Sensitivity to Cumulative Perturbations for a Class of Piecewise Constant Hybrid Systems

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Abstract—We consider a class of continuous-time hybrid dynamical systems that correspond to subgradient flows of a piecewise linear and convex potential function with finitely many pieces, and which include the fluid-level dynamics of the Max-Weight scheduling policy as a special case. We study the effect of an external disturbance/perturbation on the state trajectory, and establish that the magnitude of this effect can be bounded by a constant multiple of the integral of the perturbation. We also discuss the extent to which such a result can be extended.

I. Introduction

We consider a class of continuous-time, non-expansive, hybrid systems that are subject to an external disturbance/perturbation, and develop a bound on the effect of the perturbation on the state trajectory, in terms of the integral of the perturbation.

In order to appreciate the issues that arise, and the usefulness of such a result, let us consider a discrete-time system of the form

\[ x(t+1) = f(x(t)), \quad t = 0, 1, \ldots, \]

and its perturbed counterpart

\[ \tilde{x}(t+1) = f(\tilde{x}(t)) + u(t), \quad t = 0, 1, \ldots \] (1)

Here, \( x(t) \) and \( u(t) \) take values in \( \mathbb{R}^n \) and we assume that the mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is non-expansive, in the sense that

\[ \|f(x) - f(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n, \]

for a given norm \( \| \cdot \| \). A straightforward induction yields a bound on the distance of the perturbed trajectory from the original one: assuming the same initial conditions, \( \tilde{x}(0) = x(0) \), we have

\[ \|\tilde{x}(t) - x(t)\| \leq \sum_{\tau=0}^{t-1} \|u(\tau)\|. \] (2)

However, our goal is to derive stronger bounds, of the form

\[ \|\bar{x}(t) - x(t)\| \leq C \max_{k<t} \sum_{\tau=0}^{k} \|u(\tau)\|, \] (3)

for some constant \( C > 0 \) independent of \( u(\cdot) \).

A bound of the form (3) is not valid in general, even for non-expansive systems, or gradient fields of convex functions \( f \). Nevertheless, we show that such a bound is valid for the class of continuous-time hybrid systems driven by a piecewise constant drift, determined by the subdifferential of a piecewise linear and convex function with finitely many pieces. Within this class of systems, the dynamics are automatically non-expansive with respect to the Euclidean norm. Furthermore, this class is fairly broad, in the sense that it actually contains the seemingly larger class of non-expansive finite-partition hybrid systems\( [1] \). Finite-partition systems often arise in the context of systems that are controlled through the selection of a particular action at each time among the elements of a finite set. They have attracted broad interest, due to numerous applications to communication networks \( [3], [4] \), processing systems \( [5] \), manufacturing systems, and inventory management \( [6], [7] \), etc.

A prominent example to which our results apply are the fluid-level dynamics of the celebrated Max-Weight policy for real-time job scheduling \( [8] \). This policy is used for scheduling in queueing systems: at each time, it chooses a service vector (from a finite set) that maximizes a weighted sum of the current queue lengths. This policy and its properties like stability \( [8], [9], [10], [11] \) and state space collapse \( [12], [13] \), have been studied extensively over the last three decades.

When the Max-Weight policy is applied to a discrete-time stochastic setting, the perturbation \( u(\cdot) \) in (1) is the sample path of a stochastic process, and captures the fluctuations in job arrivals. Under usual probabilistic assumptions, \( \sum_{\tau=0}^{t-1} \|u(\tau)\| \) grows at the rate of \( t \), whereas \( \max_{k<t} \sum_{\tau=0}^{k} u(\tau) \) only grows as (roughly) \( \sqrt{t} \), with high probability. This fact, in combination with the main result of this paper, leads to tighter than earlier available probabilistic bounds on the fluctuations of the Max-Weight trajectories from their deterministic (fluid) counterparts, and opens the way for new results \( [14] \), such as strengthening the state-space collapse

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1 In a finite-partition hybrid system, the domain is partitioned into a finite number of regions, and system trajectories have a constant drift in the interior of each region.
results in [13]. More specifically, in [14], we study in detail the discrete-time Max-Weight dynamics: we use the results of this paper to prove a bound similar to (3), and also address variants of a state-space collapse conjecture posed in [13]. Furthermore, our approach also enables us to settle another open problem that was posed in [15], on delay-stability in the presence of heavy-tailed traffic, as will be reported in a forthcoming paper.

As is apparent from our discussion of the Maw-Weight policy, one may be ultimately interested in a discrete-time system, as opposed to the continuous-time systems considered in this paper. However the two settings (discrete or continuous) are closely related. For example, in many applications involving communication networks or scheduling systems, the underlying system may evolve in discrete time, but much of the analysis is often carried out in terms of related continuous-time models that are easier to analyze. For example, discrete-time models often involve uninteresting “edge effects” that can result in tedious technical details and can obscure the essence of the underlying mathematical structure. For such reasons, we found it more natural to start with the development of the core concepts and results within the more elegant continuous-time framework in this paper, and then translate them back to the discrete-time framework. As an example, [1] shows that if a continuous-time system admits a bound of the form (3), then its discrete-time counterpart obeys a similar bound.

Regarding related literature, we are not aware of any work that resembles the main result of this paper. Some seemingly related research threads deal with input-to-state stability [16], [17], [18], [19], [20], integral input-to-state stability [21], and robust input-to-state stability [22]. However we note that integral input-to-state stability [21] is concerned only with generalizations of the weak bound in (2). Furthermore, for systems with additive disturbances, \( x(t+1) = f(x(t)) + u(t) \), input-to-state stability and the bound (3) do not imply one another. The fundamental difference is that input-to-state stability concerns the growth of a single trajectory, whereas the bound in (3) concerns the size of the difference of two different trajectories. Another key difference is that input-to-state stability results usually rely on Lyapunov-type arguments [23]. However, Lyapunov functions seem to be inadequate for our purposes. This is because our bounds (as can be seen in the proof given in Section IV) rely in a delicate manner on the relative orientation of the two trajectories \( x(\cdot) \) and \( \tilde{x}(\cdot) \), in conjunction with the local “landscape” of the potential function. Furthermore, as shown in [11], the desired sensitivity bound (3) fails to carry over if the number of constant-drift regions is not finite. This means that a Lyapunov-based argument would have to make essential use of our finiteness assumption, which is a tall order.

The rest of this paper is organized as follows. In the next section we discuss some preliminaries and our notational conventions. In Section III we state our main theorem. In Section IV we provide the core of the proof, while relegating some of the details to the Appendix. Finally, in Section V we discuss possible extensions, open problems and challenges, and directions for future research.

II. Preliminaries

A. Notation

We denote by \( \mathbb{R} \) and \( \mathbb{R}^+ \) the sets of real numbers and non-negative reals, respectively. For a column vector \( v \in \mathbb{R}^n \), we denote its transpose and Euclidean norm by \( v^T \) and \( \|v\| \), respectively. For any set \( S \subseteq \mathbb{R}^n \), \( \text{span}(S) \) stands for the span of the vectors in \( S \). Furthermore, if \( p \) is a point in \( \mathbb{R}^n \), then \( p + S \) stands for the set \( \{ p + x \mid x \in S \} \), and \( d(p, S) \) for the Euclidean distance between \( p \) and \( S \), with the convention that \( d(p, S) = \infty \) if \( S \) is empty. Similarly, we let \( d(p, \{ x \}) = \| p - x \| \) for \( p, x \in \mathbb{R}^n \). We finally let \( A \setminus B = A \cap B^c \), for any two sets \( A \) and \( B \), where \( B^c \) is the complement of \( B \).

B. Perturbed Dynamical Systems

As in [24], we identify a dynamical system with a set-valued function \( F : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) and the associated differential inclusion \( \dot{x}(t) \in F(x(t)) \). We start with a formal definition, which allows for the presence of perturbations.

Definition 1 (Perturbed Trajectories). Consider a dynamical system \( F : \mathbb{R}^n \to 2^{\mathbb{R}^n} \), and let \( U : \mathbb{R} \to \mathbb{R}^n \) be a right-continuous function, which we refer to as the perturbation. Suppose that there exist measurable and integrable functions \( \overline{x}(\cdot) \) and \( \zeta(\cdot) \) of time that satisfy

\[
\overline{x}(t) = \int_0^t \zeta(\tau) \, d\tau + U(t), \quad \forall \ t \geq 0, \quad (4)
\]

\[
\zeta(t) \in F(\overline{x}(t)), \quad \forall \ t \geq 0.
\]

We then call \( U \) the perturbation. Any such \( \overline{x} \) and \( \zeta \) is called a perturbed trajectory and a perturbed drift, respectively. In the special case where \( U \) is identically zero, we also refer to \( \overline{x} \) as an unperturbed trajectory.

Note that a perturbed trajectory is automatically right-continuous. In the absence of the perturbation \( U(\cdot) \), Eq. (4) becomes the differential inclusion \( \dot{x}(t) \in F(x(t)) \) (almost everywhere). When perturbations are present, \( U \) is often absolutely continuous, of the form \( \int_0^t u(\tau) \, d\tau \), for some measurable function \( u(\cdot) \). In this case, we are essentially dealing with the differential inclusion \( \dot{x}(t) = F(\overline{x}(t)) + u(t) \). However, the integral formulation in Definition 1 is more useful because it also applies to cases where \( U \) is not absolutely continuous, e.g., if \( U \) is a sample path of a Wiener or a jump process.
Proof. It follows from Lemma 2.30 of [24] that any subgradient dynamical system is a maximal monotone map\footnote{A set valued function $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a monotone map if for any $x_1, x_2 \in \mathbb{R}^n$ and any $v_1 \in F(x_1)$ and $v_2 \in F(x_2)$, we have $(v_1 - v_2) \cdot (x_1 - x_2) \leq 0$. It is called a maximal monotone map if it is monotone, and for any monotone map $\bar{F}$, that satisfies $F(x) \subseteq \bar{F}(x)$ for all $x$, we have $F = \bar{F}$.}. The lemma then follows from Corollary 4.6 of [24].

C. Classes of Systems

We now introduce some classes of systems of interest. A dynamical system $F$ is called non-expansive if for any pair of unperturbed trajectories $x(t)$ and $y(t)$, and whenever $0 \leq t_1 \leq t_2$, we have

$$\|x(t_1) - y(t_1)\| \leq \|x(t_2) - y(t_2)\|. \quad (5)$$

For a convex function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, we denote its subdifferential by $\partial \Phi(x)$. We say that $F$ is a subgradient dynamical system if there exists a convex function $\Phi(x)$, such that for any $x \in \mathbb{R}^n$, $F(x) = -\partial \Phi(x)$. Furthermore, if $\Phi$ is of the form

$$\Phi(x) = \max_i (-\mu_i x + b_i),$$

for some $\mu_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, and with $i$ ranging over a finite set, we say that $F$ is a Finitely Piecewise Constant Subgradient (FPCS, for short) system; cf. Fig. 1.

Subgradient systems are known to have several useful properties: they are automatically non-expansive (cf. Part 5 of Theorem 4.4 in [24]), a fact that we will be using in the sequel. Existence and uniqueness results are also available [24].

Lemma 1 (Existence and Uniqueness of Solutions). For any subgradient dynamical system $F$ and any $x_0 \in \mathbb{R}^n$, there exists a unique trajectory of $F$ initialized at $x_0$.

Proof. It follows from Lemma 2.30 of [24] that any subgradient dynamical system is a maximal monotone map\footnote{A set valued function $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a monotone map if for any $x_1, x_2 \in \mathbb{R}^n$ and any $v_1 \in F(x_1)$ and $v_2 \in F(x_2)$, we have $(v_1 - v_2) \cdot (x_1 - x_2) \leq 0$. It is called a maximal monotone map if it is monotone, and for any monotone map $\bar{F}$, that satisfies $F(x) \subseteq \bar{F}(x)$ for all $x$, we have $F = \bar{F}$.}. The lemma then follows from Corollary 4.6 of [24].

III. Main Result

We now state the main result of the paper. Its proof is given in Section IV.

Theorem 1 (Input Sensitivity of FPCS Systems). Consider an FPCS system $F$. Then, there exists a constant $C$ such that for any unperturbed trajectory $x(t)$, and for any perturbed trajectory $\tilde{x}(t)$ with corresponding perturbation $U(t)$ and the same initial conditions $\tilde{x}(0) = x(0)$,

$$\|\tilde{x}(t) - x(t)\| \leq C \sup_{\tau \leq t} \|U(\tau)\|, \quad \forall t \in \mathbb{R}_+.$$ \quad (6)

Moreover, for any $\lambda \in \mathbb{R}^n$, the bound (6) applies to the (necessarily FPCS) system $F(t) + \lambda$ with the same constant $C$.

Theorem 1 is limited to FPCS systems: if any of the assumptions in the definition of FPCS systems is removed, then a similar result is no longer possible. In a forthcoming document, we discuss several examples of dynamical systems for which there exist no constant $C$ that satisfies (6); cf. Section IV.

We finally note that the vector $\lambda$ in the dynamical system $F(t) + \lambda$ can be viewed as a constant external field. Thus, the second part of the theorem asserts that the same bound holds uniformly for all constant external fields.

The proof of Theorem 1, presented in the next section, is fairly involved and so it is useful to provide some perspective on the challenges that are involved. For a constant-drift system, of the form $\dot{x}(t) = \mu + u(t)$, the result is immediate, because the state is fully determined by the integral $\int_0^t u(\tau) d\tau$. More generally, the unperturbed system goes through successive constant-drift regions, and one might expect that the result can be obtained by deriving and patching together bounds for each region encountered. There is however a difficulty, because the unperturbed trajectory often lies at the intersection of the boundaries of two or more constant drift regions. When that happens, the perturbed trajectory may chatter between different regions. As a consequence, the number of pieces and bounds that would have to be patched together can become arbitrarily large, and a bound of the desired form does not follow. For this reason, we need a much more refined analysis of the trajectories in the vicinity of the intersection of different regions, as will be seen in the next section.

IV. Proof

In this section we present the proof of Theorem 1 organized in a sequence of three subsections. In Subsection IV-A we present some notation, definitions, and lemmas, mostly concerning the geometric properties of FPCS systems and unperturbed trajectories. In particular, we define critical points (Definition 2) as the extreme points of constant-drift regions.

In Subsection IV-B we consider a time interval during which the perturbed trajectory is far from the set of critical points. Such an interval can be divided into subintervals with an important property: the set of drifts encountered is low-dimensional, in a sense to be defined below. Within each such subinterval, we show in Lemma 5 that the local dynamics...
are equivalent to the dynamics of a lower-dimensional FPCS system, and employ a suitable induction on the system dimension to obtain a certain upper bound. Then, in Proposition 1, we piece together the bounds for the different subintervals to obtain an upper bound that applies as long as the perturbed trajectory remains far from the set of critical points.

In Subsection IV-C we consider the case where the perturbed trajectory comes close to a critical point: we show, in Proposition 2, that the unperturbed trajectory stays close to the perturbed trajectory, as long as the perturbed trajectory remains sufficiently close to that critical point. Finally, in Subsection IV-D we combine the two cases and bound the distance of the trajectories at all times.

From now on, we assume that \( x(0) = \bar{x}(0) \) and that
\[
\sup_t \|U(t)\| \leq \theta. \tag{7}
\]
We will show that for any \( t \geq 0 \), we have \( \|\bar{x}(t) - x(t)\| \leq C\theta \), for some constant \( C \) independent of \( U, \theta \), and \( x(0) \). It is not hard to see that this implies the theorem in its original form.

The proof proceeds by induction on the system dimension \( n \). In particular, we make the following **induction hypothesis**, which we assume to be in effect throughout the rest of this section.

**Induction** : Theorem 1 holds for all \((n - 1)\)-dimensional FPCS systems. \tag{8}

We then rely on the induction hypothesis to prove the theorem for \( n \)-dimensional systems. For the basis of the induction we consider the case of zero-dimensional systems. In this case, the state space consists of a single point (the zero vector), we have \( x(t) = \bar{x}(t) = 0 \) at all times, and the result in Theorem 1 holds trivially.

### A. Properties of Unperturbed Dynamics

In this subsection we present some notation and definitions, and prove some properties of unperturbed trajectories. We then define and study critical points. Throughout the proof, we assume that \( F \) is an FPCS system on \( \mathbb{R}^n \), with \( F = -\partial \Phi \), where \( \Phi(x) = \max_{i=1, \ldots, m} \{-\mu_i^T x + b_i\} \). We assume that the vectors \( \mu_i \) in the definition of \( \Phi \) are distinct. This entails no loss of generality, because if \( \mu_i = \mu_j \) and \( b_i > b_j \), then \( -\mu_i^T x + b_i \) is always dominated by \( -\mu_j^T x + b_j \) and has no effect on \( \Phi(t) \).

Each vector \( \mu_i \) is called a **drift** and we define \( \mathcal{M} \) to be the set \( \{\mu_i\}_{i=1}^m \) of all drifts. For each drift \( \mu \in \mathcal{M} \), we use the notation \( b_\mu \) to refer to the corresponding constant in the expression for \( \Phi \). With these conventions, we have
\[
\Phi(x) = \max_{\mu \in \mathcal{M} \setminus b_\mu}. \tag{9}
\]

For every \( x \in \mathbb{R}^n \), we define the set \( \mathcal{M}(x) \) of active drifts at \( x \) by
\[
\mathcal{M}(x) \triangleq \{\mu \in \mathcal{M} \mid \Phi(x) = -\mu^T x + b_\mu\}. \tag{10}
\]
If at some \( x \) the corresponding set \( \mathcal{M}(x) \) consists of a single element \( \mu \), we have \( \dot{x} = \mu \). However, the dynamics become more interesting when the set \( \mathcal{M}(x) \) contains multiple elements. For that case, it follows from the definition of the subdifferential that for any \( x \in \mathbb{R}^n \), \( F(x) \) is the convex hull of \( \mathcal{M}(x) \).

For each \( \mu \in \mathcal{M} \), we define its **effective region** \( R_\mu \) by
\[
R_\mu \triangleq \{x \in \mathbb{R}^n \mid \mu \in \mathcal{M}(x)\}. \tag{11}
\]
Equivalently,
\[
R_\mu = \{x \mid -\mu^T x + b_\mu \geq -\mu^T x + b_\nu, \ \forall \nu \in \mathcal{M}\},
\]
which establishes that each region \( R_\mu \) is a polyhedron and, in particular, closed and convex. We will be using \( \mathcal{R} \) to denote the collection of all effective regions: \( \mathcal{R} \triangleq \{R_\mu \mid \mu \in \mathcal{M}\} \).

From now on, and with some abuse of traditional notation, we will use \( \dot{x}(t) \) to denote the right derivative of \( x(t) \), whenever it exists. The lemma that follows shows that for unperturbed trajectories this right derivative always exists and has some remarkable properties.

**Lemma 2** (Properties of Unperturbed Trajectories). Let \( x() \) be an unperturbed trajectory of an FPCS system \( F \). Then,
\[ (a) \text{ (Minimum Norm)} \quad \text{ For every } t \geq 0, \text{ the right derivative of } x(t) \text{ exists and is given by } \dot{x}(t) = \arg\min_{v \in F(x(t))} \|v\|, \tag{12} \]
with the minimizer being unique.
\[ (b) \text{ (Decreasing Drift Size)} \quad \text{ If } t > s, \text{ then } \|\dot{x}(t)\| \leq \|\dot{x}(s)\|, \text{ and the inequality is strict if } \dot{x}(t) \neq \dot{x}(s). \text{ Furthermore, an unperturbed trajectory traverses a connected sequence of at most } 2m - 2 \text{ line segments, possibly followed by a half-line.} \]

**Proof.** The first part of the lemma is an immediate consequence of Part 3 of Theorem 4.4 in [24]. For Part (b), we invoke Part 4 of Theorem 4.4 in [24] which states that \( \|\dot{x}(t)\| \) is a non-increasing function of time. Since for any \( x, F(x) \) is the convex hull of \( \mathcal{M}(x) \), it follows from (12) that \( \dot{x}(t) \) is uniquely determined by \( \mathcal{M}(x(t)) \). There are at most \( 2m - 2 \) non-empty subsets \( \mathcal{M}(x) \) of \( \mathcal{M} \). Hence, \( \dot{x}(t) \) can take at most \( 2m - 1 \) different values.

Fix a time \( s \geq 0 \) and let \( t \) be the minimum of the times \( \tau > s \) for which \( \dot{x}(\tau) \neq \dot{x}(s) \). The time function \( \dot{x}(\cdot) \) is piecewise constant and right-continuous (Part 4 of Theorem 4.4 in [24]). This implies that \( t > s \) and \( \dot{x}(t) \neq \dot{x}(s) \). Furthermore, from the strict convexity of the Euclidean norm we obtain
\[
\|\dot{x}(s) + \dot{x}(t)\| / 2 \leq \max_{\mu} \|\dot{x}(s)\|, \|\dot{x}(t)\| \tag{13}.
\]
Since every region \( R_\mu \) is closed, there exists a sufficiently small neighbourhood \( B \) of \( x(t) \) such that \( x(t) \notin R_\mu \), then \( B \) does not intersect \( R_\mu \). Equivalently, for any \( y \in B \), we have \( \mathcal{M}(y) \subseteq \mathcal{M}(x(t)) \), and \( F(y) \subseteq F(x(t)) \). In particular, consider a \( \tau \in [s, t] \) such that \( x(\tau) \in B \). Then,
\[
\dot{x}(s) = \dot{x}(\tau) \in F(x(\tau)) \subseteq F(x(t)).
\]
Since \( F(x(t)) \) is convex, \( (\dot{x}(s) + \dot{x}(t)) / 2 \in F(x(t)) \). Therefore, (12) implies that \( \|\dot{x}(t)\| \leq \|\dot{x}(s) + \dot{x}(t)\| / 2 \). Together with (13), this shows that \( \|\dot{x}(t)\| < \|\dot{x}(s)\| \).

For the last statement in Part (b) of the lemma, note that there are only finitely many different possible sets \( \mathcal{M}(x) \) (at
most $2^m - 1$ of them), and therefore as many choices for $F(x)$. Using (12), there are at most $2^m - 1$ possible values for $\dot{z}(t)$. As we have already shown that $\|\dot{z}(t)\|$ decreases strictly each time that it changes, an unperturbed trajectory must consist of at most $2^m - 1$ pieces, with a constant derivative on each piece. It can be seen that this implies that the trajectory traverses a connected sequence of at most $2^m - 2$ line segments, possibly followed by a half-line.

$$t \leftarrow t + 1$$

For any $x \in \mathbb{R}^n$, consider the unperturbed trajectory $z(\cdot)$ initialized with $z(0) = x$. We define the actual drift at point $x$ as

$$\xi(x) \triangleq \dot{z}(0),$$

where we continue using the convention that $\dot{z}$ stands for the right derivative. According to Lemma 2(a), the actual drift always exists and is uniquely determined by $x$.

We now proceed to define critical points, which will play a central role in the sequel.

**Definition 2 (Critical Points).** A point $p \in \mathbb{R}^n$ is called a critical point if $\text{span}(\{\mu - \mu' | \mu, \mu' \in M(p)\}) = \mathbb{R}^n$. The set of critical points is denoted by $C$.

An equivalent condition is that for a critical point $p$, the affine span of $M(p)$, i.e., the smallest affine space that contains $M(p)$, is equal to the entire set $\mathbb{R}^n$. For this to happen, $M(p)$ must have at least $n+1$ elements, and therefore $p$ must lie at the intersection of at least $n+1$ regions (although the converse is not always true). For the example in Fig. 1: $p = 0$ is the only candidate and is in fact a critical point because the affine span condition is satisfied. Furthermore, it will be shown in Lemma 3(a) that the critical points are the extreme points of the regions $R_\mu$.

**Definition 3 (Basin of a Critical Point).** Consider some $\rho \in \mathbb{R}_+ \cup \{\infty\}$ and a critical point $p$, with actual drift $\xi(p)$ equal to $\xi$. The closed ball $B$ of radius $\rho$ centered at $p$ is called a basin of $p$ (and $p$ is called a basin radius for $p$) if for every $x \in B$ and every $y \in F(x)$, we have $\xi^T y \geq \|\xi\|^2$.

Note that the inequality $\xi^T y \geq \|\xi\|^2$ implies that $\|\xi\| \leq \|y\|$. As a result, $\xi(p)$ has the minimum norm among all possible drifts within the basin of a critical point $p$. Also note that basins of a critical point $p$ are not necessarily unique: if the radius $\rho$ is positive, another basin is obtained by reducing the radius. Figure 2 shows an example of a two-dimensional system with three critical points and one associated basin.

Before moving to study the properties of critical points, we introduce one last definition.

**Definition 4 (Conic Neighbourhood Constant).** We define the Conic Neighbourhood Constant (CNC), denoted by $\rho_{\min}$, as

$$\rho_{\min} \triangleq \frac{1}{2} \min \left\{ d(p, R) \mid p \in C, R \in \mathcal{R}, p \notin R \right\},$$

i.e., $\rho_{\min}$ is half of the minimum over all critical points, of the distance of a critical point from the regions that do not contain it. We use the convention that the minimum of an empty set is infinite.

Note that $\rho_{\min}$ is always positive (and possibly infinite). We say that a dynamical system $F$ is conic if $F = -\partial \Phi$, where $\Phi$ is of the form $\Phi(x) = \max_i \{-\mu_i^T (x - p)\}$, for some $p \in \mathbb{R}^n$. It is not hard to see that for such a conic system, either $p$ is the only critical point or no critical points exist. It turns out that the “local” dynamics in the CNC-neighbourhood of a critical point of a general system are conic, hence the name CNC.

The lemma that follows lists a number of useful properties of critical points.

**Lemma 3 (Properties of Critical Points).** Consider an FPCS system $F$, with an associated set of critical points $C$.

(a) A point in a region $R_\mu$ is a critical point if and only if it is an extreme point of $R_\mu$. In particular, there are finitely many critical points.

(b) Consider a critical point $p \in C$ and a basin radius $\rho$ for $p$. Let $z(\cdot)$ be the unperturbed trajectory with initial point $z(0) = p$, and let $\xi = \dot{z}(0)$ be the actual drift at $p$. Then, before the time that $z(\cdot)$ exits the basin, $\dot{z}(t)$ is constant, and $z(t) = p + t \xi$, for all $t \in [0, \rho/\|\xi\|]$.

(c) If $C$ is non-empty, then there exists a critical point $p \in C$ such that the entire set $\mathbb{R}^n$ is a basin of $p$. In the special case where $F$ is conic with a unique critical point $p$, the entire set $\mathbb{R}^n$ is a basin of $p$.

(d) The CNC, $\rho_{\min}$, defined in (15), is a basin radius for every critical point.

(e) Consider a basin radius $\rho$ of a critical point $p \in C$, an unperturbed trajectory $x(\cdot)$, and times $t_1 < t_2$. Suppose that $\|x(t_1) - p\| \leq \rho/3$ and $\|x(t_2) - p\| > \rho$. Then, for any $t \geq t_2$, $\|x(t) - p\| > \rho/3$.

(f) Fix some $\lambda \in \mathbb{R}^n$ and consider $F'(\cdot) \triangleq F(\cdot) + \lambda$, which is also an FPCS system. Then, $F$ and $F'$ have the same set of regions $\mathcal{R}$, the same set of critical points $C$, and the same CNC $\rho_{\min}$.
In words, part (e) states that an unperturbed trajectory that starts near a critical point $p$ and later goes sufficiently far from $p$, will never come back close to $p$. The proof of Lemma 3 is given in Appendix A.

B. Bounding the Deviation when the Trajectories are Far from the Set of Critical Points

In this subsection we bound the distance between perturbed and unperturbed trajectories, for the case where the perturbed trajectory stays far from the set of critical points. To do this, we will show that when far from the set of critical points, the local dynamics are similar to those of a lower-dimensional system, and then use the induction hypothesis (8).

We start with some definitions. For any $x \in \mathbb{R}^n$ and $r > 0$, let

$$
\mathcal{U}_r(x) \triangleq \bigcup_{y : \|y-x\|_2 \leq r} \mathcal{M}(y),
$$

which is the set of all possible drifts in the $r$-neighbourhood of $x$.

**Definition 5 (Low-Dimensional Sets).** We call a subset $\mathcal{U} \subseteq \mathbb{R}^n$ low-dimensional if

$$
\text{span}\{x-y \mid x,y \in \mathcal{U}\} \neq \mathbb{R}^n. \quad (17)
$$

Equivalently, a set is low-dimensional if its affine span is not the entire space. If $x$ is a critical point, then, by definition, the vectors in $\{\mu_i - \mu_j \mid \mu_i, \mu_j \in \mathcal{M}(x)\}$ span $\mathbb{R}^n$ and the set $\mathcal{U}_r(x)$ is not low-dimensional, for any $r > 0$. On the other hand, as asserted by the next lemma, which is proved in Appendix B (available in supplementary materials), $\mathcal{U}_r(x)$ is low-dimensional when $x$ is sufficiently far from critical points.

**Lemma 4.** Consider an FPCS system with an associated set of critical points $C$. There exists $\gamma \geq 1$ such that if $r > 0$ and $d(x,C) > \gamma r$, then $\mathcal{U}_r(x)$ is low-dimensional.

In the sequel, it will be convenient to compare the perturbed trajectory with an unperturbed trajectory that starts at the same state at some intermediate time. This motivates the following terminology.

**Definition 6 (Coupled Trajectories).** Let $\theta, T \geq 0$ be some constants. Let $x(\cdot)$ be an unperturbed trajectory. Let $\tilde{x}(\cdot)$ be a perturbed trajectory with a perturbation $U(\cdot)$ that satisfies (7). If in addition we have $\tilde{x}(T) = x(T)$, we then say that $x(\cdot)$ and $\tilde{x}(\cdot)$ are $\theta$-coupled at time $T$.

The proof will now continue along the following lines. When far enough from the set of critical points, the set $\mathcal{U}_r(x)$ is low-dimensional (Lemma 4). This turns out to imply that we can describe the dynamics as the superposition of an essentially $(n-1)$-dimensional FPCS system and a constant drift.

**Lemma 5.** Consider an FPCS system. There exits a constant $\sigma \geq 1$ such that the following statement holds for all $T, \theta > 0$. Let $x(\cdot)$ and $\tilde{x}(\cdot)$ be a pair of $\theta$-coupled trajectories at time 0. Suppose that $\mathcal{U} \subseteq \mathcal{M}$ is low-dimensional, and that $\mathcal{U}_{\sigma\theta}(x(t)) \subseteq \mathcal{U}$, for all $t \in [0,T]$. Then,

$$
\|\tilde{x}(t) - x(t)\| \leq \sigma \theta, \quad \forall t \in [0,T].
$$

Moreover, for any $\lambda \in \mathbb{R}^n$, the same constant $\sigma$ also applies to the FPCS system $F(\cdot) + \lambda$.

**Proof.** Let us fix some $\mu \in \mathcal{U}$. Let $\mathcal{U}$ be the affine span of $\mathcal{U}$:

$$
\mathcal{U} \triangleq \mu + \text{span}\{\nu - \mu \mid \nu \in \mathcal{U}\}. \quad (19)
$$

Note that any choice of $\mu \in \mathcal{U}$ leads to the same set $\mathcal{U}$. Let $w$ be the projection of 0 onto $\mathcal{U}$, i.e., the smallest norm element of $\mathcal{U}$; cf. Fig. 3(a). Since $w \in \mathcal{U}$, we have

$$
w - \mu \in \text{span}\{\nu - \mu \mid \nu \in \mathcal{U}\}. \quad (20)
$$

Furthermore, by the orthogonality properties of projections, $w$ is orthogonal to the difference of any two elements of $\mathcal{U}$. In particular,

$$
w^T (\mu - \nu) = 0, \quad \forall \nu \in \mathcal{U},
$$

$$
w^T (\mu - w) = 0. \quad (21)
$$

Since $\mathcal{U}$ is low-dimensional, $\text{span}\{\nu - \mu \mid \nu \in \mathcal{U}\}$ is a proper subset of $\mathbb{R}^n$. Let $Y$ be a subspace of dimension $n-1$ that contains $\text{span}\{\nu - \mu \mid \nu \in \mathcal{U}\}$ and is orthogonal to $w$.\footnote{If $w \neq 0$, then $Y$ must be the orthogonal complement of the one-dimensional space spanned by $w$. If $w = 0$, then any $(n-1)$-dimensional subspace that contains $\text{span}\{\nu - \mu \mid \nu \in \mathcal{U}\}$ will do, and the choice of $Y$ need not be unique.} Note that by the definition of $\mathcal{U}$,

$$
\mathcal{U} - w = \mu - w + \text{span}\{\nu - \mu \mid \nu \in \mathcal{U}\}
$$

$$
= \text{span}\{\nu - \mu \mid \nu \in \mathcal{U}\} \subseteq Y, \quad (22)
$$

where the second equality is due to (20).

Any vector has an orthogonal decomposition as the sum of its projections on $Y^\perp$ (the orthogonal complement of $Y$) and $Y$; we use the subscripts $w$ and $Y$ to indicate the corresponding components, e.g.,

$$
x = x_w + x_Y, \quad \tilde{x} = \tilde{x}_w + \tilde{x}_Y, \quad (23)
$$

$$
U = U_w + U_Y.
$$

We will now show that the $w$ and $Y$ components of a trajectory evolve without interacting, according to a one-dimensional system with drift $w$, and an $(n-1)$-dimensional system $F_Y$, respectively; see Fig. 3(c) for an illustration.

**Claim 1.** Consider some $x \in \mathbb{R}^n$, and suppose that $\mathcal{M}(x) \subseteq \mathcal{U}$. Then, $F(x) = w + F_Y(x_Y)$, where $F_Y : Y \to Y$ is an FPCS system on the $(n-1)$-dimensional subspace $Y$.

**Proof of Claim.** Let $F_Y(x) = F(x) - w$. Since $F(x)$ is contained in the convex hull of $\mathcal{U}$, it is also in the affine span of $\mathcal{U}$, i.e., $F(x) \subseteq \mathcal{U}$. Then, (22) implies that

$$
F_Y(x) = F(x) - w \subseteq \mathcal{U} - w \subseteq Y. \quad (24)
$$

On the other hand, $\overline{F}_Y(x)$, being equal to $F(x) - w$, is the negative of the subdifferential of

$$
\overline{F}_Y(x) \triangleq \max_{\mu \in \mathcal{U}} \left[\left((\mu - w)^T x + b_\mu\right)\right]; \quad (25)
$$
we have used here the assumption $\mathcal{M}(x) \subseteq U$. For any $x \in \mathbb{R}^n$,

$$
\Phi_Y(x) = \Phi_Y(x_w + x_Y) \\
= \max_{\mu \in \tilde{U}} \left[ - (\mu - w)^T (x_w + x_Y) + b_\mu \right] \\
= \max_{\mu \in \tilde{U}} \left[ - (\mu - w)^T x_Y + b_\mu \right] \\
\triangleq \Phi_Y(x_Y),
$$

(26)

where the third equality is due to (21). Let $F_Y : Y \to 2^Y$ be equal to $-\Phi_Y(\cdot)$. Then, $F_Y$ is an FPCS system on the $(n-1)$-dimensional subspace $Y$. It follows from (26) that $\Phi_Y(x)$ only depends on $x_Y$. Therefore, its negative subdifferential $\overline{F}_Y(x)$ also only depends on $x_Y$, and $\overline{F}_Y(x) = F_Y(x_Y)$. Hence, the definition $\overline{F}_Y(x) = F(x) - w$ implies that $F(x) = w + \overline{F}_Y(x) = w + F_Y(x_Y)$, which establishes the claim. \( \square \)

We now appeal to the induction hypothesis [8], and let $C_{\overline{U}}$ be equal to the constant $C_Y$ of Theorem 1 for the $(n-1)$-dimensional FPCS system $F_Y$. Let $\sigma$ be the maximum of all such constants $C_{\overline{U}}$ plus 4, over all low-dimensional subsets $\tilde{U} \subseteq \mathcal{M}$:

$$
\sigma \triangleq \max \left\{ C_{\overline{U}} \mid \tilde{U} \subseteq \mathcal{M} \text{ and } \tilde{U} \text{ is low-dimensional} \right\} + 4.
$$

(27)

Suppose now that we add a constant drift $\lambda$ to $F$. We observe that for any given low-dimensional $U$, the resulting set-valued mapping $F_Y$ only changes through the addition of a constant drift $\lambda_Y$; its structure remains otherwise the same. Hence, according to the induction hypothesis [8], $C_{\overline{U}}$ is not affected when we add a constant drift $\lambda \in \mathbb{R}^n$ to the dynamics. As a consequence, the value of $\sigma$ associated with a system $F'(\cdot)$ remains the same when we consider the system $F'(\cdot) + \lambda$.

We now return to the main part of the proof of the lemma. We argue by contradiction, and assume that (18) fails to hold. Then, from the right-continuity of $x(t)$ and $\tilde{x}(t)$, there exists a time $\tilde{T} \leq T$ such that

$$
\| \tilde{x}(\tilde{T}) - x(\tilde{T}) \| \geq \sigma \theta, \\
\| \tilde{x}(t) - x(t) \| < \sigma \theta, \quad \forall \; t < \tilde{T}.
$$

(28)

It follows from (28) and the assumption $U_{\sigma \theta}(x(t)) \subseteq U$ that, for any $t < \tilde{T}$, $\mathcal{M}(x(t)) = U_0(x(t)) \subseteq U_{\sigma \theta}(x(t)) \subseteq U$. Furthermore, $\mathcal{M}(\tilde{x}(t)) \subseteq U_{\sigma \theta}(\tilde{x}(t)) \subseteq U$.

Consider some $t < \tilde{T}$ and let $\zeta(t) \in F(\tilde{x}(t))$ be a perturbed drift associated with the perturbed trajectory $\tilde{x}(\cdot)$ (cf. Definition [1]). It follows from Claim [1] that $\zeta(t) - w \in F(\tilde{x}(t)) - w = F_Y(\tilde{x}(t)) \subseteq Y$. Thus, the orthogonal decomposition of $\zeta(t)$ yields $\zeta_Y(t) = \zeta(t) - w$. This allows us
Suppose now that the perturbed trajectory stays far from the set of critical points throughout the time interval $[0, T]$. In light of Lemma \ref{lem:existence}, we can divide $[0, T]$ into a finite number of subintervals during which the unperturbed system has a constant drift, use Lemma \ref{lem:bound} to obtain bounds on the distance of the perturbed and unperturbed trajectories during each subinterval, and then combine them to obtain a bound over the entire interval $[0, T]$.

**Proposition 1.** Fix an FPCS system $F$. Consider the constant $\gamma$ in Lemma \ref{lem:existence} the constant $\sigma$ in Lemma \ref{lem:bound} and let $\eta = m^{2m+1}\sigma$, where $m$ is the number of elements of the set $\mathcal{M}$ of drifts. Let $x(\cdot)$ and $\bar{x}(\cdot)$ be a pair of $\theta$-coupled trajectories at time 0. If $d(\bar{x}(t), \mathcal{C}) \geq \gamma\eta\theta$ for all $t \in [0, T]$, then $\|\bar{x}(t) - x(t)\| \leq \eta\theta$ for all $t \in [0, T]$.

**Proof.** As already mentioned, we will divide the interval $[0, T]$ into at most $2m^3$ subintervals. We will then use Lemma \ref{lem:bound} to show that the distance between the two trajectories can only increase by an additive factor of $\sigma\theta$ in each subinterval.

We define a sequence of times $\tau_k$ by letting $\tau_1 = 0$ and

$$\tau_{k+1} = \inf \left\{ t \in (\tau_k, T) \Big| \mathcal{U}_{\theta\sigma\theta}(x(t)) \not\subseteq \mathcal{U}_{\theta\sigma\theta}(x(\tau_k)) \right\}.$$  

(33) for $k \geq 1$, with the convention that $\tau_{k+1} = T$ if the set on the right-hand side of (33) is empty. In words, $\tau_{k+1}$ is the time that the $(\theta\sigma\theta)$-neighbourhood of the unperturbed trajectory touches a new region, which does not intersect with the $(\theta\sigma\theta)$-neighbourhood of $x(\tau_k)$. Let $K_{\text{max}}$ be the maximum $k$ such that $\tau_k < T$, so that $\tau_{K_{\text{max}}+1} = T$. For $k \leq K_{\text{max}}$, we refer to the interval $[\tau_k, \tau_{k+1}]$ as phase $k$; see Fig. 4 for an illustration of different phases.

First, we show that the number of phases, $K_{\text{max}}$, is less than $m^2$. According to Lemma \ref{lem:existence}, the time interval $[0, T]$ can be partitioned into at most $2^m - 1$ subintervals $[z_j, z_{j+1}]$, $1 \leq j \leq 2^m - 1$, during each of which the unperturbed trajectory $x(t)$ is a line segment; that is, there exists a sequence of vectors $\xi_j$, $1 \leq j \leq 2^m - 1$ such that

$$x(t) = x(z_j) + (t - z_j)\xi_j, \quad \forall t \in [z_j, z_{j+1}] .$$

(34)

We argue that at most $m$ phase changes are possible during a subinterval $[z_j, z_{j+1}]$, i.e., at most $m$ of the times $\tau_k$s lie in the interval $[z_j, z_{j+1}]$. Suppose that there are $l$ phase changes (for some $l \geq 0$), at times $\tau_{k_j+1}, \ldots, \tau_{k_j+l} \in (z_j, z_{j+1})$. For each $k \in \{k_j+1, \ldots, k_j+l\}$, let $\mu_k \in \mathcal{M}$ be a drift that caused the phase change at time $\tau_k$, i.e.,

$$\mu_k \in \mathcal{U}_{(k_j-1)\sigma\theta}(x(\tau_k)) \setminus \mathcal{U}_{(k_j-1)\sigma\theta}(x(\tau_{k-1})) .$$

(35)

Equivalently,

$$d(x(\tau_{k-1}), R_k) > d(x(\tau_k), R_k) = (k-1)\sigma\theta ,$$

(36)

where $R_k \equiv R_{\mu_k}$ is the effective region of $\mu_k$. We will now show that these regions $R_k$, for $k \in \{k_j+1, \ldots, k_j+l\}$, are distinct for different $k$. In order to draw a contradiction, suppose that there are $k_1, k_2 \in \{k_j+1, \ldots, k_j+l\}$ with $k_1 < k_2$ such that $\mu_{k_1} = \mu_{k_2}$, or equivalently $R_{\mu_{k_1}} = R_{\mu_{k_2}}$. Let $f : [z_j, z_{j+1}] \to \mathbb{R}_+$ be the distance between $x(t)$ and the region $R$:

$$f(t) \equiv d(x(t), R), \quad \forall t \in [z_j, z_{j+1}] .$$

(37)
The region $R$ is a convex set. Therefore $f(\cdot)$ is the composition of a convex function (the distance from $R$) and an affine function $x(t) : [\tau_{j}, \tau_{j+1}] \to \mathbb{R}^{n}$ (see (34)). Hence, $f$ is also convex. Moreover, it follows from (36) and the assumption $k_{1} < k_{2}$ that

$$f(\tau_{k_{2}-1}) > f(\tau_{k_{2}}) = (k_{2} - 1)\sigma \theta > (k_{1} - 1)\sigma \theta = f(\tau_{k_{1}}).$$  \quad (38)

However, since $f$ is convex and $\tau_{k_{1}} \leq \tau_{k_{2}-1} \leq \tau_{k_{2}}$, we must have $f(\tau_{k_{2}-1}) \leq \max \{f(\tau_{k_{1}}), f(\tau_{k_{2}})\}$, which contradicts (38). Hence, each distinct $k_{i} \in \{k_{j} + 1, \ldots, k_{j} + l\}$ is associated with a distinct region $R_{k_{i}}$. On the other hand, since the number of different regions is at most $m$, there are at most $m$ phase changes during each of the at most $2m - 1$ line segments in the trajectory of $x(\cdot)$, and the total number of phases, $K_{\text{max}}$, is smaller than $m^{2m}$.

In our next step, we use induction on the phases to show that if $k \leq K_{\text{max}}$, then

$$\|\tilde{x}(t) - x(t)\| \leq k\sigma \theta, \quad \forall t \in [\tau_{k}, \tau_{k+1}].$$ \quad (39)

For any $k \geq 1$, we consider the induction hypothesis

$$\|\tilde{x}(\tau_{k}) - x(\tau_{k})\| \leq (k - 1)\sigma \theta.$$ \quad (40)

Note that (40) is automatically true for $k = 1$, because $\tau_{1} = 0$ and $\tilde{x}(0) - x(0)$ has been assumed to be zero. This provides the basis of the induction. Using the triangle inequality and the inequalities $\gamma \geq 1$ and $\eta = m2^{m+1}\sigma \geq 2K_{\text{max}}\sigma \geq 2k\sigma$, we obtain

$$d(x(\tau_{k}), C) \geq d(\tilde{x}(\tau_{k}), C) - \|x(\tau_{k}) - \tilde{x}(\tau_{k})\|
\geq \gamma\eta \theta - (k - 1)\sigma \theta
\geq 2\gamma k\sigma \theta - (k - 1)\sigma \theta
\geq \gamma k\sigma \theta.$$ \quad (41)

Let $U = U_{k\sigma \theta}(x(\tau_{k}))$. It follows from (41) and Lemma 4 with $r = k\sigma \theta$, that $U$ is low-dimensional. Furthermore, the definition of $\tau_{k+1}$ in (33) implies that

$$U_{k\sigma \theta}(x(t)) \subseteq U, \quad \forall t \in [\tau_{k}, \tau_{k+1}).$$ \quad (42)

Let $y(\cdot)$ be an unperturbed trajectory with initial condition $y(\tau_{k}) = \tilde{x}(\tau_{k})$. Since the unperturbed dynamics are non-expansive, for any $t \geq \tau_{k}$, we have

$$\|x(t) - y(t)\| \leq \|x(\tau_{k}) - y(\tau_{k})\|
\leq \|x(\tau_{k}) - \tilde{x}(\tau_{k})\|
\leq (k - 1)\sigma \theta.$$ \quad (43)

It is not hard to see that (43) and (42) imply that

$$U_{\sigma \theta}(y(t)) \subseteq U, \quad \forall t \in [\tau_{k}, \tau_{k+1}).$$ \quad (44)

Hence, the conditions of Lemma 5 hold, with the initial time being $\tau_{k}$ instead of zero. Therefore, $\|\tilde{x}(t) - y(t)\| \leq \sigma \theta$, for all $t \in [\tau_{k}, \tau_{k+1}]$. As a result, for $t \in [\tau_{k}, \tau_{k+1}]$,

$$\|\tilde{x}(t) - x(t)\| \leq \|\tilde{x}(t) - y(t)\| + \|y(t) - x(t)\|
\leq \sigma \theta + (k - 1)\sigma \theta
= k\sigma \theta,$$ \quad (45)

where the second inequality is due to (43). This establishes (39) and, in particular, that (40) holds with $k$ replaced by $k + 1$ (the induction step). Finally, the proposition follows from (39) and the fact that $k \leq K_{\text{max}} < m^{2m}$.

C. Proof of the Bound when Close to a Critical Point

In Proposition 1 we presented a bound on the distance between the trajectories when there are no nearby critical points. The next proposition deals with the other extreme, where the trajectories are in a basin of a critical point.
Proposition 2. Consider two constants $\theta, T > 0$, a critical point $p \in C$ and a basin $B_p$ of radius $\rho$ for $p$. Let $x(\cdot)$ and $\tilde{x}(\cdot)$ be a pair of $\theta$-coupled trajectories at time $0$, with $\tilde{x}(t) \in B_p$, for all $t \in [0, T]$. Suppose that $0 < r < \rho$, with $\rho$ possibly infinite, and that for the ball $B_r$ of radius $r$ centered at $p$,
\[
d(B_p \setminus B_r, C) \geq (\gamma + 1) \eta \theta,
\]
where $\eta = m^{2^{m+1}} \sigma \gamma$ and $\gamma$ are the constants defined in Proposition 1 and Lemma 4, respectively. Then, \[
\|\tilde{x}(t) - x(t)\| \leq 4r,
\]
for all $t \in [0,T]$.

Proof. Since the critical point $p$ belongs to $B_p$, (44), implies that $p$ must belong to $B_r$, and its distance from $B_p \setminus B_r$ is therefore at most $r$. Hence, \[d(B_p \setminus B_r, C) \leq r\text{ and, in particular, } r > \eta \theta.\]
Let
\[
r_1 \triangleq r - \eta \theta, \quad r_2 \triangleq r_1 + 3\theta,
\]
and consider two balls $B_{r_1}$ and $B_{r_2}$ centered at $p$, with radii $r_1$ and $r_2$, respectively. Since $p$ is a critical point, it is in the intersection of at least two regions. Therefore, the number $m$ of elements of the set $\mathcal{M}$ of drifts is at least two, and
\[
\eta = m^{2^{m+1}} \sigma \geq m^{2^{m+1}} \geq 16.
\]
As a result, $r_2 \leq r$ and $B_{r_1} \subset B_{r_2} \subset B_p$. The idea here is to look at the perturbed solution, and at certain times that it hits the boundary of $B_{r_2}$, consider an auxiliary unperturbed trajectory that is coupled with $\tilde{x}(t)$ at that time. Using Proposition 1, we can then show that these coupled trajectories stay close to each other, as long as the perturbed trajectory stays in $B_p \setminus B_{r_2}$. As a result, and using also the fact that the dynamics are non-expansive, the auxiliary trajectory $\tilde{x}(t)$ will remain close to $x(t)$, and the distance \[
\|\tilde{x}(t) - x(t)\| \text{ bounded.}
\]
The various parameters and trajectories are illustrated in Fig. 5.

Let $T_1^{\text{out}} = 0$, and for any $i \geq 1$ let
\[
T_i^{\text{in}} \triangleq \inf \left\{ t \in (T_{i-1}^{\text{out}}, T) \mid \tilde{x}(t) \in B_{r_1} \right\}, \quad T_i^{\text{out}} \triangleq \inf \left\{ t \in (T_{i-1}^{\text{in}}, T) \mid \tilde{x}(t) \notin B_{r_2} \right\}.
\]
If either set is empty, we let the left hand side be equal to $T$. We consider a number of rounds. Round $i$ starts at time $T_{i-1}^{\text{out}}$ and ends at time $T_i^{\text{in}}$. Note that the union of these rounds does not necessarily cover $[0, T]$. Also, note that since there is a gap of size $3\theta > \theta$ between the boundaries of $B_{r_1}$ and $B_{r_2}$, it takes some lower-bounded positive time for the perturbed trajectory to travel from one boundary to the other, and hence the length of each round is lower bounded by a positive constant. So, the number of rounds during $[0,T]$ is finite.

To each round $i$ we associate an unperturbed trajectory, denoted by $x^i(t), t \in [T_i^{\text{out}}, T_{i+1}^{\text{in}}]$, with initial point $x^i(T_{i}^{\text{out}}) = \tilde{x}(T_{i-1}^{\text{out}})$, i.e., $x^i(\cdot)$ is coupled with the perturbed trajectory at time $T_i^{\text{out}}$. For any $t \in [T_i^{\text{out}}, T_{i+1}^{\text{in}}]$, since $\tilde{x}(t) \in B_p \setminus B_{r_1}$, (49) asserts that $d(\tilde{x}(t), C) \geq \gamma \eta \theta$. Therefore, it follows from Proposition 1 that for any $t \in [T_i^{\text{out}}, T_{i+1}^{\text{in}}]$, \[
\|\tilde{x}(t) - x^i(t)\| \leq 4r \eta \theta.
\]
Note that $x^i(0) = x(t)$, for all $t \geq 0$. Thus, if $T_{i+1}^{\text{in}} = T$, the inequality (52) together with the fact $\eta \theta < r < 4r$ imply that \[
\|\tilde{x}(t) - x(t)\| < 4r, \quad \text{for all } t \in [0, T],
\]
as desired. In the following we assume that $T_{i+1}^{\text{in}} < T$. Note that the right-continuity of $\tilde{x}(\cdot)$ implies that \[
\|\tilde{x}(T_{i+1}^{\text{in}}) - p\| < r_1 + \eta \theta = r,
\]
where in the last inequality we used (52) with $i = 0$ and $t = T_{i+1}^{\text{in}}$. We now proceed to derive a bound on $\|\tilde{x}(t) - x(t)\|$, for $t \geq T_{i+1}^{\text{in}}$, by developing a bound on $\|\tilde{x}(t) - z(t)\|$. Let $\xi = \xi(p)$ be the actual drift at $p$. We consider two cases.

Case 1. ($p$ is an equilibrium point, i.e., $\xi = 0$). In this case, $z(t) = p$, for all $t \geq T_{i+1}^{\text{in}}$. Comparing with the unperturbed trajectory $x^i(\cdot)$ and using the non-expansive property and the definition of $T_{i+1}^{\text{in}}$, it follows for any round $i \geq 1$ and any $t \in [T_i^{\text{out}}, T_{i+1}^{\text{in}}]$, we have
\[
\|x^i(t) - z(t)\| \leq \|x^i(T_{i}^{\text{out}}) - z(T_{i}^{\text{out}})\| = \|\tilde{x}(T_{i}^{\text{out}}) - p\| = r_2 \leq 3r_2.
\]
Combining this with (52) and (53), we get the following bound for all $i \geq 1$ and for all $t \in [T_i^{\text{out}}, T_{i+1}^{\text{in}}]$,
\[
\|\tilde{x}(t) - x(t)\| \leq \|\tilde{x}(t) - x^i(t)\| + \|x^i(t) - z(t)\| + \|z(t) - x(t)\| \\
\leq \eta \theta + 3r_2 + r = \eta \theta + 3(r - \eta \theta + 3\theta) + r \\
= 4r + 2(4.5 - \eta)\theta < 4r,
\]
where the second inequality is due to (52), (54), and (53), and the last inequality is due to (53).

Furthermore, for any $t \in [T_i^{\text{in}}, T_{i+1}^{\text{out}}]$, our definitions imply that $\|\tilde{x}(t) - z(t)\| = \|\tilde{x}(t) - p\| < r_2$. Hence, for such $t$,
\[
\|\tilde{x}(t) - x(t)\| \leq \|\tilde{x}(t) - z(t)\| + \|z(t) - x(t)\| \leq r_2 + r < 4r,
\]
where the second inequality is due to (53). Thus, the proposition holds in this case.
The following claim suggests that if $\tilde{x}(\cdot)$ is an unperturbed trajectory that is coupled with the perturbed trajectory $\tilde{x}(\cdot)$ at time 0, $z(\cdot)$ is an unperturbed trajectory that starts at $p$ at time $T_1^{\text{in}}$, and each $x^i(\cdot)$ is an unperturbed trajectory that is coupled with $\tilde{x}(\cdot)$ at time $T_i^{\text{out}}$.

**Case 2.** ($p$ is not an equilibrium point, i.e., $\xi \neq 0$). The dynamics in this case are illustrated in Fig. 5. Here, we need to find an alternative derivation of (54), and also derive a new bound for $\|\tilde{x}(t) - z(t)\|$ when $t \in [T_1^{\text{in}}, T_1^{\text{out}}]$.

Let $\zeta(\cdot)$ be a perturbed drift associated with the perturbed trajectory $\tilde{x}(\cdot)$. Since $B_p$ is a basin and $\zeta(t) \in F(\tilde{x}(t))$, it follows from the definition of basins that for any $t \in [0, T]$, $\xi^T \zeta(t) \geq \|\xi\|^2$. Hence, for any $t \in [T_1^{\text{in}}, T]$, $\|\tilde{x}(t) - p\| \geq \frac{1}{\|\xi\|} \xi^T (\tilde{x}(t) - p)$

\[
\geq \frac{1}{\|\xi\|} \left( \xi^T (\tilde{x}(T_1^{\text{in}}) - p) + \int_{T_1^{\text{in}}}^{t} \xi^T \zeta(\tau) \, d\tau \right) + \xi^T (U(t) - U(T_1^{\text{in}})) \geq -\|\tilde{x}(T_1^{\text{in}}) - p\| + \frac{1}{\|\xi\|} \int_{T_1^{\text{in}}}^{t} \|\xi\|^2 \, d\tau - 2\theta \geq -r_1 + (t - T_1^{\text{in}})\|\xi\| - 2\theta.
\]

The first inequality above is the Cauchy-Schwarz inequality; the second equality follows from the definition of perturbed trajectories (cf. Definition [1]); the next inequality uses the Cauchy-Schwarz inequality for the first term, (57) for the second, and the bounds on $U(\cdot)$ for the third; the last inequality uses the defining property $\|\tilde{x}(T_1^{\text{in}}) - p\| = r_1$ of $T_1^{\text{in}}$.

We define an escape time $T_1^{\text{esc}} = T_1^{\text{in}} + (r_1 + r_2 + 3\theta) / \|\xi\|$. The following claim suggests that if $\tilde{x}(t)$ ever escapes $B_{r_2}$, it happens before time $T_1^{\text{esc}}$.

**Claim 2.** If $T_i^{\text{in}} < T$ for some $i \geq 1$, then $T_i^{\text{out}} \leq T_1^{\text{esc}}$.

**Proof of Claim.** If $T_1^{\text{esc}} \geq T$, then $T_i^{\text{out}} \leq T \leq T_1^{\text{esc}}$. Suppose now that $T_1^{\text{esc}} < T$. It follows from (58) and the definition of $T_1^{\text{esc}}$ that $\|\tilde{x}(T_1^{\text{esc}}) - p\| \geq r_2 + \theta > r_2$. Hence $\tilde{x}(t)$ is outside of $B_{r_2}$ at time $T_1^{\text{esc}}$, and by definition, $T_i^{\text{out}} \leq T_1^{\text{esc}}$.

Since $z(T_1^{\text{in}}) = p$, we have $\|\tilde{z}(0)\| = \xi$. According to Lemma 2b, $\|\tilde{z}(t)\| \leq \|\xi\|$, for $t \geq T_1^{\text{in}}$. Hence, if $t \in [T_1^{\text{in}}, T_1^{\text{esc}}]$, then

\[
\|z(t) - p\| \leq (t - T_1^{\text{in}})\|\xi\| \leq (T_1^{\text{esc}} - T_1^{\text{in}})\|\xi\| = r_2 + r_1 + 3\theta = 2r_2.
\]

We now proceed by considering two cases: $t \in [T_i^{\text{out}}, T_i^{\text{in}}]$ and $t \in [T_i^{\text{in}}, T_i^{\text{out}}]$. We first suppose that $T_i^{\text{out}} < T$, and consider $t \in [T_i^{\text{out}}, T_i^{\text{in}}]$. Since $T_i^{\text{in}} \leq T_i^{\text{out}} < T$, Claim 2 implies that $T_i^{\text{out}} \leq T_1^{\text{esc}}$. It then follows from (59) that $\|z(T_i^{\text{out}}) - p\| \leq 2r_2$. Then, $\|x^i(0) - z(t)\| \leq \|x^i(T_i^{\text{out}}) - z(T_i^{\text{out}})\| \leq \|x^i(T_i^{\text{out}}) - p\| + \|p - z(T_i^{\text{out}})\| \leq r_2 + 2r_2 = 3r_2$.

where the first inequality is due to the non-expansive property, and the last inequality is due to the definition of $T_i^{\text{out}}$. Thus, the bound (54) and the subsequent derivation of (55) remain valid for this case as well, so that for every round $i$, $\|\tilde{x}(t) - x(t)\| < 4r$, $\forall t \in [T_i^{\text{out}}, T_i^{\text{in}}]$. (61)
We now discuss the case where $t$ does not belong to a round, i.e., $t \in [T_{i}^{in}, T_{i}^{out})$, for some $i$ such that $T_{i}^{in} < T$. It follows from Claim 2 that $t < T_{i}^{out} \leq T_{\text{exc}}$. Therefore, (59) implies that for any $t \in [T_{i}^{in}, T_{i}^{out})$,
\[
\| p - z(t) \| \leq 2r_{2}.
\] (62)
Then,
\[
\| \tilde{x}(t) - x(t) \| \leq \| \tilde{x}(t) - p \| + \| p - z(t) \| + \| z(t) - x(t) \|
\leq r_{2} + 2r_{2} + r
\leq 4r,
\] (63)
where the second inequality is due to $t \in [T_{i}^{in}, T_{i}^{out})$, (62), and (53). Together with (61), this completes the argument for Case 2, and the proof of the proposition.

\[\Box\]

D. Completing the Proof of the Theorem

We now use the machinery developed in this section and combine the results for the various cases to complete the proof of Theorem 1.

If there are no critical points, then the perturbed trajectory never gets close to a critical point, and the theorem follows from Proposition 1. In the following, we assume that the set of critical points is non-empty. Let $M$ be the number of critical points, let $\rho_{\min}$ be the CNC, and let $D_{C}^{c}$ be the diameter of the set of critical points:
\[
D_{C}^{c} \triangleq \max_{p,q \in C} \| p - q \|. (64)
\]
According to Lemma 3(a) there are finitely many critical points, so that $D_{C}^{c}$ is well-defined and finite. We define a threshold parameter $\theta^{*}$ as follows:
\[
\theta^{*} \triangleq \frac{\rho_{\min}}{4(M + 2)(\gamma + 1)\eta}, (65)
\]
where $\gamma$ and $\eta$ are the constants defined in Lemma 4 and Proposition 1 respectively. In what follows, we use Proposition 2 to prove that the following constant satisfies Theorem 1:
\[
C = \frac{4D_{C}^{c}}{\theta^{*}} + 5(M + 2)(\gamma + 1)\eta. (66)
\]
We consider two cases, depending on whether the perturbation bound $\theta$ is larger or smaller than the threshold $\theta^{*}$.

**Case 1** ($\theta \geq \theta^{*}$). According to Lemma 3(c) there exists a critical point $p^{*}$, for which the entire set $\mathbb{R}^{n}$ is a basin. We let $r = D_{C}^{c} + (\gamma + 1)\eta\theta$ and $\rho = \infty$. This choice of $p^{*}$, $r$, and $\rho$ satisfies the conditions of Proposition 2. It follows from Proposition 2 that
\[
\| \tilde{x}(t) - x(t) \| \leq 4r < C\theta, \quad \forall t \geq 0, (67)
\]
which establishes the desired result.

**Case 2** ($\theta < \theta^{*}$). Once more, we rely on Proposition 2 but in a local manner. We consider a “small” basin of size $\rho_{\min}/2$ for each critical point, and define a number of phases $[T_{i}^{in}, T_{i}^{out}]$ so that throughout any particular phase, the perturbed trajectory lies in one of these basins. We then use Proposition 2 to bound the distance between the two trajectories in each phase, and use Proposition 1 to bound their distance while outside the basins. In the end, we use Lemma 3(c) to show that each basin is visited at most once, in a certain sense, and finally put everything together to prove the desired bound on the distance of the two trajectories. Figure 6 shows an illustration of the different trajectories and variables that we use in the argument that follows.

Let
\[
r = (\gamma + 1)\eta\theta, \quad \rho = \rho_{\min}/2. (68)
\]
It follows from (65) and the assumption $\theta < \theta^{*}$ that $r < \rho$. Moreover, based on Lemma 3(d) $\rho$ is a basin radius for each one of the critical points. For any critical point $p \in C$, let $B_{r}(p)$ and $B_{\rho}(p)$ be the balls of radii $r$ and $\rho$, respectively, centered at $p$. We define two sequences of times $T_{i}^{in}$ and $T_{i}^{out}$ as follows. Let $T_{0}^{out} = 0$, and for any $i \geq 1$ let
\[
T_{i}^{in} \triangleq \inf \left\{ t > T_{i-1}^{out} \mid \exists p \in C : \tilde{x}(t) \in B_{\rho}(p) \right\}. (69)
\]
We denote by $p^{i}$ the critical point $p$ in the right-hand side of (69), so that $\tilde{x}(T_{i}^{in}) \in B_{\rho}(p^{i})$. Note that the different balls $B_{r}(p)$ do not intersect and therefore $p^{i}$ is uniquely defined; we refer to it as the effective critical point at phase $i$. We then define
\[
T_{i}^{out} \triangleq \inf \left\{ t > T_{i}^{in} \mid \tilde{x}(t) \notin B_{\rho}(p^{i}) \right\}. (70)
\]
In the above, we let $T_{i}^{in}$ or $T_{i}^{out}$ be infinite in case the set on the right-hand side of (69) or (70) is empty.

Fix an $i \geq 1$. We first derive a bound on $\| \tilde{x}(t) - x(t) \|$ for $t \in [T_{i-1}^{in}, T_{i}^{in}]$. Let $y^{i}(\cdot)$ be an unperturbed trajectory with $y^{i}(T_{i-1}^{in}) = \tilde{x}(T_{i-1}^{in})$. By definition, for any $t \in [T_{i-1}^{out}, T_{i}^{in}]$,
\[
d(\tilde{x}(t), C) \geq \frac{t}{T_{i}^{in}} \geq (\gamma + 1)\eta\theta. (71)
\]
Hence, from the non-expansive property of the unperturbed dynamics, for any $t \in [T_{i-1}^{out}, T_{i}^{in}]$, we obtain
\[
\| \tilde{x}(t) - x(t) \| \leq \| x(t) - y^{i}(t) \| + \| \tilde{x}(t) - y^{i}(t) \|
\leq \| x(T_{i-1}^{out}) - y^{i}(T_{i-1}^{out}) \| + r\]
\[
= \| x(T_{i-1}^{out}) - \tilde{x}(T_{i-1}^{out}) \| + r. (72)
\]
On the other hand, if $t \in [T_{i}^{in}, T_{i}^{out}]$, for some $i \geq 1$, we have for any critical point $p \in C$ other than $p^{i}$,
\[
d(p, B_{\rho}(p^{i})) \geq \| p - p^{i} \| - \rho
\geq \rho_{\min} - \rho
= \frac{\rho_{\min}}{2}
\geq (\gamma + 1)\eta\theta^{*}
> r, (73)
\]
where the second and third inequalities follow from the definitions of $\rho_{\min}$ and $\theta^{*}$ in (15) and (65), respectively. Using also the fact $d(p^{i}, B_{\rho}(p^{i}) \setminus B_{\rho}(p^{i})) \geq r$, we obtain
\[
d(B_{\rho}(p^{i}), B_{\rho}(p^{i}) \setminus B_{\rho}(p^{i})) \geq r = (\gamma + 1)\eta\theta. (74)
\]
Moreover, for any $t \in [T_{i}^{in}, T_{i}^{out}]$, $\tilde{x}(t) \in B_{\rho}(p^{i})$. Hence, the conditions of Proposition 2 are observed. Let $x^{i}(t)$ be an
four regions and two critical points, $p^1$ and $p^2$. There are two balls of radii $r$ and $p$, defined in (69), centered at each critical point. The solid curved line $\tilde{x}(\cdot)$ is a perturbed trajectory, coupled with an unperturbed trajectory $x(\cdot)$ at time $0$. Times $T_i^{\text{in}}$ and $T_i^{\text{out}}$, defined in (69) and (70), are the first times that the perturbed trajectory hits the ball of radius $r$ or leaves the larger ball of radius $p$, respectively. In this example, $T_2^{\text{out}} = \infty$, because the perturbed trajectory never leaves the $p$-neighbourhood of $p^2$. For each $i$, $x(\cdot)$ and $y(\cdot)$ are unperturbed trajectories coupled with $\tilde{x}(\cdot)$ at times $T_i^{\text{in}}$ and $T_i^{\text{out}}$, respectively. These unperturbed trajectories are shown by dashed lines.

The second inequality is due to (77) and the definition of $T_i^{\text{in}}$. The same bound also holds for $\|x(T_i^{\text{out}}) - p\|$. On the other hand,

\[ \|x(T_i^{\text{in}}) - p\| \geq \|p - \tilde{x}(T_i^{\text{out}})\| - \|x(T_i^{\text{out}}) - \tilde{x}(T_i^{\text{out}})\| \geq \rho - 5ir \geq \frac{\rho_{\text{min}}}{2} - 5(M + 1)r = 20(M + 2)(\gamma + 1)\eta^{\theta^*} - 5(M + 1)r \geq 20(M + 2)(\gamma + 1)\eta\theta - 5(M + 1)r = 20(M + 2)r - 5(M + 1)r > 15(M + 2)r, \]

(79)

where the second and third inequalities are due to (77) and the definition of $T_i^{\text{in}}$. The same bound also holds for $\|x(r) - p\|$. On the other hand,

\[ \|x(T_i^{\text{out}}) - p\| \geq \|p - \tilde{x}(T_i^{\text{out}})\| - \|x(T_i^{\text{out}}) - \tilde{x}(T_i^{\text{out}})\| \geq \rho - 5ir \geq \frac{\rho_{\text{min}}}{2} - 5(M + 1)r = 20(M + 2)(\gamma + 1)\eta^{\theta^*} - 5(M + 1)r \geq 20(M + 2)(\gamma + 1)\eta\theta - 5(M + 1)r = 20(M + 2)r - 5(M + 1)r > 15(M + 2)r, \]

(79)

where the second and third inequalities are due to (77) and the definition of $T_i^{\text{in}}$. The same bound also holds for $\|x(T_i^{\text{out}}) - p\|$. On the other hand,

\[ \|x(T_i^{\text{out}}) - p\| \geq \|p - \tilde{x}(T_i^{\text{out}})\| - \|x(T_i^{\text{out}}) - \tilde{x}(T_i^{\text{out}})\| \geq \rho - 5ir \geq \frac{\rho_{\text{min}}}{2} - 5(M + 1)r = 20(M + 2)(\gamma + 1)\eta^{\theta^*} - 5(M + 1)r \geq 20(M + 2)(\gamma + 1)\eta\theta - 5(M + 1)r = 20(M + 2)r - 5(M + 1)r > 15(M + 2)r, \]

(79)
defined in Lemmas 4 and 5 respectively, are also identical for the two systems, and the same is true for the constant $\eta = m^{2m+1} \sigma$, defined in Proposition 1, the constant $\theta^*$ defined in (65), and finally for the constant $C$ in (66). In other words, the same constant $C$ also works for the dynamical system $F'$. This completes the proof of Theorem 1.

### V. Discussion

In this section we review our main results and their implications, and also discuss the extent to which they can or cannot be generalized to broader classes of systems.

We have established a bounded input sensitivity property of FPCS (finely piecewise constant subgradient) systems, in a strong sense. In particular, we have shown that the increase in the distance between perturbed and unperturbed trajectories is upper-bounded by a constant multiple of the magnitude of the integral of the instantaneous perturbations; cf. (6). As discussed in the introduction, this is much stronger than the elementary upper bounds which involve the integral of the magnitude of the instantaneous perturbations. Furthermore, our definitions are broad enough to include as possible perturbations the sample paths of jump or Brownian motion processes.

### A. Implications

FPCS systems arise in many contexts. As discussed in Section I, a prominent example is the celebrated Max-Weight policy for scheduling in queueing networks. Having made this connection, we can (cf. (14)) apply a variant of our result to the Max-Weight policy, establish bounds on the distance between the actual discrete-time stochastic system and its fluid approximation, and also obtain state space collapse results that are stronger than available ones [13], [25].

More broadly, flows or algorithms that evolve along the subgradient of a potential function are a fairly natural model, likely to arise in many other contexts. Recall also that, as mentioned in Section I the FPCS class has been shown [2] to contain all non-expansive finite-partition hybrid systems that obey some minimal well-formedness and uniqueness properties.

### B. Generalizations

Broad generalizations that assume only a subset of the properties of FPCS systems are not possible. In [11] we provide (counter)examples that show that a sensitivity bound of the form (6) does not hold for various classes of systems. Our counterexamples include:

1) A non-expansive system; hence the non-expansiveness property is not sufficient by itself.

2) A system that moves along the gradient of a twice continuously differentiable strictly convex function; hence the subgradient property is not sufficient by itself.

3) A system that moves along the subgradient of a piecewise linear convex function with infinitely many number of pieces; hence the finiteness of the number of pieces is essential.

Even though our main result cannot be extended by weakening its assumptions, it may still be possible to derive similar sensitivity bounds for other classes of systems. For example, [1] provides necessary and sufficient conditions for linear systems $\dot{x} = Ax$, in terms of the spectrum of $A$. It will be interesting to explore whether there are some other natural classes of systems that do not have the non-expansiveness property but for which the conclusions in Theorem 1 are valid.

### C. Some open problems

Besides attempts to obtain bounded sensitivity results for other types of systems, there are some interesting open problems for FPCS systems specifically.

1) The bound in Theorem 1 involves a constant $C$ which grows exponentially with the number of regions. It is not known whether this is unavoidable or whether a smaller (polynomial) constant is possible.

2) Theorem 1 studies the distance between a perturbed and an unperturbed trajectory, but this does not necessarily provide a strong bound on the distance between two perturbed trajectories. Consider an FPCS system and two different perturbations $U_1(\cdot)$ and $U_2(\cdot)$ that are close at all times. We conjecture that in such a case the perturbed trajectories are also close.

### References


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**APPENDIX A**

**Proof of Lemma 3**

Proof of Lemma 3(a) We fix some ν and p ∈ Rn = \{x | −νTx + b ≥ −µTx + b, ∀ µ\}. Suppose that p is a critical point. By the definition of M(p), we have −νTx + bν = −µTx + bµ for every µ ∈ M(p), i.e., these constraints are all active at p. Furthermore, by the definition of critical points, the vectors \{µ − ν | µ ∈ M(p)\} span Rn. It is not hard to see that this implies that the vectors \{µ − ν | µ ∈ M(p)\} also span Rn, so that n of them are linearly independent. Using linear programming terminology, out of the constraints that define Rν, there are n linearly independent active constraints at p, and p is a “basic feasible solution” in Rν. This is equivalent to p being an extreme point of Rν, cf. Theorem 2.3 in [25].

For the converse implication, suppose that p is an extreme point of Rν. Using again Theorem 2.3 in [25], n of the vectors µ − ν, associated with active constraints at p (i.e., with µ ∈ M(p)) are linearly independent. It follows that the vectors µ − µ′, for µ, µ′ ∈ M(p) span Rn, and (by the definition), p is a critical point.

Proof of Lemma 3(b) In order to draw a contradiction, consider a time t > 0 where z(t) is in the basin and ˙z(t) = ξ. It follows from Lemma 3(b) that ∥z(t)∥ < ∥ξ∥. Hence, ξ Tz(t) ≤ ∥ξ∥∥z(t)∥ < ∥ξ∥2, which contradicts the definition of a basin.

Proof of Lemma 3(c) We assume that the set C of critical points is non-empty. We will first show that there exists a critical point p such that ∥ξ(p)∥ ≤ ∥ξ(x)∥, for all x ∈ Rn. We will then show that Rn is a basin for this particular p.

Since there exists a critical point, Part (a) implies that some Rν has an extreme point. Using linear programming theory (cf. Theorem 2.6 in [25]) it follows that all of the non-empty regions Rν also have extreme points.

4This is because the regions are defined in terms of constraints aT x ≤ b or aT x ≥ b, where each a is of the form a = µ − µ′, for some µ, µ′; different regions correspond to different choices in the direction of the inequalities, but the vectors a are the same or every region.
Consider some $x$ in some region $R_n$. Let $x'$ be an extreme point of that region, chosen so that all constraints that were active at $x$ are also active at $x'$. (This can be done by moving inside $R_n$ while respecting active constraints, until additional constraints are made active, exactly as in the proof of Theorem 2.6 in [20].) The resulting extreme point $x'$ satisfies $M(x') \supseteq M(x)$. Since $F(x')$ is the convex hull of $F(x)$, it follows that $F(x') \supseteq \bar{F}(x)$.

From Lemma 2(a), $\xi(x)$ is the minimum norm element of $F(x)$, which implies that $\|\xi(x')\| \leq \|\xi(x)\|$. We conclude that when we minimize the function $\|\xi(x)\|$ over all $x \in \mathbb{R}^n$, it suffices to restrict to the (finite) set of extreme points of the different regions, or equivalently the set of critical points (cf. Part (a)). This concludes the proof that there exists a critical point, $p$, such that $\|\xi(p)\| \leq \|\xi(x)\|$, for all $x \in \mathbb{R}^n$. Let $\xi^* = \xi(p)$.

We now proceed to show that $\mathbb{R}^n$ is a basin of $p$. Let $z(t)$ be an unperturbed trajectory with initial point $z(0) = p$. Similar to the proof of Part (b), if for some $t > 0$, $z(t) \neq \xi^*$, then it follows from Lemma 2(b) that $\|\xi(z(t))\| = \|z(t)\| < \|z(0)\| = \|\xi^*\|$, which contradicts the definition of $\xi^*$. Hence, for any $t \geq 0$, $z(t) = p + t\xi^*$, which implies that $-\xi^* \in \partial \Phi(p + t\xi^*)$. Hence, $\Phi$ is the convex function for which $\Phi$ is the subdifferential.

For any $x \in \mathbb{R}^n$, let $\tilde{\Phi}(x) = \Phi(x) + t\xi^*T(x - p)$, so that $\partial \tilde{\Phi}(x) = \partial \Phi(x) + \xi^*$. Since $-\xi^* \in \partial \Phi(p + \xi^*)$, we have $0 \in \partial \tilde{\Phi}(p + t\xi^*)$, which implies that $p + t\xi^*$ is a minimizer of $\tilde{\Phi}$. Consider an $x \in \mathbb{R}^n$ and a $y \in \bar{F}(x)$. Then, $y - \xi^* \in -\partial \tilde{\Phi}(x)$. It follows from the supporting hyperplane theorem that, for any $t \geq 0$, $(\xi^* - y)T(p + t\xi^* - x) \leq \tilde{\Phi}(p + t\xi^*) - \tilde{\Phi}(x) \leq 0$, where the last inequality is because $p + t\xi^*$ is a minimizer of $\tilde{\Phi}$. Then, by letting $t$ go to infinity, we obtain

$$
(\xi^* - y)T \xi^* = \lim_{t \to \infty} \frac{1}{t} (\xi^* - y)T (p + t\xi^* - x) \leq 0. 
$$

Hence, $\|\xi^*\|^2 \leq \xi^*T y$, which shows that $\mathbb{R}^n$ is a basin of $p$. In the special case where $F$ is conic and has a critical point, then this is the only critical point and therefore has $\mathbb{R}^n$ for a basin.

Proof of Lemma 3(d). Consider a critical point $p \in C$, and let

$$
\bar{\Phi}(x) = \max_{\mu \in M(p)} \{ -\mu^T(x - p) \}, \quad \forall x \in \mathbb{R}^n. \tag{82}
$$

Hence, the dynamical system $\dot{x} \in \bar{F}(x) \triangleq -\partial \bar{\Phi}(x)$ is conic. Since the vectors $\{\mu - \mu^* | \mu \in M(p)\}$ span $\mathbb{R}^n$, it follows that $p$ is also a critical point of the system $\dot{x} \in \bar{F}(x)$. Lemma 3(c) then implies that the entire set $\mathbb{R}^n$ is a basin for $p$, for the system $\dot{x} \in \bar{F}(x)$.

Let $B$ be the ball of radius $\rho_{\text{min}}$ centred at $p$, where $\rho_{\text{min}}$ is the definition of the CNC. By the definition of the CNC, if $B \cap R_\mu$ is non-empty for some $\mu \in M$, then $p \in R_\mu$. Hence, for any $x \in B$, we must have $M(x) \subseteq M(p)$. Therefore, for any $x \in B$,

$$
\Phi(x) = \max_{\mu \in M} \{ -\mu^T(x + b_\mu) \} = \max_{\mu \in M(p)} \{ -\mu^T(x + b_\mu) \} = \max_{\mu \in M(p)} \{ -\mu^T(x - p) - \mu^Tp + b_\mu \} \tag{83}
$$

where the second equality is because the set $M(x)$ of maximizers of $-\mu^Tx + b_\mu$ is a subset of $M(p)$. Hence, for any $x \in B$, $\bar{F}(x) = \bar{F}(x)$. As a result, for $x \in B$, $\xi(x)$ for the system $\dot{x} \in \bar{F}(x)$ is equal to $\xi(x)$ for the system $\dot{x} \in \bar{F}(x)$. Since $\mathbb{R}^n$ is a basin of $p$ for the system $\dot{x} \in \bar{F}(x)$, it follows that for any $x \in B$ and any $y \in \bar{F}(x) = \bar{F}(x)$, we have $y^T \xi(p) \geq \|\xi(p)\|$. Hence, $B$ is a basin for the system $\dot{x} \in \bar{F}(x)$.

Proof of Lemma 3(e). The result will be derived by comparing the trajectory $x(t)$ of interest to another unperturbed trajectory, $z(t)$, initialized with $z(t_1) = p$. According to the non-expansive property of the dynamics, we have $\|x(t) - z(t)\| \leq \|x(t_1) - z(t_1)\| \leq \rho/3$, for every $t \geq t_1$. Hence, $\|z(t_2) - p\| \leq \|x(t_2) - p\| - \|x(t_2) - z(t_2)\| > \rho - \frac{\rho}{3} = \frac{2\rho}{3}$. (84)

In order to draw a contradiction, suppose that there is a time $t_3 > t_2$ such that $\|x(t_3) - p\| \leq \rho/3$. In this case, $\|z(t_3) - p\| \leq \|z(t_3) - x(t_3)\| + \|x(t_3) - p\| \leq \rho + \frac{\rho}{3} = \frac{2\rho}{3}$. (85)

Hence, $z(t_3)$ is in the basin of $p$, which implies that $\xi(p)^T \xi(z(t_3)) \geq \|\xi(p)\|$ and

$$
\|\xi(z(t_3))\| \geq \|\xi(p)\|. \tag{86}
$$

The trajectory $z(t)$ starts inside the $2\rho/3$-neighbourhood of $p$ at time $t_1$, leaves this neighbourhood before time $t_2$, and returns back to it by time $t_3$. Since the $2\rho/3$-neighbourhood is convex, $z(t)$ must have changed its direction in the meanwhile, and there exists a time $t' \in (t_1, t_3)$ such that $\dot{z}(t') \neq \dot{z}(t_1) = -\xi(p)$. Then, using Lemma 2(b)

$$
\|\xi(z(t_3))\| \leq \|\dot{z}(t')\| < \|\dot{z}(t_1)\| = \|\xi(p)\|. \tag{87}
$$

This contradicts (86) and concludes the proof.

Proof of Lemma 3(f). For every drift $\mu \in M$ of $F$, $\mu + \lambda$ is a drift of $F'$. The associated effective region $R'_{\mu + \lambda}$ of $F'$ is given by $R'_{\mu + \lambda} = \{x \in \mathbb{R}^n | (-\mu + \lambda)^T x + b_\mu \geq -\nu T x + b_\nu, \forall \nu \in M\} = \{x \in \mathbb{R}^n | -\mu^T x + b_\mu \geq -\nu T x + b_\nu, \forall \nu \in M\} = R_\mu$. Hence, the regions associated with $F$ and $F'$ are the same. Consider a point $p \in \mathbb{R}^n$ and let $M'(p) = M(p) + \lambda$ be the set of active drifts of $p$ in system $F'$. The affine span of $M'(p)$ is $\mathbb{R}^n$ if and only if the affine span of $M(p)$ is $\mathbb{R}^n$. Hence, $p$ is a critical point for the system $F'$ if and only if it is a critical point for the system $F$. Finally, by the definition of the CNC, since $F'$ and $F'$ have the same set of regions and the same set of critical points, they also have the same CNC.
**Supplementary Material**

**APPENDIX B**

**Proof of Lemma 4**

We provide here the proof of Lemma 4. We will make use of an auxiliary result, proved in [27], which states that if a point is close to each of several half-spaces, then that point is also close to the intersection of those half-spaces.

**Lemma 6** ([27], Lemma 5.1). Given a finite collection of half-spaces $W_i \subset \mathbb{R}^n$, with non-empty intersection, there exists a finite constant $c > 0$ such that

$$d \left( x, \bigcap_i W_i \right) \leq c \cdot \max_i d(x, W_i), \quad \forall x \in \mathbb{R}^n.$$  \hfill (88)

**Proof of Lemma 4** For any $x \in \mathbb{R}^n$, let $r(x) = \sup \{ r : U_r \text{ is low-dimensional} \}$. By definition, if $x$ is not a critical point, then $r(x) > 0$, and if $r \geq r(x)$, then $U_r$ is not low-dimensional. We will show that

$$\gamma \triangleq \inf_{x \notin C} \frac{r(x)}{d(x, C)} > 0.$$  \hfill (89)

In order to draw a contradiction, suppose that there exists a sequence of points $y_k \in \mathbb{R}^n \setminus C$ such that

$$\frac{r(y_k)}{d(y_k, C)} \to 0.$$  \hfill (90)

Since $U_{r(y_k)}(y_k)$ is not low-dimensional, there exist $n + 1$ drifts $\mu_1, \ldots, \mu_{n+1} \in U_{r(y_k)}(y_k)$ such that

$$\text{span}\{\mu_i - \mu_j | i, j \leq n + 1\} = \mathbb{R}^n.$$  \hfill (91)

Because the set $M$ of all drifts is finite, there exists an infinite subsequence $\{x_k\}$ of $\{y_k\}$ for which (91) holds for the same set of drifts. We fix this set of drifts $\{\mu_i\}_{i=1}^{n+1}$. Then, for any $k$, $\{x_k\}_{i=1}^{n+1} \subseteq U_{r(x_k)}(x_k)$. It follows from the definition of $r(x)$ that for any $k$,

$$r(x_k) = \max_{i \leq n+1} d(x_k, R_i),$$  \hfill (92)

where $R_i = R_{\mu_i}$ is the effective region of $\mu_i$. We define $n(n+1)$ half-spaces $W_{i,j}$ as follows. For any $i, j \leq n + 1$ with $i \neq j$, let

$$W_{i,j} \triangleq \left\{ x \in \mathbb{R}^n \mid - (\mu_i - \mu_j)^T x + b_i - b_j \geq 0 \right\},$$  \hfill (93)

where $b_i$ is a shorthand for $\mu_i$. Then, for any $i \leq n + 1,$

$$R_i = \bigcap_{j \neq i} W_{i,j}.$$  \hfill (94)

Hence, for any $i \leq n + 1$ and any $x \in \mathbb{R}^n$, $d(x, R_i) \geq \max_{j \leq n + 1} d(x, W_{i,j})$. Then, it follows from (92) that for any $k \geq 1$,

$$r(x_k) = \max_{i \leq n+1} d(x_k, R_i) \geq \max_{i \leq n+1} d(x_k, W_{i,j}).$$  \hfill (95)

It follows from (91) that the following system of $n$ linear equations is non-degenerate:

$$-(\mu_i - \mu_{n+1})^T x + b_i - b_{n+1} = 0, \quad i = 1, \ldots, n.$$  \hfill (96)

Hence, it has a unique solution, which we denote by $p$. Note that $W_{i,j}$ and $W_{j,i}$ are different, and their intersection is $\{ x \mid - (\mu_i - \mu_j)^T x + b_i + b_j = 0 \}$. Therefore,

$$\{ p \} = \bigcap_{i \neq j, i,j \leq n+1} W_{i,j}. \hfill (97)$$

It follows from Lemma 6 with $\delta = 1/c$, that there exists a constant $\delta > 0$ such that for any $x \in \mathbb{R}^n$,

$$\max_{i,j \leq n+1} d(x, W_{i,j}) \geq \delta d(x, \bigcap_{i \neq j, i,j \leq n+1} W_{i,j}) = \delta d(x, p).$$  \hfill (98)

Combining (95) and (98), we have for any $k$,

$$r(x_k) \geq \max_{i \leq n+1} d(x_k, W_{i,j}) \geq \delta d(x_k, p).$$  \hfill (99)

Back to the hypothesis (90), there are two possible cases: (a) $\{ x_k \}$ has a subsequence $\{ z_k \}$ with $d(z_k, C) \to \infty$, or (b) $x_k$ has a subsequence $z_k$ with $r(z_k) \to 0$.

In the first case, where $d(z_k, C) \to \infty$, it follows from (99) that

$$\lim_{k \to \infty} \frac{r(z_k)}{d(z_k, C)} = \lim_{k \to \infty} \frac{d(z_k, p)}{d(z_k, C)} \geq \delta \lim_{k \to \infty} \frac{d(z_k, C) - d(p, C)}{d(z_k, C)} \geq \delta > 0,$$

which contradicts (90).

In the second case, where $r(z_k) \to 0$, it follows from (99) and (92) that for any $i \leq n + 1$,

$$d(p, R_i) \leq d(p, z_k) + d(z_k, R_i) \leq \frac{r(z_k)}{\delta} + r(z_k) \to 0.$$  \hfill (100)

Then, since each $R_i$ is a closed set, we must have $p \in R_i$. Hence, $p \in \bigcap_{i \leq n+1} R_i$, which together with (91) implies that $p$ is a critical point. Using this fact, and then (99), we obtain

$$\lim_{k \to \infty} \frac{r(z_k)}{d(z_k, C)} \geq \lim_{k \to \infty} \frac{r(z_k)}{d(z_k, p)} \geq \delta > 0,$$

which again contradicts (90). Hence, (90) is contradicted in both cases, and (89) follows.

Let $\gamma = \max \{ 1, 1/\gamma \}$. It follows from (89) that $\gamma r(x) \geq d(x, C)$, for all $x \notin C$. Hence, if $\gamma r < d(x, C)$, then $r < r(x)$, and by the definition of $r(x)$, $U_r(x)$ is low-dimensional. \hfill \Box