Numerics: B. Mahault $\chi \simeq 0.95$ TT $(\chi = 0.60)$

$z \simeq 1.33$ TT $(z = 1.20)$

$\alpha \simeq 1.67$ TT $(\alpha \simeq 1.60)$

$\Rightarrow$ still some mystery!

But at least it is clear why moving helps $\Rightarrow N(t)$ grows faster with $t \Rightarrow$ better averaging of the noise.

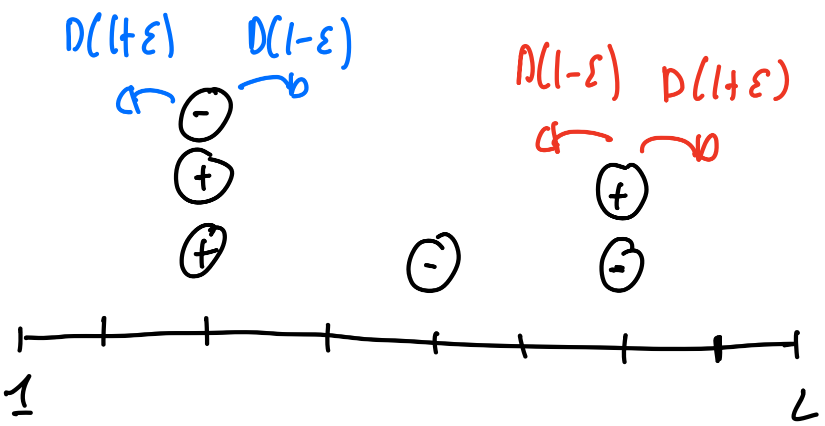
3) The transition to collective motion

15 years after the introduction of the VM $\to$ Nature of transition still unclear

$\Rightarrow$ simpler model: Heisenberg spins $\Rightarrow$ Ising spins

3.1) The active Ising Model

① Be wise discretize ② Scalar is easier



- L^d sites
- periodic boundary conditions

- Symmetric diffusion in $(d-1)$ dimensions
- Diffusion biased by spins along $\vec{e}_x$.

$\Rightarrow$ Self propulsion

- Aligning rates $s = \pm 1$; $W_f(s \to -s) = \Gamma e^{-\beta s \frac{m_i}{\rho_i}}$

where $\rho_i = n_i^+ + n_i^-$ → occupancy of site i.

$m_i = n_i^+ - n_i^-$ → magnetisation ——.

Comment: W_f satisfies detailed balance with respect to

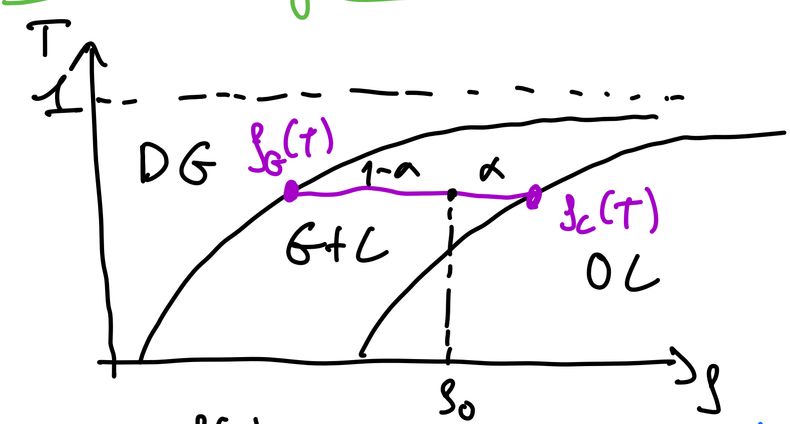
$$H = -J \sum_{\text{sites } i} \frac{1}{n_i} \sum_{S_j, S_k \text{ at } i} S_j \cdot S_k$$

⇒ L^d independent, fully connected Ising models at temperature $T = \beta^{-1}$

But hopping dynamics insensitive to H

⇒ global dynamics out of equilibrium (even at $\varepsilon = 0$)

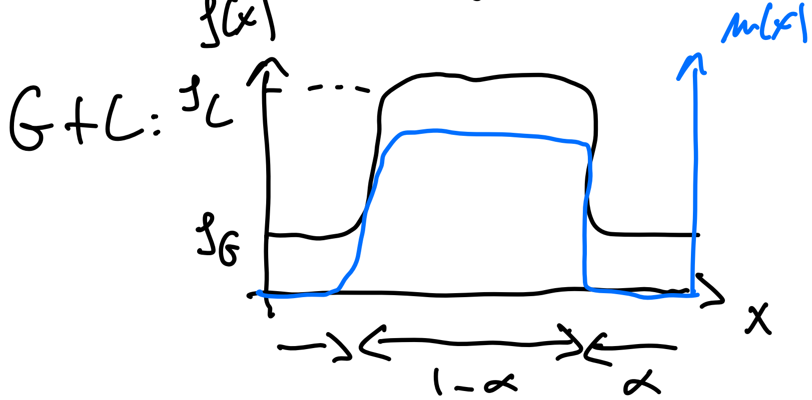
Phase diagram in 2D



DG: disordered gas (SRC)

OL: ordered liquid (SRC)

G+L: liquid-gas coexistence phase

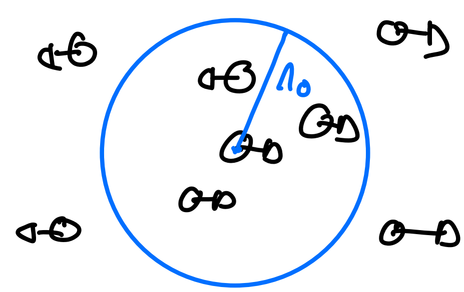


lever rule

Q: How can we understand this phase transition?

3.2) Off-lattice Ising model

$$\dot{\vec{r}}_i = v_0 \sigma_i \vec{u}_x + \sqrt{2D}\,\vec{\xi}_i \quad ; \quad \sigma_i \in \{\pm 1\}$$



Alignment: $r_0 > 0$ $\quad i \mathcal{V} j$ iff $|\vec{r}_i - \vec{r}_j| < r_0$

$$\rho_i = \sum_{j \mathcal{V} i} 1 \quad ; \quad m_i = \sum_{j \mathcal{V} i} \sigma_j$$

$$w(\sigma_i \to -\sigma_i) = \Gamma_0\, e^{-\beta \sigma_i \cdot \bar{m}_i} \quad ; \quad \bar{m}_i = \frac{m_i}{\rho_i}$$

Goal: Account for emerging physics

3.2.1) A single particle in a field $\bar{m}(\vec{r})$

$$\partial_t P(\vec{r}, \sigma=1, t) = D\Delta P - \vec{\nabla}\cdot\left[v_0 \vec{e}_x P\right] - \Gamma_0\, e^{-\beta \bar{m}(\vec{r})} P(\vec{r},1,t) + \Gamma_0\, e^{\beta \bar{m}(\vec{r})} P(\vec{r},-1,t) \qquad (1)$$

$$\partial_t P(\vec{r}, \sigma=-1, t) = D\Delta P + \vec{\nabla}\cdot\left[v_0 \vec{e}_x P\right] + \Gamma_0\, e^{-\beta \bar{m}(\vec{r})} P(\vec{r},1,t) - \Gamma_0\, e^{\beta \bar{m}(\vec{r})} P(\vec{r},-1,t) \qquad (2)$$

$$\rho(\vec{r}) = P(\vec{r},1,t) + P(\vec{r},-1,t) \quad ; \quad m(\vec{r}) = P(\vec{r},1,t) - P(\vec{r},-1,t);$$

$$P(\vec{r},\pm 1) = \frac{\rho \pm m}{2}$$

$$(1+2) \qquad \boxed{\partial_t \rho(\vec{r}) = D\Delta\rho - \partial_x\left[v_0 m\right]} \qquad (3)$$

$$(1-2) \qquad \partial_t m(\vec{r}) = D\Delta m - \partial_x\left[v_0 \rho\right] + 2\Gamma_0 \rho\, \mathrm{sh}(\beta \bar{m}) - 2\Gamma_0 m\, \mathrm{ch}(\beta \bar{m})$$

Small m Landau expansion:

$$\partial_t m(\vec{r}) = D\Delta m - \partial_x\left[v_0 \rho\right] + 2\Gamma_0 \rho\left(\beta \bar{m} + \frac{\beta^3 \bar{m}^3}{6}\right) - 2\Gamma_0 m\left(1 + \frac{\beta \bar{m}^2}{2}\right)$$

ρ & m in Landau terms $\Rightarrow$ counting statistics.
$\bar{m} \Rightarrow$ aligning field.

3.2.2) Off lattice AIM at mean field level

$\bar{m}(r)$: local intensive magnetisation $\rightarrow \bar{m}(r) \hat{=} \frac{m(r)}{\rho(r)}$

Comment: $\bar{m}$ could be anything: averaged over k nearest neighbors,
— Voronoi neighbors
etc.

$\rightarrow$ generic recipe to get mean-field dynamics.

$$\boxed{\partial_t \rho(\vec{r}) = D\Delta\rho - \partial_x[v_0 m]} \qquad (3)$$

$$\Rightarrow \boxed{\partial_t m = D\Delta m - \partial_x[v_0\rho] + 2\Gamma_0 m(\beta-1) - \frac{m^3}{\rho^2}\Gamma_0\left(\beta^2 - \frac{\beta^3}{3}\right) + m\,O\!\left(\frac{m^2}{\rho^2}\right)} \qquad (4)$$

(3+4) : MF description of the AIM.

Fixed points: $\times$ 1st homogeneous solution $\rho = \rho_0$, $m = 0$ (disordered profile)

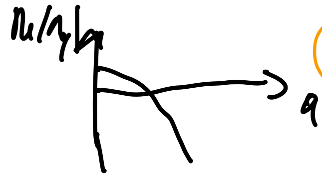
linear stability: $\rho(\vec{r},t) = \rho_0 + \sum_{\vec{q}} \delta\rho_q e^{i\vec{q}\cdot\vec{r}}$; $m(\vec{r},t) = 0 + \sum_{\vec{q}} \delta m_q e^{i\vec{q}\cdot\vec{r}}$

$$\begin{cases} \partial_t \delta\rho_{\vec{q}} = -Dq^2\delta\rho_{\vec{q}} - iv_0 q_x \delta m_{\vec{q}} \\ \partial_t \delta m_{\vec{q}} = -Dq^2 \delta m_{\vec{q}} - iv_0 q_x \delta\rho_{\vec{q}} + 2\Gamma_0(\beta-1)\delta m_{\vec{q}} \end{cases}$$

$$\partial_t \begin{pmatrix} \delta\rho_{\vec{q}} \\ \delta m_{\vec{q}} \end{pmatrix} = M_{\vec{q}} \begin{pmatrix} \delta\rho_{\vec{q}} \\ \delta m_{\vec{q}} \end{pmatrix} \quad ; \quad M_{\vec{q}} = \begin{pmatrix} -Dq^2 & -iv_0 q_x \\ -iv_0 q_x & -Dq^2 + 2\Gamma_0(\beta-1) \end{pmatrix}$$

Eigenvalues: $\lambda^{\pm}_{\vec{q}} = -Dq^2 + \Gamma_0(\beta-1) \pm \sqrt{\Gamma_0^2(\beta-1)^2 - v_0^2 q_x^2}$

instab whenever $\mathrm{Re}(\Lambda_{\vec{q}}^{+}) > 0 \Leftrightarrow \Gamma_0(\beta-1) > 0$



"zero $\vec{q}$" instability

takes a homogeneous $\vec{m}_0 = 0$ profile towards an ordered phase $\vec{m}_0 \neq 0$

$\beta > 1 \Leftrightarrow T < 1$

* 2^{nd} homogeneous solution $\rho = \rho_0$, $m = m_0 \neq 0$, ordered profile

$$2\Gamma_0 m_0 (\beta-1) = \frac{m_0^3}{\rho_0^2} \Gamma_1 \left(\beta^2 - \frac{\beta^3}{3}\right) \Rightarrow m_0^2 = \rho_0^2 \frac{2(\beta-1)}{\beta^2 - \beta^3/3}$$

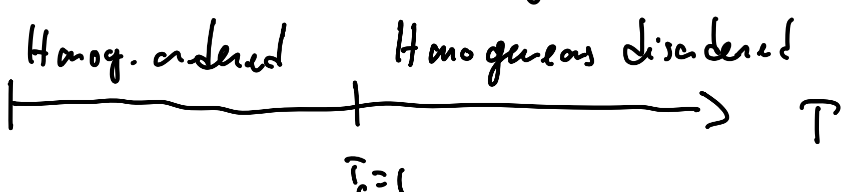
$\beta \in [0,3] \Rightarrow \beta^2 - \frac{\beta^3}{3} > 0 \Leftrightarrow T > \frac{1}{3}$

① $T > 1$, $\beta < 1$, $(\beta-1) < 0 \Rightarrow$ no solution

② $\frac{1}{3} < T < 1 \Rightarrow m_0 = \rho_0 \sqrt{\frac{2(\beta-1)}{\beta^2 - \beta^3/3}}$

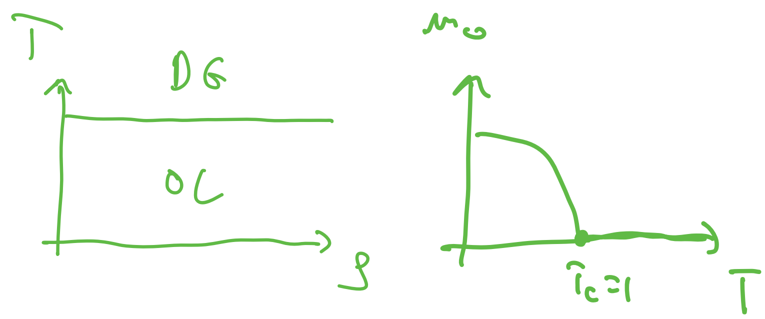
③ $T < \frac{1}{3} \Rightarrow$ need to Taylor expand $\cosh(\beta \frac{m}{\rho})$ & $\sinh(\beta \frac{m}{\rho})$ further

linear stability of ②: $\Rightarrow$ homogeneous solution stable



$\Rightarrow$ No inhomogeneous solution

$\Rightarrow$ Wrong phase diagram



Q: What's missing? $\Rightarrow$ Fluctuations

3.2.3) Beyond mean-field: fluctuation-induced first order transition

P1

1st simplification: work 1d.

$$\dot\rho = \partial_x\left[D\partial_x\rho - v_0 m + \sqrt{2\sigma_\rho}\,\Lambda\right] \;;\; \dot m = D\partial_{xx}m - \partial_x(v_0\rho) - F(\rho,m) + \sqrt{2\sigma_m}\,\zeta$$

$$F(\rho,m) = 2\Gamma_0\, m(1-\beta) + \frac{m^3}{\rho^2}\Gamma_0\left(\beta^2 - \frac{\beta^3}{3}\right) \equiv \alpha m + \tilde\Gamma\,\frac{m^3}{\rho^2}$$

$\Lambda, \zeta \Rightarrow$ Gaussian fluctuations

$\sigma_\rho, \sigma_m \Rightarrow$ amplitude $\propto \sqrt{\rho}$ (CLT)

2nd simplification: $\partial_x\left[\sqrt{2\sigma_\rho}\,\Lambda\right]$ conserved noise $\sim q[\dots]$ in Fourier $\xrightarrow[q\to 0]{} 0 \Rightarrow$ neglect.

3rd simplification: work in high temperature phase so that $\left|\frac{m(r)}{\rho(r)}\right| \ll 1$.

Idea: $\rho = \infty \Rightarrow$ MF should be valid $\Rightarrow$ small noise expansion

$$\tilde\rho \equiv \langle\rho\rangle \;;\; \tilde m(\vec r,t) = \langle m(\vec r,t)\rangle;\; \delta\rho \equiv \rho - \tilde\rho;\; \delta m \equiv m - \tilde m$$

$\dot\rho$ is linear $\Rightarrow \dot{\tilde\rho} = \partial_x\left[D\partial_x\tilde\rho - v_0\tilde m\right]$

$$\dot{\tilde m} = D\partial_{xx}\tilde m - \partial_x(v_0\tilde\rho) - \underbrace{\langle F(\rho,m)\rangle}_{\text{express in terms of } \tilde\rho,\,\tilde m} \;;\; F = \alpha m + \tilde\Gamma\,\frac{m^3}{\rho^2}$$

$$\langle F(\rho,m)\rangle = F(\tilde\rho,\tilde m) + \frac{\partial F}{\partial m}\Big|_{\tilde\rho,\tilde m}\underbrace{\langle(m-\tilde m)\rangle}_{0} + \frac{\partial F}{\partial\rho}\underbrace{\langle(\rho-\tilde\rho)\rangle}_{0} + \frac{1}{2}\frac{\partial^2F}{\partial\rho^2}\Big|_{\tilde\rho,\tilde m}\langle(\rho-\tilde\rho)^2\rangle + \frac{\partial^2F}{\partial\rho\,\partial m}\langle(\rho-\tilde\rho)(m-\tilde m)\rangle + \frac{1}{2}\frac{\partial^2F}{\partial m^2}\langle(m-\tilde m)^2\rangle$$

$$= F(\tilde\rho,\tilde m) + \frac{1}{2}\frac{\partial^2F}{\partial\rho^2}\langle\delta\rho^2\rangle + \frac{1}{2}\frac{\partial^2F}{\partial m^2}\langle\delta m^2\rangle + \frac{\partial^2F}{\partial\rho\,\partial m}\langle\delta\rho\,\delta m\rangle$$

$$= F(\tilde\rho,\tilde m) + 3\,\frac{\tilde m^3}{\tilde\rho^4}\tilde\Gamma\langle\delta\rho^2\rangle + 3\tilde\Gamma\,\frac{\tilde m}{\tilde\rho^2}\langle\delta m^2\rangle - 3\tilde\Gamma\,\frac{\tilde m^2}{\tilde\rho^3}\langle\delta\rho\,\delta m\rangle$$

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Leading order: $\langle F(\rho,m)\rangle \simeq \alpha\tilde{m} + 3\Gamma\frac{\tilde{m}}{\rho^2}\underbrace{\langle \delta m^2\rangle}_{=?}$

Linearized dynamics

$$\partial_t \delta\rho = \partial_x\left[D\partial_x\delta\rho - v_0\delta m\right];\quad \partial_t\delta m = D\partial_{xx}\delta m - v_0\partial_x\delta\rho - \alpha\delta m + \sqrt{2\sigma_m}\,\xi + \mathcal{O}\left(\frac{m^2}{\rho^2}\right)$$

$\sigma_m \simeq \tilde{\rho}\,\sigma$;

ρ, m slow field, relaxing on larger length scale.

$\delta\rho, \delta m$ fast fields $\Rightarrow$ treat $\tilde{\rho}, \tilde{m}$ as constant first

Parseval Plancherel: $\langle \delta m^2(x)\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dq\, \langle \delta m_q \delta m_{-q}\rangle$ $(*)$

Fourier-Transform:

$$\partial_t\delta\rho_q = -Dq^2\delta\rho_q - iqv_0\delta m_q;\quad \partial_t\delta m_q = -Dq^2\delta m_q - iqv_0\delta\rho_q - \alpha\delta m_q + \sqrt{2\sigma\tilde{\rho}}\,\eta_q$$

$$\langle \eta_q(t)\eta_{q'}(t')\rangle = \delta_{q+q'}\,\delta(t-t')$$

Idea: use Ito formula to get $\frac{d}{dt}\langle\delta\rho_q\delta\rho_{-q}\rangle$, $\frac{d}{dt}\langle\delta m_q\delta m_{-q}\rangle$, $\frac{d}{dt}\langle\delta m_q\delta\rho_{-q}\rangle$. Solve in steady state

$\Rightarrow$ get $\langle\delta m^2(x)\rangle$ for $(*)$

Let's do it for $v_0 = 0$: $\begin{cases}\partial_t\delta\rho_q = -Dq^2\delta\rho_q\\ \partial_t\delta m_q = -(Dq^2+\alpha)\,\delta m_q + \sqrt{2\sigma\tilde{\rho}}\,\eta_q\end{cases}$

$$\frac{d}{dt}\langle \delta m_q \delta m_{-q}\rangle = -2(Dq^2+\alpha)\langle \delta m_q \delta m_{-q}\rangle + 2\sigma\tilde{\rho}$$

$$\Rightarrow \langle \delta m_q \delta m_{-q}\rangle = \frac{\sigma\tilde{\rho}}{Dq^2+\alpha}$$

$$\langle \delta m^2(x)\rangle = \frac{1}{2\pi}\frac{\sigma\tilde{\rho}}{\alpha}\underbrace{\int_{-\infty}^{+\infty} dq\,\frac{1}{1+\left(\sqrt{\frac{D}{\alpha}}q\right)^2}}_{\pi} = \frac{\sigma\tilde{\rho}}{2\pi\alpha}\sqrt{\frac{\alpha}{D}}\cdot\pi = \frac{\sigma\tilde{\rho}}{2\sqrt{D\alpha}}$$

All in all:

$$\langle F(\rho,m)\rangle \simeq \alpha\tilde{m} + 3\Gamma\frac{\tilde{m}}{\rho^2}\langle\delta m^2\rangle = \tilde{m}\left(\alpha + \frac{3\Gamma\sigma}{2\tilde{\rho}\sqrt{\alpha D}}\right)$$

Going back to the dynamics of $\tilde{m}, \tilde{\rho}$

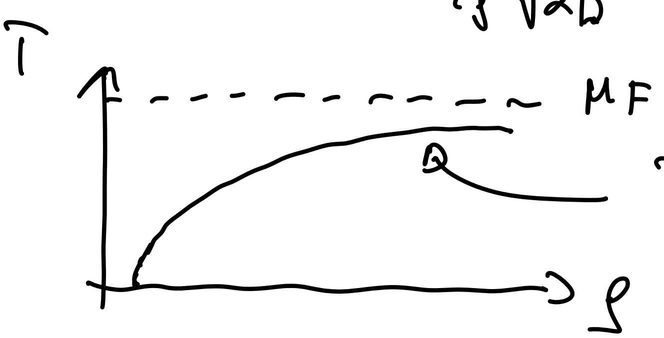
$$\partial_t\tilde{\rho} = D\partial_{xx}\tilde{\rho} - \partial_x v_0\tilde{m}$$

$$\partial_t\tilde{m} = D\partial_{xx}\tilde{m} - v_0\partial_x\tilde{\rho} - \tilde{m}\left(2\Gamma(1-\beta) + \frac{3\Gamma\sigma}{2\tilde{\rho}\sqrt{\alpha D}}\right) - \tilde{m}\,O\!\left(\frac{\tilde{m}^2}{\tilde{\rho}^2}\right)$$

What happened to the transition?

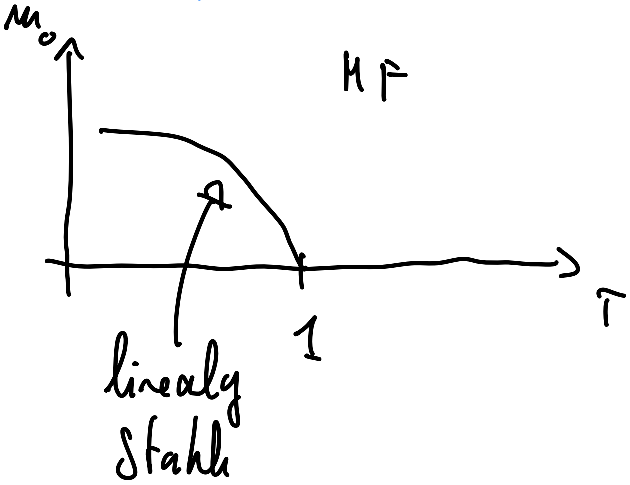
before, α changes sign at $\beta = 1$

now $\quad 2\Gamma(\beta-1) = \dfrac{3\Gamma\sigma}{2\tilde{\rho}\sqrt{\alpha D}} \Rightarrow \beta_c = 1 + \dfrac{\Omega}{\tilde{\rho}} \Rightarrow T_c^G \equiv \dfrac{1}{\beta_c} < 1$

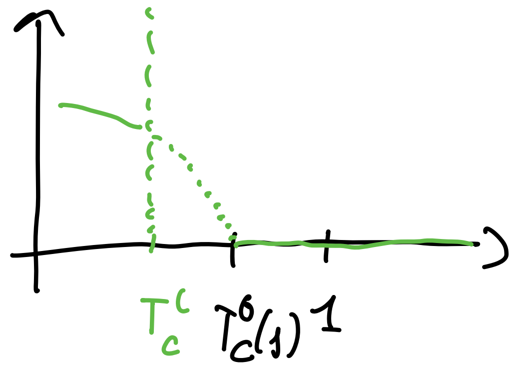


$T_c^G(\rho) < T_c^{MF}$ as in the microscopic simulations.

What happens to the ordered homogeneous solution?



$\Rightarrow$



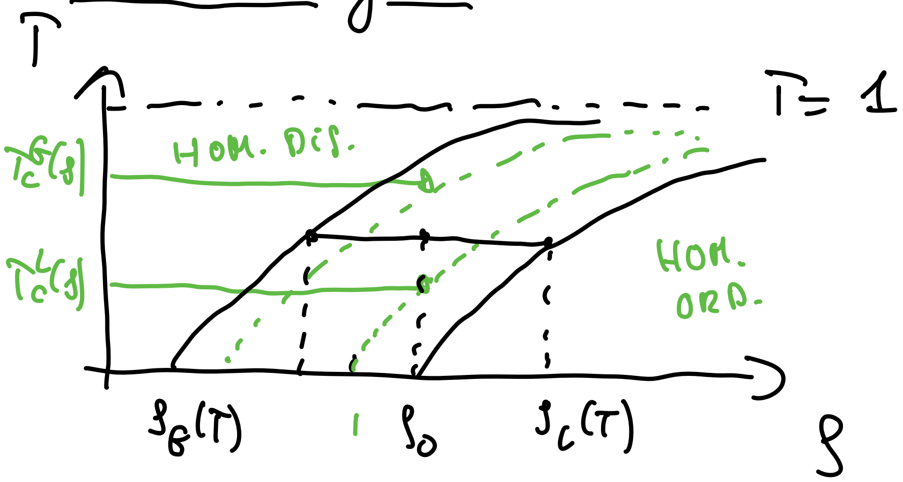
for $T_c^L(\rho) < T < T_c^G(\rho)$

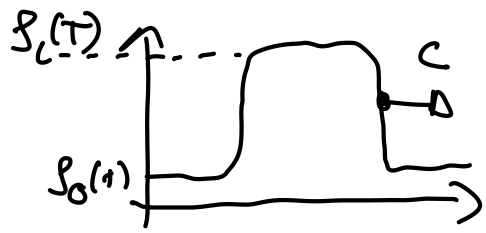
$\Rightarrow$ $\rho = \rho_0$, $m = m_0$ linearly **UNSTABLE**

↳ leads to band

for $T < T_c^L(\rho) \Rightarrow \rho = \rho_0, m = m_0$, **stable**

Phase Diagram





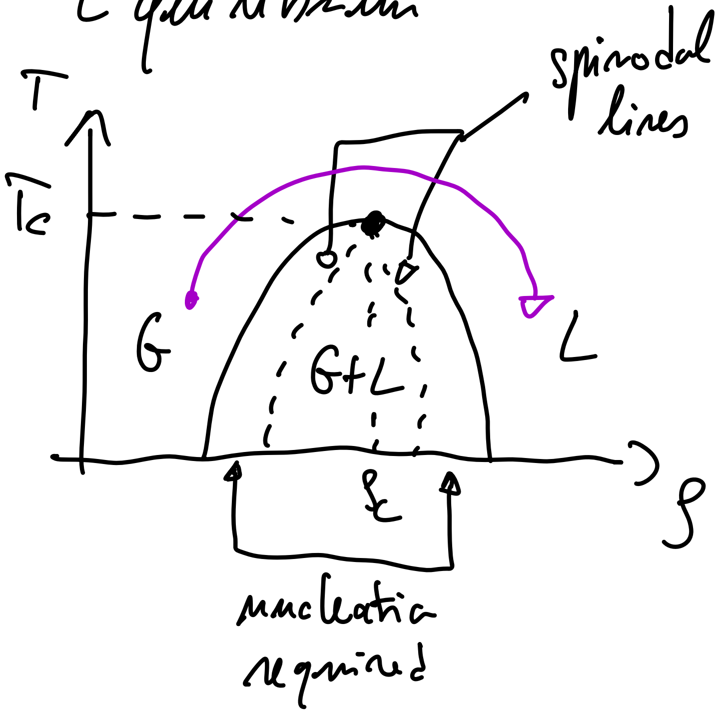
Inverting the relations $T^c_{G,L}(\rho) \Leftrightarrow \rho^c_{G,L}(\rho)$

in $[\rho_G^c, \rho_L^c]$, $\rho = \rho_0$, $m = m_0$ linearly unstable
"spinodal region"

in $[\rho_G, \rho_G^c] \cup [\rho_L^c, \rho_L]$, $\rho = \rho_0$, $m = m_0$ linearly stable but globally unstable to nucleation

$\Rightarrow$ liquid-gas phase transition!

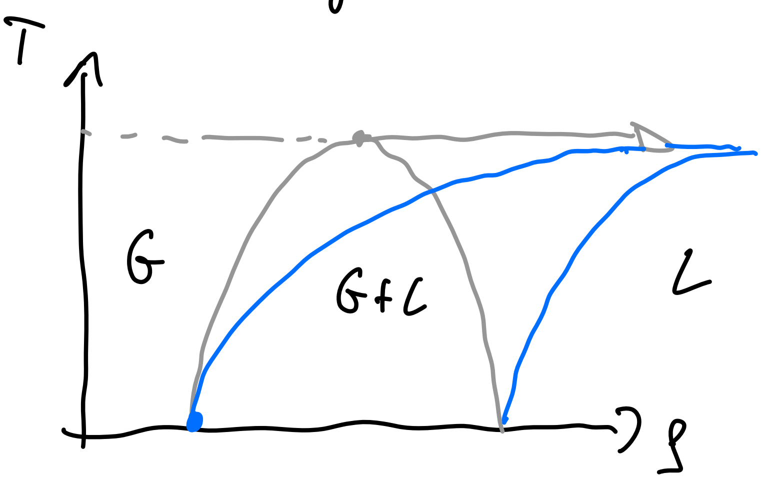
Equilibrium



Gas & liquid have the same symmetry. The system can go from G to L without crossing a transition line

Critical point at (T_c, ρ_c)

Active Ising Model



$\rho_c = +\infty, T_c = T_c^{MF} = 1$

Gas & liquid have different symmetries $\Rightarrow$ cannot go from G to L without crossing transition line.