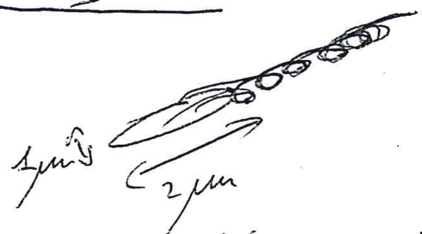
2017 | Chapter 7: Active Particles | 7.

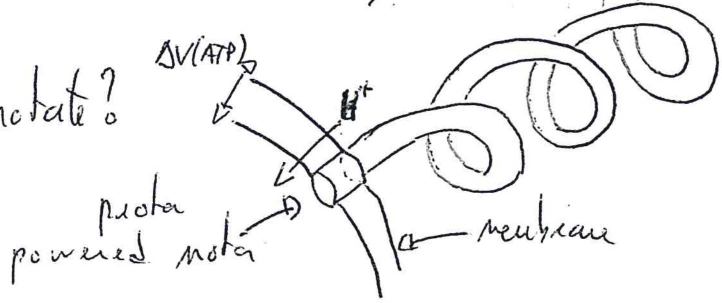
7.1) Run-and-tumble bacteria

7.1.1) Runs



- E coli 5-10 flagella
- More for other bacteria
- Sometimes 1, sometimes 0 (pili)

Why do flagella rotate?



Navier-Stokes

$$\underbrace{\rho D_t v}_{\text{inertia}} = \underbrace{\rho \partial_t v}_{1'} + \underbrace{\rho\, v \cdot \vec{\partial} v}_{1} = -\vec{\partial} p + \rho \vec{f} + \underbrace{\rho \nu \Delta \vec{v}}_{2}$$

(2) → viscous forces

Typical length L

Typical speed U

$$1 \sim 1' \sim \frac{\rho U^2}{L} \quad , \quad 2 \sim \rho \nu \frac{U}{L^2} \Rightarrow \frac{1}{2} \sim \frac{UL}{\nu} \equiv Re$$

The Reynolds number measures the ratio between viscous and inertial forces.

Bacteria $L = 10^{-6}\,m$, $U = 10^{-5}\,m/s$, $\nu = 10^{-6}\,m^2/s$ $\Big\}$ $Re = 10^{-5}$

Human $L = 1\,m$, $U = 1\,m/s$, $\nu = 10^{-6}\,m^2/s$ $\Big\}$ $Re = 10^{6}$

Inertia is 11 orders of magnitude more important for us than for a bacterium! To experience the same Re: swim in honey ($\nu = 10^{-3} m^2/s$)

$$\frac{U}{10^{-3}} = 10^{-5} \Rightarrow U = 10^{-8}\,m/s = \frac{1\,cm}{10\text{ days}} \quad :-)$$

Coasting length: Human $\sim 1\,m$ ⇒ bacteria $\sim 0{,}1$ Å!

[Life at low Reynolds, Purcell 1977, Am. J. Phys. 45, p3]

Comment: The flow around a swimming bacterium is given by Stokes equation

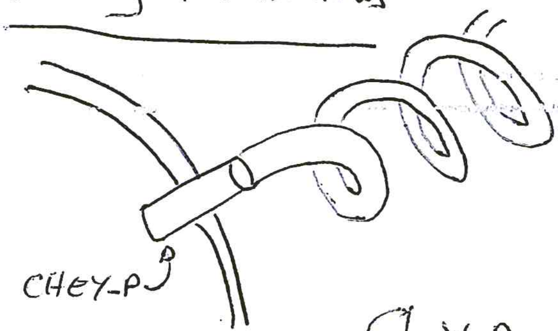
$$\eta \Delta \vec{r} = \vec{\nabla} p - \cancel{\vec{f}} \quad \text{(force)} \Rightarrow \text{reversible, no arrow of time}$$

Reciproqual motion in configuration space cannot generate a net motion, even if backward & forward motion take place at different paces.

Proof: Ishimoto & Yamada, http://arxiv.org/abs/1107.5938

How to swim? Rotation of a helix $\Rightarrow$ chiral object $\Rightarrow$ non-reciproqual motion $\Rightarrow$ runs in (quasi) straight lines.

7.1.2) The tumbles

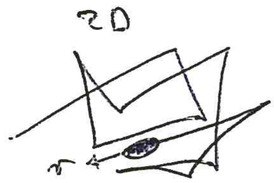


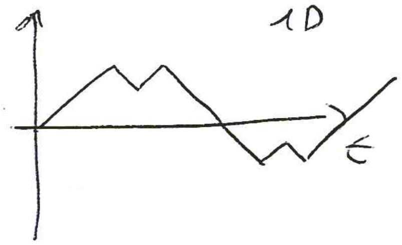
A phosphorilated CheY binds to the motor and change its rotation direction ($CCW \to CW$) $\Rightarrow$ split the flagellar bundle $\Rightarrow$ runs

CheY-P unbinds (because CheZ steals the P): CCW rotation again $\Rightarrow$ new run.

Tumbles | occur at rate $\alpha \simeq 1$ Hz for wild type E. coli
| last ~ 0.1 s $\Rightarrow$ assumed instantaneous in most models

$\Rightarrow$ Typical trajectories:





7.1.3 Modelling

1D: $\dot{x}_0 = v_0 \sigma$; $\sigma = \pm 1$ $\quad \sigma \xrightarrow{\alpha/2} -\sigma$

Master equation: $P_R(x,t)$ the proba to find the bacterium going to the right at x,t
$P_L(x,t)$ ———————————— left ———

2017 | Master equation:

$$\begin{cases} \partial_t P_R(x,t) = -\partial_x\left[v_0 P_R(x,t)\right] - \frac{\alpha}{2} P_R(x,t) + \frac{\alpha}{2} P_L(x,t) \\ \partial_t P_L(x,t) = \partial_x\left[v_0 P_L(x,t)\right] + \frac{\alpha}{2} P_R(x,t) - \frac{\alpha}{2} P_L(x,t) \end{cases}$$

2D $P(\vec{r},\theta)$: proba to find the bacterium at $\vec{r}$ going in the direction θ

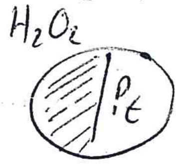
$$\partial_t P(\vec{r},\theta) = -\vec{\nabla}\cdot\left[v_0 P(\vec{r},\theta)\vec{u}_\theta\right] - \alpha P(r,\theta) + \frac{\alpha}{2\pi}\underbrace{\int_0^{2\pi} d\theta' P(r,\theta')}_{=\rho(r)}$$

$\rho(r)$ density of probability in r, whatever θ.

$$= -\partial_x\left[v_0\cos\theta P(x,y,\theta)\right] - \partial_y\left[v_0\sin\theta P(x,y,\theta)\right] - \alpha P + \frac{\alpha\rho}{2\pi}$$

7.2) Active Brownian Particles

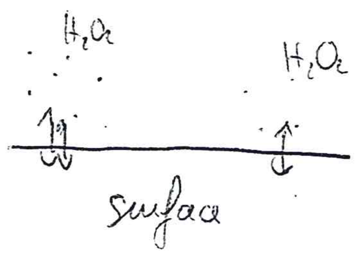
7.2.1) Self-Phoresis



Gold or latex

Idea: Pt catalyzes the transmutation of H_2O_2 into $H_2O + O_2$

Asymmetry of surface $\Rightarrow$ gradient of $[H_2O_2]$



Interactions solute - surface $\Rightarrow$ pressure

non-uniform solute $[\ ]^\circ$ $\Rightarrow$ pressure gradient $\Rightarrow$ flow

In practice, many more ingredients: van der Waals, electrostatic, hydrophylic, etc $\Rightarrow$ sign and modulus of speed hard to predict but they self-propel!

Exp: Latex/Pt, light colloid, 3d swimmers, slow $v_0 \sim 1$-$3\,\mu m/s$ [Palacci et al, PRL 105, 088304, 2010]

Au/Pt, heavy particles, sediment + 2d swimmers, faster 5-20 $\mu m/s$ [Theurkauff et al, PRL 108, 268303, 2012]

7.2.2) Model

2D $\dot{\vec{r}} = v_0 \vec{u}_\theta + \sqrt{2D_t}\,\vec{\eta}$, $\vec{u}_\theta = (\cos\theta, \sin\theta)$; $\dot{\theta} = \sqrt{2D_r}\,\xi$

η_x, η_y, ξ three independent Gaussian-White noise

Fokker-Planck $\partial_t P(\vec{r},\theta) = -\vec{\nabla}\left[v_0 \vec{u}_\theta P(\vec{r},\theta)\right] + \Delta\left[D_t P(\vec{r},\theta)\right] + \partial_{\theta\theta}\left[D_r P(\vec{r},\theta)\right]$

2017 ~~VIII~~ Some simple cases

7.3.1 Free diffusion of ABP in 2d

$$x(t)-x(0)=\int_0^t v_0\cos[\theta(s)]ds+\sqrt{2D_t}\int_0^t \eta_x(s)ds\ ;\ y(t)-y(0)=\int_0^t v_0\cos[\theta(s)]ds+\sqrt{2D_t}\int_0^t \eta_y(s)ds$$

$$\theta(t)-\theta(0)=\int_0^t\sqrt{2D_r}\,\xi(s)ds \Rightarrow P[\theta(t)-\theta(0)]=\frac{1}{\sqrt{4\pi D_r t}}e^{-\frac{[\theta(t)-\theta(0)]^2}{4D_r t}}$$

a) $\langle x(t)-x(0)\rangle = v_0\int_0^t\langle\cos\theta(s)\rangle ds$

① without Itō $\langle\cos\theta(s)\rangle=\frac{1}{\sqrt{4\pi D_r s}}\int_{-\infty}^{\infty}d\theta(s)\,\frac{e^{i\theta(s)}+e^{-i\theta(s)}}{2}\,e^{-\frac{1}{2}\frac{[\theta(s)-\theta(0)]^2}{2D_r s}}$; $\theta(s)=\theta(0)+u$

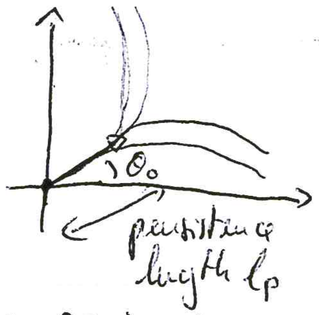
$$\langle\cos\theta(s)\rangle=\frac{1}{\sqrt{4\pi D_r s}}\int_{-\infty}^{+\infty}du\,\frac{e^{i\theta(0)+iu}+e^{-i\theta(0)-iu}}{2}\,e^{-\frac{1}{2}\frac{u^2}{2D_r s}}$$

$$\int_{-\infty}^{+\infty}du\,e^{\pm iu-\frac{1}{2}\frac{u^2}{2D_r s}}=\sqrt{4\pi D_r s}\,e^{-D_r s}\Rightarrow\langle\cos\theta(s)\rangle=\langle\cos\theta(0)\rangle e^{-D_r s}$$

② Itō $\partial_t\langle\cos\theta\rangle=\langle-\sin\theta\cdot\dot\theta\rangle-\frac{1}{2}2D_r\langle\cos\theta\rangle=-D_r\langle\cos\theta\rangle\Rightarrow\langle\cos\theta(s)\rangle=\langle\cos\theta(0)\rangle e^{-D_r s}$

$$\langle x(t)-x(0)\rangle=v_0\langle\cos\theta(0)\rangle\frac{1-e^{-D_r t}}{D_r}\xrightarrow[t\to\infty]{}\frac{v_0}{D_r}\langle\cos\theta_0\rangle$$

exercise $\langle y(t)-y_0\rangle=\frac{v_0}{D_r}\langle\sin\theta_0\rangle$



$l_p\sim\frac{v_0}{D_r}$. Persistence time $t\sim\frac{1}{D_r}$

b) $P(\theta_0)=\frac{1}{2\pi}$; $\langle(x(t)-x(0))^2\rangle=\int_0^t du\int_0^t ds\,v_0^2\langle\cos\theta(s)\cos\theta(u)\rangle+2v_0\sqrt{2D_t}\int_0^t du\int_0^t ds\,\underbrace{\langle\cos\theta(s)\eta_x(u)\rangle}_{=\langle\cos\theta(s)\rangle\langle\eta_x\rangle=0}$

$$+2D_t\underbrace{\underbrace{\int_0^t du\int_0^t ds\underbrace{\langle\eta_x(u)\eta_x(s)\rangle}_{\delta(u-s)}}_{1}}_{t}$$

2017

7

$s>u \quad \partial_s \langle \cos\theta(s)\cos\theta(u)\rangle = -\underbrace{\langle \sin\theta(s)\cdot\dot\theta(s)\cos\theta(u)\rangle}_{\langle \sin\theta(s)\cos\theta(u)\rangle \underbrace{\langle \sqrt{2D_r}\,\xi(s)\rangle}_{=0}} - D_r\langle \cos\theta(s)\cos\theta(u)\rangle$

$\Rightarrow \langle \cos\theta(s)\cos\theta(u)\rangle = e^{-D_r(s-u)}\langle \cos^2\theta(u)\rangle$

$P(\theta_0) = \frac{1}{2\pi} \Rightarrow$ isotropic $\langle \cos^2\theta(u)\rangle = \frac{1}{2}$ [or compute $\partial_u\langle \cos^2\theta(u)\rangle$ with Itô]

$s>u \quad \langle \cos\theta(s)\cos\theta(u)\rangle = \frac{1}{2}e^{-D_r(s-u)}$

$u>s \quad \langle \cos\theta(u)\cos\theta(s)\rangle = \frac{1}{2}e^{-D_r(u-s)}$

$$\langle (x(t)-x(0))^2\rangle = 2D_t t + \frac{v_0^2}{2}\int_0^t du\left[\int_0^u ds\, e^{-D_r(u-s)} + \int_u^t ds\, e^{-D_r(s-u)}\right]$$

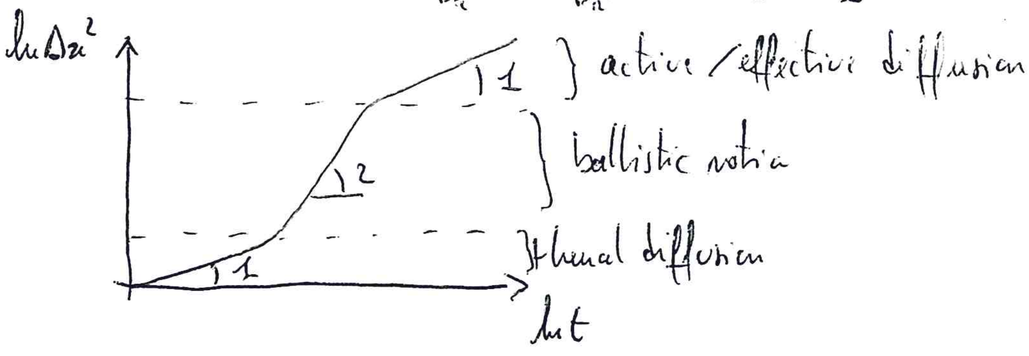
$$= 2D_t t + \frac{v_0^2}{2}\int_0^t du\left[e^{-D_r u}\frac{e^{D_r u}-1}{D_r} + e^{D_r u}\frac{e^{-D_r t}-e^{-D_r u}}{-D_r}\right]$$

$$= 2D_t t + \frac{v_0^2}{2D_r}\int_0^t du\left[1-e^{-D_r u} + 1 - e^{D_r(u-t)}\right]$$

$$= 2\left[D_t + \frac{v_0^2}{2D_r}\right]t - \frac{v_0^2}{2D_r}\left[\frac{e^{-D_r t}-1}{-D_r} + \frac{1-e^{-D_r t}}{D_r}\right] = 2\left[D_t + \frac{v_0^2}{2D_r}\right]t + \frac{v_0^2}{D_r^2}\left[e^{-D_r t}-1\right]$$

$t\to\infty \quad \langle \Delta x^2\rangle \sim 2D_{eff}t \quad ; \quad D_{eff} = D_t + \frac{v_0^2}{2D_r}$

$t\to 0 \quad \Delta x^2 \sim 2D_t t + \frac{v_0^2}{D_r}t + \frac{v_0^2}{D_r^2}\left[-D_r t + \frac{1}{2}D_r^2 t^2\right] \sim 2D_t t + \frac{1}{2}V_0^2 t^2$



3.2) RTP in 1D

$$\begin{cases} \partial_t P_R(x,t) = -\partial_x\left[v P_R(x,t)\right] - \frac{\alpha}{2} P_R(x,t) + \frac{\alpha}{2} P_L(x,t) \\ \partial_t P_L(x,t) = \partial_x\left[v P_L(x,t)\right] + \frac{\alpha}{2} P_R(x,t) - \frac{\alpha}{2} P_L(x,t) \end{cases}$$

$$\rho = P_R + P_L \Rightarrow \partial_t \rho = -\partial_x J \,;\; J = v P_R - v P_L$$

$$\partial_t J = v\partial_t P_R - v\partial_t P_L = -v\partial_x\left[v(P_R - P_L)\right] - \alpha v P_R + \alpha v P_L$$
$$= -v\partial_x\left[v\rho\right] - \alpha J$$

$$\Rightarrow J_H = J_0 e^{-\alpha t} \,;\; J_p = J_0 e^{-\alpha t} + e^{-\alpha t}\int_{-\infty}^{t} ds\, e^{\alpha s}(-v\partial_x v\rho)$$

$$= J_0 e^{-\alpha t} - \int_{-\infty}^{t} ds\, \underbrace{e^{-\alpha(t-s)}}_{\simeq 0 \text{ when } t-s \gg \alpha^{-1}} \left[v\partial_x v\rho\right]$$

ρ is a conserved field $\Rightarrow$ evolve slowly on time-scales $\gg 1/\alpha$

$$J \simeq J_0 e^{-\alpha t} - \left[v\partial_x v\rho\right]_t \underbrace{\int_{-\infty}^{t} ds\, e^{-\alpha(t-s)}}_{\simeq \frac{1}{\alpha}}$$

$$t \gg \frac{1}{\alpha} \quad J \simeq -\frac{v}{\alpha}\partial_x(v\rho)$$

$$\Rightarrow \partial_t \rho = \partial_x\left[\frac{v}{\alpha}\partial_x(v\rho)\right]$$

① α, v uniform $\Rightarrow \partial_t \rho = \frac{v^2}{\alpha}\partial_{xx}\rho$; diffusion eq° with effective $D = \frac{v^2}{\alpha}$

② $\alpha(x), v(x)$; $J=0$ steady-state $\Rightarrow \rho v = C^{te} \Rightarrow \rho \propto \frac{1}{v}$

Particles accumulate where they go slowly.

4) Motility-induced Phase separation

Phenomenon: auto-propulsion + slowdown at high density $\rightarrow$ phase separation.

Very general:

* RTP: [Tailleur, Cates, PRL 100, 218103 (2008)]; Thompson et al., J. Stat. Mech P02029, (2011)] $\rightarrow$ Quorum-Sensing interactions
* ABP: [Fily, Marchetti, PRL 108, 235702 (2012)]; Redner, Hagan, Baskaran, PRL 110, 055701 (2013)] $\rightarrow$ pairwise forces
  $v(\rho)$ [Cates, Tailleur, EPL 20010, (2013)]
* Experiments showing enhanced clustering: [Theurkauff et al. PRL 2012, 108, 268303; Buttinoni et al., PRL 2013, 110, 238301; Palacci et al, Science 339, 936 (2013)]

4.1) Linear Stability analysis

An intuitive explanation relies on a feedback between two ingredients

(i) Non-uniform speed $v(r)$

$$\partial_t P(r,t) = -\vec{\nabla}\cdot\left[v(r)\vec{u}(\theta)P(r,t)\right] + \text{isotropic reorientation terms}$$

→ e.g. $D_r \partial_{\theta\theta} P$ (rotational diff°)

$-\alpha P + \frac{\alpha}{2\pi}\int d\theta' P(r,\theta')$ (tumble)

$$P_s(r,\theta) = \frac{c}{v(r)} \text{ is a steady-state solution.}$$

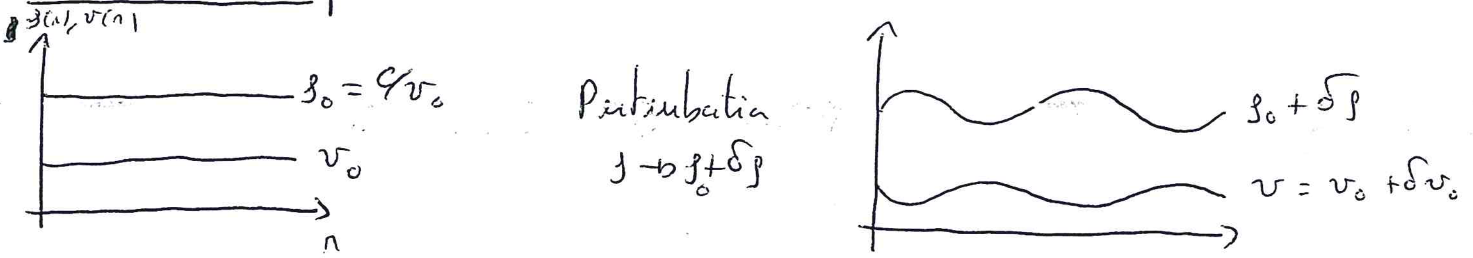
Makes sense: P_s measures time spent somewhere. Go slower → spend more time.

(ii) Repulsive forces/crowding: modelled as $v(\rho)$ with $v'(\rho)<0$

→ OKish for bacteria

→ mean-field approximation for pairwise force particles

Feedback loop:



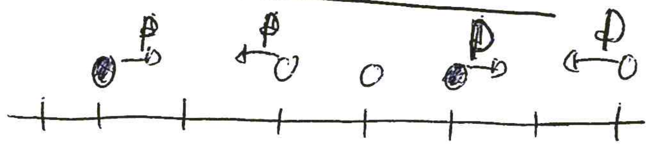
$\delta v_0 = v'(\rho_0)\delta\rho_0$ $\quad\rho$ relaxes towards $\frac{c}{v_0+\delta v_0} = \frac{c}{v_0}\cdot\frac{1}{1+\frac{v'}{v}\delta\rho} \simeq \rho_0\left(1-\frac{v'}{v}\delta\rho\right)$

If $\left|\rho_0\frac{v'}{v}\right| > 1$, the fluctuation of ρ is amplified ⇒ instability

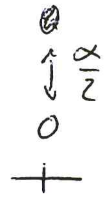
$$\Leftrightarrow \boxed{\frac{v'}{v} < -\frac{1}{\rho}}$$

⇒ Hand-waving argument, can we prove this for a real model?

4.2) RTP on lattice in 1D



+



+ periodic boundary conditions

$\varphi \Leftrightarrow \{n_i^+, n_i^-\}$ the list of numbers of particles on site i going to the right or to the left.

Tumbling rate $\frac{\alpha}{2}$

Hopping rate $i \to i+1$ for a $\oplus$ particle $p\left(1 - \frac{n_{i+1}}{n_M}\right)$ when $n_i = n_i^+ + n_i^-$

$i \to i-1$ —— $\ominus$ —— $p\left(1 - \frac{n_{i-1}}{n_M}\right)$

Idea: Understand the evolution of $\rho_i^\pm(t) = \langle n_i^\pm(t) \rangle$

Master equation: $\partial_t P(\varphi) = \sum_{\varphi' \neq \varphi} W(\varphi' \to \varphi) P(\varphi') - W(\varphi \to \varphi') P(\varphi)$

$\varphi = \{n_i^\pm\}$; who are the φ' from which the system can reach φ?

$\forall i$ $\varphi' \Leftrightarrow n_{i-1}^+ + 1, n_i^+ - 1 \to \varphi$ at rate $W = p(n_{i-1}^+ + 1)\left(1 - \frac{n_i - 1}{n_M}\right)$

$\varphi' \Leftrightarrow n_{i+1}^- + 1, n_i^- - 1 \to \varphi$ —— $W = p(n_{i+1}^- + 1)\left(1 - \frac{n_i - 1}{n_M}\right)$

$\varphi' \Leftrightarrow n_i^- + 1, n_i^+ - 1 \to \varphi$ —— $W = \frac{\alpha}{2}(n_i^- + 1)$

$\varphi' \Leftrightarrow n_i^+ + 1, n_i^- - 1 \to \varphi$ —— $W = \frac{\alpha}{2}(n_i^+ + 1)$

Conversely $\varphi \to \varphi'$

$\forall i$: $\varphi' = n_i^+ - 1, n_{i+1}^+ + 1$ at rate $W = p\, n_i^+ \left(1 - \frac{n_{i+1}}{n_M}\right)$

$\varphi' = n_i^- - 1, n_{i-1}^- + 1$ at rate $W = p\, n_i^- \left(1 - \frac{n_{i-1}}{n_M}\right)$

$\varphi' = n_i^- + 1, n_i^+ - 1$ —— $W = \frac{\alpha}{2} n_i^+$

$\varphi = n_i^- - 1, n_i^+ + 1$ —— $W = \frac{\alpha}{2} n_i^-$

$$\partial_t P(\{n_i^+, n_i^-\}) = \sum_i p(n_{i-1}^+ + 1)\left(1 - \frac{n_i - 1}{n_M}\right) P(\dots, n_{i-1}^+ + 1, n_i^+ - 1) + p(n_{i+1}^- + 1)\left(1 - \frac{n_i - 1}{n_M}\right) P(\dots, n_i^- - 1, n_{i+1}^- + 1, \dots)$$

$$+ \frac{\alpha}{2}(n_i^- + 1) P(\dots, n_i^- + 1, n_i^+ - 1, \dots) + \frac{\alpha}{2}(n_i^+ + 1) P(\dots, n_i^- - 1, n_i^+ + 1, \dots)$$

$$- \left[p n_i^+ \left(1 - \frac{n_{i+1}}{n_M}\right) + p n_i^- \left(1 - \frac{n_{i-1}}{n_M}\right) + \frac{\alpha}{2} n_i^+ + \frac{\alpha}{2} n_i^-\right] P(\dots, n_i^+, n_i^-, n_{i+1}^+, n_{i+1}^-, \dots) \quad (*)$$

Then use $\langle n_i^\pm \rangle = \sum_{\{n_i\}} n_i^\pm P(\dots, n_i^+, n_i^-)$ to get $\partial_t \langle n_i^\pm \rangle$ from $(*) \Rightarrow$ good luck

Evolution of ~~operator~~ observables: $\langle O(q) \rangle = \sum_q O(q) P(q) = \langle O \rangle_q$

$$\partial_t \langle O \rangle_q = \sum_q O(q)\, \partial_t P(q) = \sum_{q,q'} O(q) \left[W(q' \to q) P(q') - W(q \to q') P(q)\right]$$

$$= \sum_{q,q'} \left[O(q') W(q \to q') - O(q) W(q \to q')\right] P(q)$$

$$= \sum_q \left\{\sum_{q'} \left[O(q') - O(q)\right] W(q \to q')\right\} P(q)$$

$$\partial_t \langle O \rangle_q = \Big\langle \sum_{q'} \underbrace{\Delta O(q \to q')}_{\text{change in } O \text{ from } q \text{ to } q'} \cdot \underbrace{W(q \to q')}_{\text{rate at which the hop occurs}} \Big\rangle_q$$

Dynamics of the average occupancies $\partial_t \langle n_i^\pm \rangle = ?$

Consider all q' you can reach from $q = \{n_1^+, n_1^-, \dots, n_L^+, n_L^-\}$ such that $W(q \to q') \neq 0$ and $\Delta n_i^\pm = n_i^\pm(q') - n_i^\pm(q) \neq 0$

$\to$ tumbles in i and arrival / departure in i.

$$\partial_t \langle n_i^+ \rangle = \left\langle p\, n_{i-1}^+ \left(1 - \frac{n_i}{n_M}\right) \times (+1) + p\, n_i^+ \left(1 - \frac{n_{i+1}}{n_M}\right) \times (-1) + \frac{\alpha}{2} n_i^+ \times (-1) + \frac{\alpha}{2} n_i^- \times (+1) \right\rangle$$

$$= p \left\langle n_{i-1}^+ \left(1 - \frac{n_i}{n_M}\right)\right\rangle - p \left\langle n_i^+ \left(1 - \frac{n_{i+1}}{n_M}\right)\right\rangle + \frac{\alpha}{2} \langle n_i^- - n_i^+ \rangle$$

$$\partial_t \langle n_i^- \rangle = -p \left\langle n_i^- \left(1 - \frac{n_{i-1}}{n_M}\right)\right\rangle + p \left\langle n_{i+1}^- \left(1 - \frac{n_i}{n_M}\right)\right\rangle + \frac{\alpha}{2} \langle n_i^+ - n_i^- \rangle$$

Comment: Not closed $\to \partial_t \langle n_i^\pm \rangle$ involves the 2-point functions $\langle n_i\, n_{i\pm1}^\pm \rangle$; in turn $\partial_t \langle n_i^\pm n_j^\pm \rangle$ involves $\langle n_i^\pm n_j^\pm n_k^\pm \rangle$ etc.

Mean-field approximation $\langle n_i^{\pm} n_j^{\pm}\rangle = \langle n_i^{\pm}\rangle\langle n_j^{\pm}\rangle$

In general, quantitatively wrong but qualitatively ok away from critical points.

$\rho_i^{\pm} \equiv \langle n_i^{\pm}\rangle \; ; \; \rho_i = \rho_i^+ + \rho_i^-$

$$\begin{cases} \partial_t \rho_i^+ = p\,\rho_{i-1}^+\left(1-\frac{\rho_i}{\rho_M}\right) - p\,\rho_i^+\left(1-\frac{\rho_{i+1}}{\rho_M}\right) - \frac{\alpha}{2}\rho_i^+ + \frac{\alpha}{2}\rho_i^- \\ \partial_t \rho_i^- = p\,\rho_{i+1}^-\left(1-\frac{\rho_i}{\rho_M}\right) - p\,\rho_i^-\left(1-\frac{\rho_{i-1}}{\rho_M}\right) + \frac{\alpha}{2}\rho_i^+ - \frac{\alpha}{2}\rho_i^- \end{cases}$$

Typical microscopic length $\frac{p}{\alpha}$ (run-length) $\Rightarrow$ we expect variations of $\rho_i^{\pm}$ on scales $\frac{p}{\alpha}$. If $\frac{p}{\alpha} \gg 1 \Rightarrow$ smooth fields locally $\Rightarrow$ Taylor expansion

$$O_{i\pm 1} = O(x) \pm \frac{1}{L}\nabla O(x) + \frac{1}{2L^2}\Delta O(x) \quad \text{where } x = \frac{i}{L} \in [0,1]$$

$$\partial_t \rho^+(x) = p\left(\rho^+ - \frac{1}{L}\nabla\rho^+ + \frac{1}{2L^2}\Delta\rho^+\right)\left(1-\frac{\rho}{\rho_M}\right) - p\,\rho^+\left(1 - \frac{\rho + \frac{1}{L}\nabla\rho + \frac{1}{2L^2}\Delta\rho}{\rho_M}\right) - \frac{\alpha}{2}\rho^+ + \frac{\alpha}{2}\rho^-$$

$$= -\frac{p}{L}(\partial_x\rho^+)\left(1-\frac{\rho}{\rho_M}\right) + \frac{p}{L}\rho^+\frac{\nabla\rho}{\rho_M} + \frac{p}{2L^2}(\Delta\rho^+)\left(1-\frac{\rho}{\rho_M}\right) + \frac{p\,\rho^+}{2L^2}\frac{\Delta\rho}{\rho_M} - \frac{\alpha}{2}\rho^+ + \frac{\alpha}{2}\rho^- \quad (1)$$

$$\partial_t \rho^-(x) = p\left(\rho^- + \frac{1}{L}\partial_x\rho^- + \frac{1}{2L^2}\partial_{xx}\rho^-\right)\left(1-\frac{\rho}{\rho_M}\right) - p\,\rho^-\left(1 - \frac{\rho - \partial_x\rho\cdot\frac{1}{L} + \frac{1}{2L^2}\partial_{xx}\rho}{\rho_M}\right) + \frac{\alpha}{2}\rho^+ - \frac{\alpha}{2}\rho^-$$

$$= \frac{p}{L}\partial_x\rho^-\left(1-\frac{\rho}{\rho_M}\right) - \frac{p}{L}\rho^-\frac{\partial_x\rho}{\rho_M} + \frac{p}{2L^2}(\partial_{xx}\rho^-)\left(1-\frac{\rho}{\rho_M}\right) + \frac{p}{2L^2}\rho^-\frac{\partial_{xx}\rho}{\rho_M} + \frac{\alpha}{2}\rho^+ - \frac{\alpha}{2}\rho^- \quad (2)$$

$m(x) = \rho^+(x) - \rho^-(x)$

$$(1)+(2) \Rightarrow \partial_t\rho(x) = -\frac{p}{L}(\partial_x m)\left(1-\frac{\rho}{\rho_M}\right) + \frac{p}{L}m\frac{\partial_x\rho}{\rho_M} + \frac{p}{2L^2}(\partial_{xx}\rho)\left(1-\frac{\rho}{\rho_M}\right) + \frac{p}{2L^2}\frac{\rho}{\rho_M}\Delta\rho$$

$$\boxed{\partial_t\rho(x) = -\partial_x\left[\frac{p}{L}m\left(1-\frac{\rho}{\rho_M}\right) - \frac{p}{2L^2}\partial_x\rho\right] \quad (3)}$$

$$(1)-(2) \Rightarrow \partial_t m(x) = -\frac{p}{L}(\partial_x\rho)\left(1-\frac{\rho}{\rho_M}\right) + \frac{p}{L}\rho\frac{\partial_x\rho}{\rho_M} + \frac{p}{2L^2}\Delta m\cdot\left(1-\frac{\rho}{\rho_M}\right) + \frac{p}{2L^2}m\frac{\Delta\rho}{\rho_M} - \alpha\rho^+ + \alpha\rho^-$$

$$\boxed{\partial_t m(x) = -\alpha m - \partial_x\left[\frac{p}{L}\rho\left(1-\frac{\rho}{\rho_M}\right)\right] + \frac{p}{2L^2}(\partial_{xx}m)\left(1-\frac{\rho}{\rho_M}\right) + \frac{p}{2L^2}m\left(\frac{\partial_{xx}\rho}{\rho_M}\right) \quad (4)}$$

Analysis of (3)-(4)

1. $L\to\infty$; (3) $\Rightarrow \partial_t\rho = 0 \Rightarrow$ Nothing happens in a time of order 1 for a conserved field at arbitrary large scales.

$\Rightarrow$ speed up time $t = L\tau$

$$(3) \Rightarrow \dot\rho = -\partial_x\left[p\,m\left(1-\frac{\rho}{\rho_M}\right) - \frac{p}{2L}\partial_x\rho\right]$$

$$(4) \Rightarrow \dot{m} = -\alpha L m - p\, \partial_x\left[\rho\left(1-\frac{\rho}{\rho_M}\right)\right] + \frac{p}{2L}(\partial_{xx} m)\left(1-\frac{\rho}{\rho_M}\right) + \frac{p}{2L}\, m\, \partial_{xx}\rho$$

For finite $\partial_{xx}\rho$, $\partial_{xx} m$, the last two terms are negligible in the large L limit.

$$\Rightarrow \dot{m} = -\alpha L m - p\, \partial_x\left[\rho\left(1-\frac{\rho}{\rho_M}\right)\right]$$

m relaxes in a time $\Delta\tau \sim \frac{1}{L}$ towards $m = -\frac{p}{\alpha L}\partial_x\left[\rho\left(1-\frac{\rho}{\rho_M}\right)\right]$

This yields

$$\dot{\rho} = \partial_x\left[\frac{p^2}{\alpha L}\left(1-\frac{\rho}{\rho_M}\right)\partial_x\left[\rho\left(1-\frac{\rho}{\rho_M}\right)\right] + \frac{p}{2L}\partial_x \rho\right]$$

Again, nothing happens in a time $\tau \sim 1 \Rightarrow \tau = L\tilde{\tau}$

$$\Rightarrow \dot{\rho} = \partial_x\left[\left(\frac{p}{2} + \frac{p^2}{\alpha}\left(1-\frac{\rho}{\rho_M}\right)\left(1-\frac{2\rho}{\rho_M}\right)\right)\partial_x\rho\right] \equiv \partial_x\left[D_{eff}(\rho)\,\partial_x(\rho)\right]$$

$$D_{eff}(\rho) = \frac{p}{2} + \frac{p^2}{\alpha}\left(1-\frac{\rho}{\rho_M}\right)\left(1-\frac{2\rho}{\rho_M}\right)$$

If $D_{eff}(\rho_0) > 0$, small fluctuations around ρ_0 relax to ρ_0.

If $D_{eff}(\rho_0) < 0$; $\dot{\rho} = \nabla\left[D_{eff}\nabla\rho\right] \Leftrightarrow \frac{d\rho}{d(-t)} = \nabla\left[(-D_{eff})\nabla\rho\right]$ in reversed-time

$\Rightarrow$ small fluctuations get amplified $\Rightarrow$ linear instability.

$$D_{eff} < 0 \Leftrightarrow \left(\frac{2\rho}{\rho_M}-1\right)\left(1-\frac{\rho}{\rho_M}\right) \geqslant \frac{\alpha}{2p} = \frac{1}{2\ell_p}$$

persistence length $\ell_p = \frac{p}{\alpha}$

$\ell_p \gg 1 \Leftrightarrow$ instab for $\rho > \frac{\rho_M}{2}$

Comment: Here $v(\rho) \simeq p\left(1-\frac{\rho}{\rho_M}\right)$; $v'(\rho) = -\frac{p}{\rho_M}$

$$\frac{v'}{v} < -\frac{1}{\rho} \Leftrightarrow -\frac{p\rho}{\rho_M} < -p\left(1-\frac{\rho}{\rho_M}\right) \Leftrightarrow \frac{\rho}{\rho_M} > 1-\frac{\rho}{\rho_M} \Leftrightarrow \rho > \frac{\rho_M}{2}$$

$\Rightarrow$ Consistent with hand-waving argument.