasing linearly between c and $\max\{c,p\bar{F}(k)\}$. Furthermore, since $\left(p\bar{F}(q^r(w))\,(1-g(q^r(w)))-c\right)<0$ for $w<w^e$ (because $q^r(w)>q^e$), we can deduce that $\frac{\partial \pi_s(w)}{\partial w}=\left(p\bar{F}(q^r(w))\,(1-g(q^r(w)))-c\right)\cdot\frac{\partial q^r(w)}{\partial w}>0$ for $w\in(\max\{c,p\bar{F}(k)\},w^e)$. And we know $\frac{\partial \pi_s(w)}{\partial w}<0$ for $w>\max\{w^e,p\bar{F}(k)\}=w^e$. Therefore, $w^{eq}(k)=w^e$ and equations (25) and (4) imply $q^{eq}(k)=q^e$. The inequality $p\bar{F}(k)<w^e$ is equivalent to the inequality $q^e<k$ (see equation (4)). Therefore, when $q^e<k$ holds, the inequality $w^{eq}(k)=w^e>\max\{c,p\bar{F}(k)\}=p\bar{F}(\min\{q^*,k\})$ holds and we can deduce that $w^{eq}(k)\notin\mathcal{W}(k)$ (using Theorem 1).

Next assume $w^e\leq p\bar{F}(k)$ holds. Since $\frac{\partial \pi_s(w)}{\partial w}<0$ for $w>\max\{w^e,p\bar{F}(k)\}=\max\{c,p\bar{F}(k)\}$, equation (26) implies $w^{eq}(k)=p\bar{F}(k)$ and equation (25) implies $q^{eq}(k)=k$. The inequality $w^e\leq p\bar{F}(k)$ is equivalent to the inequality $k\leq q^e$ (see equation (4)). Therefore, when $k\leq q^e$ holds, the equality $w^{eq}(k)=p\bar{F}(k)=\max\{c,p\bar{F}(k)\}=p\bar{F}(\min\{q^*,k\})$ holds and we can deduce that $w^{eq}(k)\in\mathcal{W}(k)$ (again using Theorem 1). $\square$

H. Proof: The set of Pareto-dominated contracts $\mathcal{D}(k)$ as a function of capacity

Proof of Theorem 7. Equation (25) allows us to express the retailer's objective function as follows:

$$\pi_r(w) = \begin{cases} pE[S(k)] - wk, & \text{if } c \leq w \leq p\bar{F}(k); \\ pE[S(q^r(w))] - p\bar{F}(q^r(w))q^r(w), & \text{if } p\bar{F}(k) < w \leq p. \end{cases} \tag{27}$$

Note that $\pi_r(w)$ is strictly decreasing in w, when $w \in (c, p\bar{F}(k))$. Furthermore, when $w \in (p\bar{F}(k), p)$, note that $\pi_r(w)$ is strictly decreasing in w because $\frac{\partial \pi_r(w)}{\partial w} = p\bar{F}(q^r(w))g(q^r(w)) \cdot \frac{\partial q^r(w)}{\partial w} < 0$. From the proof of Theorem 6, we know that the supplier's profit $\pi_s(w)$ is also strictly decreasing for $w > \max\{w^e, p\bar{F}(k)\}$. Therefore, any wholesale price contract in the set $(\max\{w^e, p\bar{F}(k)\}, p]$ is Pareto-dominated by $\max\{w^e, p\bar{F}(k)\}$.

Since the supplier's profit is decreasing as the wholesale price w decreases from $\max\{w^e, p\bar{F}(k)\}$ (see the proof of Theorem 6) but the retailer's profit is increasing as the wholesale price decreases, we can conclude that any wholesale price contract in the set $[c, \max\{w^e, p\bar{F}(k)\}]$ is not Pareto-dominated. Thus, the set of Pareto-dominated wholesale price contracts in $[c,p]$ is exactly $\mathcal{D}(k) = (w^e, p] = (\max\{w^e, p\bar{F}(k)\}, p]$. $\square$

I. Proof: Efficiency loss for a two-stage push channel with capacity constraint

LEMMA 1. *Assume F has the IGFR property and that the quantity q^e is defined implicitly in terms of equation* (3)*. If $q^e \leq k \leq q^s$, then*

$$\frac{p\left(\int_0^k \bar{F}(x)\,dx\right) - ck}{p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e} \leq \left(\left(\frac{k}{q^e}\right)\left(m - 1 + \frac{1}{1-m}\left(\frac{k}{q^e}\right)^{-m}\right) - \frac{1}{1-m} + 1\right)/m. \tag{28}$$

Proof of Lemma 1. Recall the generalized failure rate function $g(y)$ for c.d.f. F is defined as $g(y) \stackrel{\text{def}}{=} -y\frac{\partial \bar{F}(y)}{\partial y}/\bar{F}(y)$. Since $\bar{F}(y) = e^{-\int_0^y f(t)/\bar{F}(t)\,dt} = e^{-\int_0^y g(t)/t\,dt}$, we have

$$\frac{p\left(\int_0^k \bar{F}(x)\,dx\right) - ck}{p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e} = \frac{p\left(\int_0^k e^{-\int_0^x g(t)/t\,dt}\,dx\right) - ck}{p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e} = 1 + \frac{p\left(\int_{q^e}^k e^{-\int_0^x g(t)/t\,dt}\,dx\right) - c(k - q^e)}{p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e}. \tag{29}$$

For any $y \in [q^e, k]$, define the *profit-gain factor* $a(y)$ by

$$a(y) \stackrel{\text{def}}{=} \left(p\left(\int_{q^e}^y e^{-\int_0^x g(t)/t\,dt}\,dx\right) - c(y - q^e)\right) / \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right). \tag{30}$$

The derivative $\frac{\partial a(y)}{\partial y}$ is expressed via equation (31) below, when $y \in [q^e, k]$, leading to the following nonnegative upper bound:

$$\begin{aligned}
\frac{\partial a(y)}{\partial y} &= \left(pe^{-\int_0^y g(t)/t\,dt} - c\right) / \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right) && (31)\\
&\leq \left(pe^{-\int_0^{q^e} g(t)/t\,dt - \int_{q^e}^y g(q^e)/t\,dt} - c\right) / \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right) && (32)\\
&= \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} e^{-\int_0^{q^e} g(t)/t\,dt} - c\right) / \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right) && \\
&\leq \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} e^{-\int_0^{q^e} g(t)/t\,dt} - c\right) / \left(p\left(q^e e^{-\int_0^{q^e} g(t)/t\,dt}\right) - cq^e\right) && (33)\\
&= \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} \bar{F}(q^e) - c\right) / \left(p\bar{F}(q^e) - c\right) q^e && (34)
\end{aligned}$$

$$\leq \left(p \left(\frac{y}{q^e} \right)^{-(p-c)/p} - c \right) / \left(p-c\right) q^e \tag{35}$$
$$= \left(\left(\frac{y}{q^e} \right)^{-m} + (m-1) \right) / \left(mq^e\right). \tag{36}$$

Therefore,

$$\begin{aligned}
\frac{p\left(\int_0^k \bar{F}(x)\,dx\right) - ck}{p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e} &= 1 + \int_{q^e}^k \frac{\partial a(y)}{\partial y}\,dy \\
&\leq 1 + \int_{q^e}^k \left(\left(\frac{y}{q^e} \right)^{-m} + (m-1) \right) / \left(mq^e\right)\,dy \\
&= 1 + \left(\frac{k}{1-m} \left(\frac{k}{q^e} \right)^{-m} - \frac{q^e}{1-m} \left(\frac{q^e}{q^e} \right)^{-m} + (m-1)(k-q^e) \right) / \left(mq^e\right) \\
&= 1 + \left(\frac{1}{1-m} \left(\frac{k}{q^e} \right)^{1-m} - \frac{1}{1-m} + (m-1)(\frac{k}{q^e} - 1) \right) / m \\
&= \left(\left(\frac{k}{q^e} \right) \left(m - 1 + \frac{1}{1-m} \left(\frac{k}{q^e} \right)^{-m} \right) - \frac{1}{1-m} + 1 \right) / m. \quad \square
\end{aligned}$$

Lemma 2. *Under the same assumptions as in Lemma 1, when* $\bar{F}(k) = \delta$ *we have* $k \cdot \delta^{1/m} \leq q^e$*.*

Proof of Lemma 2. Assume $q^e < k \cdot \delta^{1/m}$. This leads to a contradiction (inequality (38)):

$$\begin{aligned}
\delta = \bar{F}(k) = e^{-\int_0^k g(t)/t\,dt} &= e^{-\int_0^{q^e} g(t)/t\,dt} \cdot e^{-\int_{q^e}^k g(t)/t\,dt} \leq 1 \cdot e^{-\int_{q^e}^k g(q^e)/t\,dt} = (k/q^e)^{-g(q^e)} \\
&\leq (k/q^e)^{-m} && (37) \\
&< (k/(k \cdot \delta^{1/m}))^m = \delta. && (38)
\end{aligned}$$

Inequality (37) holds because $p\bar{F}(1 - g(q^e)) \leq c$, implying $g(q^e) \geq m$. Inequality (38) follows from our assumption, $q^e < k \cdot \delta^{1/m}$. $\square$

Proof of Theorem 8. The case when $\beta = 1$ is equivalent to the unconstrained problem which is addressed in Perakis and Roels (2006). Therefore, fix channel capacity k and assume $\beta > 1$, so that $q^s = k$. When $\beta > 1$, the probability of excess demand, which we will denote by δ, is fixed and satisfies $\beta = \delta p / c$.

Fix a c.d.f. $F \in \mathcal{F}(k, \beta)$. The efficiency $\text{Eff}(F)$ of F satisfies the following lower bound:

$$\begin{aligned}
\text{Eff}(F) &\stackrel{\text{def}}{=} E[pS(q^{eq}) - cq^{eq}]/E[pS(k) - ck] \\
&\geq E[pS(q^e) - cq^e]/E[pS(k) - ck] && (39) \\
&= \left(p \left(\int_0^{q^e} \bar{F}(x)\,dx \right) - cq^e \right) / \left(p \left(\int_0^k \bar{F}(x)\,dx \right) - ck \right) \\
&\geq m / \left(\left(\frac{k}{q^e} \right) \left(m - 1 + \frac{1}{1-m} \left(\frac{k}{q^e} \right)^{-m} \right) - \frac{1}{1-m} + 1 \right) && (40) \\
&\geq \left(\left(\frac{1}{\beta} \cdot \frac{1}{1-m} \right)^{1/m} \left(\frac{\beta - 1 + m}{m} \right) - \left(\frac{1}{1-m} \right) \right)^{-1}. && (41)
\end{aligned}$$

In particular, inequality (39) follows because $q^e \leq q^{eq} \leq q^s$. Inequality (40) follows from Lemma 1. The function on the right-hand side of inequality (40) is decreasing as q^e decreases and from Lemma 2 we know that the equilibrium order quantity q^e satisfies $q^e \geq k \cdot \delta^{1/m}$. Therefore, inequality (41) follows when we substitute in $q^e = k \cdot \delta^{1/m} = k\left(\beta(1-m)\right)^{1/m}$.

It can be verified that the lower bound in inequality (41) is attained when the c.d.f. F is taken equal to H, where the c.d.f. H satisfies

1. $\bar{H}(x) = 1$ for $x \in [0, k \cdot \delta^{1/m}]$,
2. $\bar{H}(x) = (k/x)^m \cdot \delta$ for $x \in [k \cdot \delta^{1/m}, \infty)$.

(To verify this claim confirm that $q^e = k \cdot \delta^{1/m}$, using eq. (3), implying that we can convert the inequalities in eqs. (39) and (41) into equalities. Furthermore, since the c.d.f. F is taken equal to H, we can convert the inequalities in eqs. (32),(33), and (35) into equalities. Therefore, inequality (40) becomes an equality.) The c.d.f. H does not satisfy Assumption 1, because the corresponding density is zero for $x \leq k \cdot \delta^{1/m}$. However, it can be approximated arbitrarily closely by c.d.f.s in the class $\mathcal{F}(k, \beta)$ (in particular, that satisfy Assumption 1), with an arbitrarily small change in the resulting efficiency. □

J. Proof: m-suppliers/1-retailer, Set of wholesale prices $\mathcal{W}(k)$

Proof of Theorem 9. First, we write the Lagrangian $\mathcal{L}_s(q, \lambda)$ for *CHANNEL(k)* and the Lagrangian $\mathcal{L}_r(q, \gamma)$ for *RETAILER(k,w)*:

$$\mathcal{L}_s(q, \lambda) = \sum_{i=1}^{m} \big(p_i E[\min(q_i, D_i)] - c_i q_i\big) + \sum_{i=1}^{m} \lambda_i q_i + \lambda_{m+1}\big(k - \sum_{i=1}^{m} q_i\big),$$

$$\mathcal{L}_r(q, \gamma) = \sum_{i=1}^{m} \big(p_i E[\min(q_i, D_i)] - w_i q_i\big) + \sum_{i=1}^{m} \gamma_i q_i + \gamma_{m+1}\big(k - \sum_{i=1}^{m} q_i\big).$$

Note that $\pi_s(q)$ and $\pi_r(q)$ are strictly concave for $q \in [0, l_1) \times \cdots [0, l_m)$ because each c.d.f. F_i is strictly increasing over $[0, l_i)$. Because the feasible sets are convex and compact, *CHANNEL(k)* and *RETAILER(k,w)* have unique solutions. Furthermore, because of the concavity of the objective function and the fact that the Slater condition is satisfied, any critical point of the respective Lagrangian (that satisfies the Karush-Kuhn-Tucker conditions) is the unique maximizer in the respective constrained decision problem. Conversely, the optimal solution in the respective constrained decision problem must correspond to a unique critical point of the respective Lagrangian (Sundaram 1996, chap. 7).

The Karush-Kuhn-Tucker conditions for the channel's decision problem, *CHANNEL(k)*, are:

$$p_j \bar{F}_j(q_j) - c_j + \lambda_j - \lambda_{m+1} = 0, \quad j = 1, \ldots, m;$$

$$q_i \geq 0, \quad i = 1, \ldots, m;$$

$$k - \sum_{i=1}^{m} q_i \geq 0;$$

$$\lambda_i q_i = 0, \quad i = 1, ..m;$$

$$\lambda_{m+1}\big(k - \sum_{i=1}^{m} q_i\big) = 0;$$

$$\lambda_i \geq 0, \quad i = 1, \ldots, m+1.$$

Let (q^s, λ) denote the unique vector that satisfies these conditions.

The Karush-Kuhn-Tucker conditions for the retailer's decision problem, *RETAILER(k,w)*, are:

$$p_j \bar{F}_j(q_j) - w_j + \gamma_j - \gamma_{m+1} = 0, \quad j = 1, \ldots, m;$$

$$q_i \geq 0, \quad i = 1, \ldots, m;$$
$$k - \sum_{i=1}^{m} q_i \geq 0;$$
$$\gamma_i q_i = 0, \quad i = 1, \ldots, m;$$
$$\gamma_{m+1}\big(k - \sum_{i=1}^{m} q_i\big) = 0;$$
$$\gamma_i \geq 0, \quad i = 1, \ldots, m+1.$$

Let $(q^r(w), \gamma)$ denote the unique vector that satisfies these conditions.

Let $M = \{1, \ldots, m\}$ and $Z \stackrel{\text{def}}{=} \{i \in M \mid q_i^s = 0\}$. Therefore, $M \setminus Z$ is the set of items ordered by the system when solving its decision problem. Similarly, let $Z_r(w) \stackrel{\text{def}}{=} \{i \in M \mid q_i^r(w) = 0\}$, so that $M \setminus Z_r(w)$ is the set of items ordered by the retailer when solving its decision problem. Because of the uniqueness of the channel optimal solution, a wholesale price vector $(w_1, \ldots, w_m)$ will coordinate the retailer's portfolio decision (i.e., $q^r(w) = q^s$) if and only if $Z_r(w) = Z$ and $q_i^r(w) = q_i^s$ for every $i \notin Z$.

We claim that $q^r(w) = q^s$ if and only if conditions (42)–(44) hold:

$$\alpha \in [0, \lambda_{m+1}], \tag{42}$$

$$w_j - c_j = w_i - c_i \stackrel{\text{def}}{=} \alpha, \quad \forall i, j \notin Z, \tag{43}$$

$$w_t \geq p_t - \lambda_{m+1} + \alpha, \quad \forall t \in Z. \tag{44}$$

Suppose $q_i^r(w) = q_i^s$, for all i. Eq. (42) follows because

$$0 \leq \gamma_{m+1} \leq \lambda_{m+1}$$

which implies that there exists an $\alpha \in [0, \lambda_{m+1}]$ such that

$$0 \leq \lambda_{m+1} - \alpha = \gamma_{m+1}.$$

Necessity for condition (43) follows because $-c_j + \lambda_j - \lambda_{m+1} = -w_j + \gamma_j - \gamma_{m+1}$ and $\gamma_j = \lambda_j = 0$, when $j \notin Z$, implying

$$c_j + \lambda_{m+1} = w_j + \gamma_{m+1} \quad \forall j \notin Z.$$

Necessity for condition (44) follows because, when $t \in Z$, $p_t - w_t + \gamma_t - \gamma_{m+1} = 0$ and $\gamma_t \geq 0$ hold, implying

$$\lambda_{m+1} - \alpha = \gamma_{m+1} \geq p_t - w_t \quad \forall t \in Z.$$

Now we show sufficiency by showing that conditions (42),(43),(44) imply $Z_r(w) = Z$ and $q_i^r(w) = q_i^s$ for every $i \notin Z_r(w)$. Using conditions (43) and (44) we rewrite the KKT conditions for the retailer's decision problem, *RETAILER(k,w)*:

$$p_j \bar{F}_j(q_j) - c_j + \gamma_j - (\gamma_{m+1} + \alpha) = 0, \quad \forall j \notin Z; \tag{45}$$

$$p_t\bar{F}_t(q_t) - (p_t - \lambda_{m+1} + \alpha + \delta_t) + \gamma_t - \gamma_{m+1} = 0, \quad \forall t \in Z; \tag{46}$$

$$\delta_t = w_t - (p_t - \lambda_{m+1} + \alpha) \geq 0, \quad \forall t \in Z;$$

$$q_i \geq 0, \quad i = 1, \ldots, m;$$

$$k - \sum_{i=1}^{m} q_i \geq 0;$$

$$\gamma_i q_i = 0, \quad i = 1, \ldots, m;$$

$$\gamma_{m+1}\big(k - \sum_{i=1}^{m} q_i\big) = 0;$$

$$\gamma_i \geq 0, \quad i = 1, \ldots, m+1.$$

When $\gamma_{m+1} = \lambda_{m+1} - \alpha$, $\gamma_i = 0$ for all $i \notin Z$, and $\gamma_i = \delta_i$ for all $i \in Z$, we have that (q^s, γ) satisfies the KKT conditions for *RETAILER(k,w)*. Note that (q^s, γ) satisfies (45) because (q^s, λ) satisfies the KKT conditions for *CHANNEL(k)* and (46) is satisfied because $q_t = 0$. Therefore, $q_i^r(w) = q_i^s$ for every $i \in M$. $\square$

K. Proof: m-suppliers/1-retailer, equilibrium setting

Proof of Theorem 10. It can be shown that each supplier's payoff function is continuous and quasi-concave with respect to their own wholesale price; see Sabbaghi et al. (2007). Furthermore, the game is symmetric and the strategy space (the hypercube of possible wholesale price vectors) is compact and convex. Therefore, by Theorem 2 in Cachon and Netessine (2004), there exists at least one symmetric pure strategy Nash equilibrium (i.e., wholesale price vector), in which all the suppliers charge the same wholesale price w.

Due to the symmetry in the problem, $Z = \emptyset$ (i.e., all the products are included in the channel's optimal portfolio). Furthermore, the capacity constraint is tight for the channel, thus the channel's optimal order vector is $(k/m, \ldots, k/m)$. The symmetric equilibrium (identical wholesale prices across products) results in the retailer order vector $(k/m, \ldots, k/m)$ because the retailer's capacity constraint is also tight under the condition $k \leq m \cdot q^e$. Thus the wholesale price vector $(w, \ldots, w)$ is in the set $\mathcal{W}(k)$ by definition. $\square$

L. Proof: Buyback and revenue-sharing contracts continue to coordinate

Proof of Theorem 11. Our proof follows the proof technique given in Cachon (2003) for the 1-supplier, 1-retailer channel in the absence of a capacity constraint.

Our proof has two parts. The first part shows that buyback contracts coordinate a capacity-constrained newsvendor, allocating any fraction of the channel optimal profit among the parties. The second part shows that buyback contracts are equivalent to revenue sharing contracts in a constrained setting.

Under a buyback contract (w, b) the newsvendor pays w per unit to the supplier for each unit ordered and is compensated b per unit for any unit unsold at the end of the sales season. We show that if

$$w = b + c(p - b)/p, \quad b \in [0, p], \tag{47}$$

then the buyback contract (w, b) coordinates the capacity-constrained newsvendor's ordering decision, giving the newsvendor $(p - b)/p$ fraction of the channel-optimal profit and the supplier b/p fraction of the channel-optimal profit.

We show that under the above buyback contract, (w,b), the channel-optimal order quantity, q^c, equals the retailer-optimal order quantity, q^r, as well as the supplier-optimal order quantity (i.e., the retailer's order quantity that is optimal from the supplier's point of view), q^s. Indeed,

$$\begin{aligned} q^c &\overset{\text{def}}{=} \arg\max_{0\le q\le k} pS(q) - cq \\ &= \arg\max_{0\le q\le k} \big((p-b)/p\big)\big(pS(q)-cq\big) \\ &= \arg\max_{0\le q\le k} (p-b)S(q) - (w-b)q && \text{(Using buyback contract (47))} \\ &= \arg\max_{0\le q\le k} pS(q) - wq + b(q-S(q)) \\ &\overset{\text{def}}{=} q^r \end{aligned} \tag{48}$$

and

$$\begin{aligned} q^c &\overset{\text{def}}{=} \arg\max_{0\le q\le k} pS(q) - cq \\ &= \arg\max_{0\le q\le k} \big(b/p\big)\big(pS(q)-cq\big) \\ &= \arg\max_{0\le q\le k} bS(q) - (c-w+b)q && \text{(Using buyback contract (47))} \\ &= \arg\max_{0\le q\le k} wq - cq - b(q-S(q)) \\ &\overset{\text{def}}{=} q^s \end{aligned} \tag{49}$$

Equations (48) and (49) prove that the newsvendor and supplier receive $\big((p-b)/p\big)$ and $\big(b/p\big)$ fractions, respectively, of the channel-optimal profit.

Next, we remind the reader that buyback contracts and revenue sharing contracts are equivalent (regardless of the channel's capacity constraint). Under a revenue sharing contract the newsvendor purchases each unit from a supplier at a price of w_r per unit, keeps a fraction f of the revenue, and shares a fraction $(1-f)$ of the revenue with the supplier. A given buyback contract, (w,b), is a revenue sharing contract where the newsvendor purchases at $w-b$ per unit from the supplier and in return shares a fraction b/p of the revenue with the supplier. Similiarly, a given revenue sharing contract, (w_r,f), is a buyback contract where the newsvendor purchases at $w_r+(1-f)p$ per unit and is compensated $(1-f)p$ per unit by the supplier for any unsold items at the end of the sales season. Since there is a one-to-one mapping from buyback contracts to revenue sharing contracts and because buyback contracts coordinate a constrained newsvendor's ordering decision, we conclude that revenue sharing contracts also coordinate a constrained newsvendor's ordering decision. □

M. Proof: Necessary and suff. conditions for risk-sharing contracts to coordinate

Proof of Theorem 12. Let

$$B \overset{\text{def}}{=} \{(u,v) \mid u=(1-\lambda)v+\lambda p,\ \lambda\in[c/p,F(k)]\}$$

and

$$A \overset{\text{def}}{=} \{(u,v) \mid c\le u\le p, v\le u\}.$$

The proof has two parts. First we show every buyback contract $(w,b) \in B \subseteq A$ channel-coordinates the newsvendor's decision. Then, we show that there are no other buyback contracts in the set A that can channel-coordinate the newsvendor's decision. Before we proceed note that the optimal order quantity for the constrained channel is k (because $\bar{F}(k) > c/p$). Thus, the capacity constraint is tight.

First we show that every buyback contract $(w,b) \in B$ channel-coordinates. If $(w,b) \in B$, then $w - b = \lambda(p-b)$ for some $\lambda \in [c/p, \bar{F}(k)]$. The newsvendor orders $\min\{k, \bar{F}^{-1}(\frac{w-b}{p-b})\}$. But $\frac{w-b}{p-b} \in [c/p, \bar{F}(k)]$, therefore $\bar{F}^{-1}(\frac{w-b}{p-b}) \geq k$ and $\min\{k, \bar{F}^{-1}(\frac{w-b}{p-b})\} = k$. The newsvendor thus orders the channel-optimal order quantity for this capacity-constrained channel.

Next we show that there is no buyback contract (w,b) outside of B but in set A that channel-coordinates the newsvendor's action. Assume the contrary. Namely, assume a buyback contract $(w,b) \in A \setminus B$ channel-coordinates the newsvendor's action. Under buyback contract (w,b), the constrained newsvendor orders $\min\{k, \bar{F}^{-1}(\frac{w-b}{p-b})\}$. But since (w,b) channel-coordinates the newsvendor's decision, we have $\min\{k, \bar{F}^{-1}(\frac{w-b}{p-b})\}\} = k$, since the newsvendor's constraint is tight. Therefore, $\bar{F}^{-1}(\frac{w-b}{p-b}) \geq k$, implying $\frac{w-b}{p-b} \leq \bar{F}(k)$. Furthermore, $\min_{(w,b)\in A} \frac{w-b}{p-b} = \frac{c}{p}$, implying $\frac{w-b}{p-b} \geq \frac{c}{p}$. Thus, $(w,b) \in B$, because $w - b = \lambda(p-b)$ for some $\lambda \in [c/p, \bar{F}(k)]$. But this is a contradiction. □

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