Chapter 3: The transition to collective motion

1) Introduction

def: Ability of a large group of individual unit to move coherently over scales much larger than the typical distance between individuals.

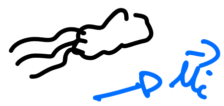
Physics: $\dot{\vec{r}}_i \to \vec{u}_i \equiv \frac{\dot{\vec{r}}_i}{\|\dot{\vec{r}}_i\|}$ & $\vec{m}(t) = \frac{1}{N}\sum_{i=1}^{N} \vec{u}_i(t)$

× collective motion ⟺ spontaneous symmetry breaking leading to

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \vec{m}(s)ds \equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{1}{N}\sum_i \vec{u}_i(t) = \vec{m}_0 \neq 0 \qquad (2)$$

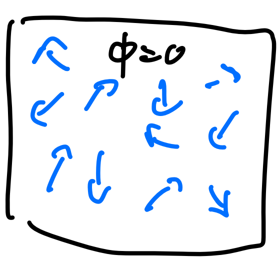
× long ranged order: $\vec{m}(\vec{r}) \equiv \sum_i \vec{u}_i \delta(\vec{r}-\vec{r}_i)$ $\quad \langle \vec{m}(\vec{r})\cdot\vec{m}(0)\rangle \sim m_0^2 \neq 0$ as $|r|\to\infty$

A generic transition: Self-propelled particles defined by their polarities $\vec{u}_i$ (physically, not as in (1))

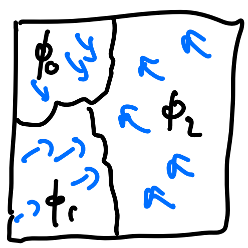


Any ordering transition of any order parameter ϕ will lead to collective motion if ϕ and $\vec{m}$ are not independent.

high T phase



Low T phase



diverse forms of "flocks": (2) corresponds to the emergence of polar flocks. Alternatively, $\vec{m}_0$ could vanish but $\langle u_i u_j - \frac{\delta_{ij}}{d}\rangle \equiv \langle q\rangle$ could be non-zero ⟹ nematic order (e.g. bacteria).

Here, focus on polar flocks. More models: [Chaté, Annual Review of Condensed matter Physics 11: 189 (2020)] 66

History:

1986: Craig Reynolds write the program "boids" for bird-oids, which is at the root of many movies

1995: [Vicsek, Czirok, Ben-Jacob, Cohen, Shochet, PRL 75, 1226, (1995)]
Microscopic simulation showing a transition to collective motion in 2D, with long-ranged order ⟹ big surprise

1995: Toner, Tu; PRL 1995: RG study of phenomenological stochastic PDE that explains why long-range order is possible.
Revisited in 1998, 2003, 2012 ⟹ no conclusive computations today

2004: Chaté, Gregoire: PRL 2004
the transition to collective motion is 1st order, with a non-uniform band phase

2006: Bertin, Droz, Gregoire, PRE 2006: first coarse-grained theory of the Vicsek model, which accounts for bands.
↓ Many more important works.

In this lectures, first discuss the general case before turning to the simpler case of the Active Ising Model:
Solon, Tailleur, PRL 2013, PRE 92, 042119 (2015)
and field theoretical description of the phase transition
Martin et al PRL 126, 148001 (2021)

The Vicsek model:

Continuous space, discrete time model in 2D.

N particles in a system of size $L_x \times L_y$ with periodic boundary conditions.

Parallel update of all particles $\vec{r}_i(t), \theta_i(t) \Rightarrow \vec{r}_i(t+1), \theta_i(t+1)$

① $\theta_i(t+1) = \langle \theta_j \rangle_{j \in \mathcal{N}_i} + \eta\, \xi_i^t$

$$\langle \theta_j \rangle_{j \in \mathcal{N}_i} = \arg\Big[\sum_{j \in \mathcal{N}_i} \vec{u}(\theta_j) \Big] \quad \text{where } j \in \mathcal{N}_i \text{ iff } |\vec{r}_i - \vec{r}_j| < r_0$$

$\xi_i^t \Rightarrow$ rdm number uniformly drawn in $[-\pi, \pi]$.

$\eta \Rightarrow$ sets the noise intensity $\Rightarrow$ like "T" in thermal equilibrium.

$(\eta = 1 \Leftrightarrow T = \infty)$

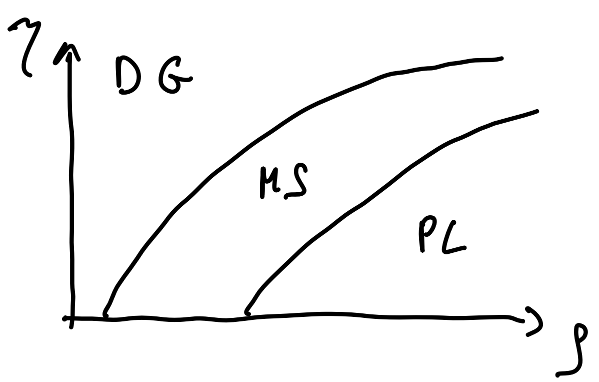
② $\vec{r}_i(t+1) = \vec{r}_i(t) + v_0\, \vec{u}(\theta_i(t+1))$

Lots of possible variations:
- add translational noise
- put the noise in $\langle \theta_j \rangle$
- use a continuous time Langevin dynamics
- etc.

Open question: what is relevant at large scale and what is not?

Phase diagram [Solon et al, PRL 114, 068101 (2015)]

Physics: $\vec{r}_i \to \vec{u}_i = \frac{\dot{\vec{r}}_i}{|\dot{\vec{r}}_i|}$



DG: . disordered gas $\vec{m}_0 = 0$

. short range correlations

$$\rho(\vec{r}) = \sum_i \delta(\vec{r} - \vec{r}_i)\ ;\ \vec{m}(\vec{r}) = \sum_i \vec{u}_i\, \delta(\vec{r} - \vec{r}_i)$$

$$\langle \vec{m}(\vec{r}) \cdot \vec{m}(0) \rangle_c \propto e^{-r/\xi}$$

finite correlation length ξ

PL: polar liquid phase

① Long-range order $\vec{m}_0 \neq 0$; $\vec{m}_0 = m_0 \vec{u}_{\parallel}$

② critical phase $\vec{m}(\vec{r}) = (m_0 + \delta m_{\parallel}) \vec{u}_{\parallel} + \delta m_{\perp} \vec{u}_{\perp}$

$\langle \delta \vec{m}_{\perp}(\vec{r}) \delta \vec{m}_{\perp}(0) \rangle \propto$ • power law decay
• anisotropic ($\vec{r} = r\vec{u}_{\parallel}$ & $\vec{r} = r\vec{u}_{\perp}$ ⇒ diff^t exponents)

Theory: Toner, PRE 86, 031918 (2012)

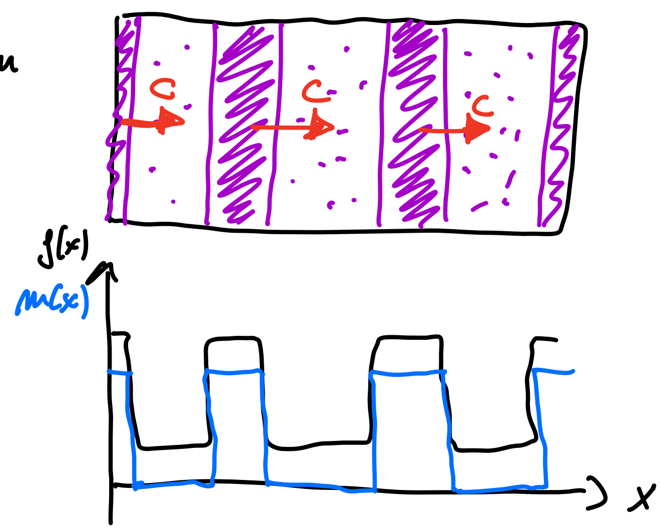
Numerics: Mahault, Ginelli, Chaté, PRL 123, 218001 (2019)

MS: Inhomogeneous, "banded" system

Physics of bands:

Caussin et al, PRL 112, 148102 (2014)

Solon et al, PRE 92, 062111 (2015)



2) The long-ranged ordered phase

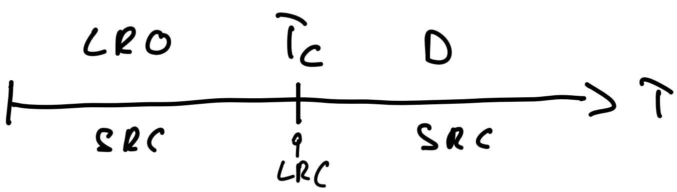
2.1) Ferromagnetic systems at equilibrium

2.1.1) The Ferromagnetic order

Ising model $S_i = \pm 1$ $d = 2, 3$

Heisenberg model $\vec{S}_i \in \mathbb{R}^3$; $|\vec{S}_i| = 1$, $d = 3$

Phase diagrams



$T > T_c$, disordered phase $\lim_{t\to\infty} \frac{1}{t}\int_0^t ds\, S_i(s) = 0$ (D)

short-range correlations $\langle \vec{S}(r)\cdot\vec{S}(0)\rangle_c \underset{r\to\infty}{\sim} e^{-r/\xi}$

$T < T_c$, ordered phase $\lim_{t\to\infty} \frac{1}{t}\int_0^t ds\, S_i(s) = m_0 \neq 0$

$$\langle \vec{S}(\vec{r})\cdot\vec{S}(\vec{0})\rangle \underset{r\to\infty}{\sim} e^{-r/\xi}$$

$T = T_c$. critical point

. diverging correlation length $\xi \sim |T-T_c|^{-\nu}$

2.1.2) The XY model

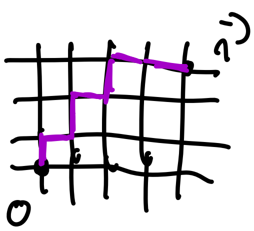
$S_i = (\cos\theta_i, \sin\theta_i)$ on a 2D lattice

$H = -J\sum_{i\sim j} \vec{S}_i\cdot\vec{S}_j \Rightarrow P_s \propto e^{+K\sum_{i\sim j}\cos(\theta_i-\theta_j)}$; $K = \frac{J}{T}$

A) High temperature phase

$K\to 0$ $\quad P_s \propto \prod_{i\sim j}\left[1 + K\cos(\theta_i-\theta_j)\right]$

$$\langle \vec{S}(0)\cdot\vec{S}(\vec{r})\rangle = \frac{1}{Z}\int \prod_i \frac{d\theta_i}{2\pi}\cos(\theta_0-\theta_r)\underbrace{\prod_{i\sim j}\left[1+K\cos(\theta_i-\theta_j)\right]}_{\text{expand}}$$



— $\equiv K\cos(\theta_i-\theta_j)$

— $\equiv 1$

$\Rightarrow$ if θ_i enters an odd # of times $\int d\theta_i \Rightarrow 0$

$\langle\ \rangle \Rightarrow$ sum over all path that connects $\vec{0}$ & $\vec{r}$

contribution of a path $\sim K^{\text{length of path}} \Rightarrow$ shortest path dominates

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$$\int \frac{d\theta_i}{2\pi} \cos(\theta_k - \theta_i)\cos(\theta_i - \theta_j) = \frac{1}{2}\int \frac{d\theta_i}{2\pi}\left[\cos(\theta_k - \theta_j) + \underbrace{\cos(\theta_k + \theta_j - 2\theta_i)}_{\int d\theta_i \to 0}\right]$$

$$= \frac{1}{2}\cos(\theta_k - \theta_j)$$

$$\langle \vec{S}(0)\cdot\vec{S}(\vec{r})\rangle \sim \left(\frac{K}{2}\right)^{|\vec{r}|} = e^{-r \ln\frac{2}{K}} = e^{-r/\xi} \; ; \; \xi = \left(\ln\frac{2}{K}\right)^{-1}$$

$\Rightarrow$ finite correlation length

B) Low temperature phase

Spin wave Hamiltonian in d dimension

$T = 0 \Rightarrow \theta_i = \theta_0 \;\; \forall i \; ; \; T = 0^+ \;\; \theta_i = \theta_0 + \delta\theta_i$

$$\cos(\theta_i - \theta_j) = \cos(\delta\theta_i - \delta\theta_j) \simeq 1 + \frac{(\delta\theta_i - \delta\theta_j)^2}{2} = 1 + \frac{(\theta_i - \theta_j)^2}{2} \to \sim a^2 \frac{\nabla\theta^2}{2}$$

$$H \simeq -\frac{\tilde{J}}{2}\int d^d\vec{r}\,[\nabla\theta]^2 \Rightarrow P[\{\theta(\vec{r})\}] \propto e^{-\frac{\tilde{K}}{2}\int d^d r\,(\vec{\nabla}\theta)^2}$$

$\tilde{J} = J a^{2-d}$ so that $\sum_{i,j} \sim \int d^d r$

Correlation function: $\langle \vec{S}(0)\cdot\vec{S}(\vec{r})\rangle = \langle \cos(\theta(\vec{0}) - \theta(\vec{r}))\rangle = \mathrm{Re}\langle e^{i(\theta(\vec{0}) - \theta(\vec{r}))}\rangle$

Random variable A $\quad \ln\langle e^{tA}\rangle = \Psi(t)$ $\quad$; $\Psi(t)$ cumulant generating function

Gaussian variable $\Rightarrow \Psi(t) = t\langle A\rangle + \frac{1}{2}t^2\langle A\rangle_c$

$$\Rightarrow \langle \vec{S}(\vec{0})\cdot\vec{S}(\vec{r})\rangle = e^{\underbrace{\langle\theta(0) - \theta(\vec{r})\rangle}_{=0 \text{ (translational invariance)}} - \frac{1}{2}\langle[\theta(0) - \theta(\vec{r})]^2\rangle_c} = e^{-\frac{1}{2}\langle(\theta(0) - \theta(r))^2\rangle}$$

$$\langle(\theta(\vec{r}) - \theta(\vec{r}'))^2\rangle = 2\langle\theta(r)^2\rangle - 2\underbrace{\langle\theta(\vec{r})\,\theta(\vec{r}')\rangle}_{= f(|\vec{r} - \vec{r}'|)} \text{ in SS}$$

$$\langle \theta(\vec{r})\,\theta(\vec{r}\,')\rangle = \int \frac{d^d\vec{q}\, d^d\vec{q}\,'}{(2\pi)^{2d}} \langle \theta_{\vec{q}}\,\theta_{\vec{q}\,'}\rangle\, e^{i\vec{q}\cdot\vec{r}+i\vec{q}\,'\cdot\vec{r}\,'}$$

Parseval - Plancherel $\Rightarrow$ $P[\{\theta_q\}] \propto e^{-\frac{\tilde{K}}{2}\int \frac{d^d\vec{q}}{(2\pi)^d} q^2 \underbrace{|\theta_q|^2}_{\theta_{\vec{q}}\theta_{-\vec{q}}}}$

$$\Rightarrow \langle \theta_{\vec{q}}\,\theta_{\vec{q}\,'}\rangle = \frac{(2\pi)^d}{q^2\tilde{K}}\,\delta_{\vec{q}+\vec{q}\,'}$$

$$\langle \theta(\vec{r})\,\theta(\vec{r}\,')\rangle = \frac{1}{\tilde{K}}\int \frac{d^d\vec{q}}{(2\pi)^d}\,\frac{e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}}{\vec{q}^{\,2}} \equiv -\frac{1}{\tilde{K}}\,C(\vec{r}-\vec{r}\,')$$

Direct algebra shows that $\nabla^2 C(\vec{r}) = \int \frac{d^d\vec{q}}{(2\pi)^d}\, e^{i\vec{q}\cdot\vec{r}} = \delta(\vec{r})$

Gauss Theorem: $\int_{\text{in } Sph(\vec{r})} d^d\vec{r}\; \vec{\nabla}\cdot(\vec{\nabla}C) = 1 = \oint_{\text{on } Sph} \vec{\nabla}C\cdot d\vec{S}$

isotropic solution: $\oint d\vec{S}\cdot\vec{\nabla}C = S_d\cdot r^{d-1}\frac{dC}{dr}$; $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$

$$\Rightarrow \frac{dC}{dr} = \frac{1}{S_d\, r^{d-1}} = \frac{r^{1-d}}{S_d}$$

$d>2$ $\quad C(r) = C(0) + \frac{r^{2-d}}{S_d(2-d)} \xrightarrow[r\to\infty]{} C(0)$ finite correlation at ∞

$\Rightarrow$ long-range order (like Ising, Heisenberg)

$d=2$ $\quad C(r) = C_0 + \frac{1}{2\pi}\ln r$

$$\Rightarrow \langle \theta(\vec{r})\,\theta(\vec{r}\,')\rangle = \frac{C_0}{\tilde{K}} - \frac{1}{2\pi\tilde{K}}\ln|\vec{r}-\vec{r}\,'|$$

$$\langle [\mathcal{O}(0)-\mathcal{O}(\vec r)]^2\rangle = \underbrace{2\langle \mathcal{O}(0)^2\rangle - 2\frac{C_0}{\tilde K}}_{b} + \frac{1}{\pi\tilde K}\ln|\vec r|$$

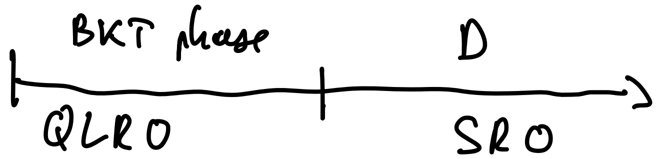
$\vec r \to \vec 0 \Rightarrow$ has to vanish; lattice cut-off a $\quad b = -\frac{1}{\pi \tilde K}\ln a$

$$\langle [\mathcal{O}(0)-\mathcal{O}(\vec r)]^2\rangle = \frac{1}{\pi\tilde K}\ln\frac{r}{a}$$

Finally: $\langle \vec S(0)\cdot\vec S(\vec r)\rangle = e^{-\frac{1}{2\pi\tilde K}\ln\frac{r}{a}} \simeq \left(\frac{a}{r}\right)^{\frac{1}{2\pi\tilde K}}$

$\Rightarrow$ power-law decay $\Rightarrow$ critical phase

Phase diagram:



$\hookrightarrow \langle \vec S(0)\cdot\vec S(R)\rangle \propto \left(\frac{a}{|R|}\right)^{\frac{1}{2\pi K_{eff}}}$

$$K_{eff} \overset{RG}{=} \frac{2}{\pi} + \frac{4b}{\pi^2}\sqrt{T_c - T}$$

high-T: $\xi \sim a\, e^{\frac{\pi^2}{8b\sqrt{T-T_c}}} \Rightarrow$ essential singularity.

Conclusion: Static birds cannot align!

But they can if they move! $\Rightarrow$ Why?

2.2) The Toner-Tu phase

Argument adapted from Toner's lecture at 2018 Les Houches school on active matter, arxiv: 1812.00310

2.2.1) Mermin wagner theorem

No LRO for diffusive spins.

Alignment rule $\theta_i(t+1) = \langle \theta_j \rangle_n + \eta$

$$\langle \theta_j \rangle_n \simeq \frac{1}{2d} \sum_{j \sim i} \theta_j \quad \text{when } \theta_j = \theta_0 + \delta\theta_j \ \forall j$$

$$\theta_i(t+1) - \theta_i(t) = \frac{1}{2d} \underbrace{\left[\sum_{j \sim i} \theta_j - \theta_i \cdot 2d \right]}_{\Delta_d \theta_i} + \eta$$

$\partial_t \theta(\vec{r}) \simeq D \Delta_d \theta + \eta$; same for $\delta\theta(\vec{r})$

If $\delta\theta(\vec{r}) = \delta\theta_0 \, \delta(\vec{r} - \vec{r}_0) \Rightarrow \delta\theta(\vec{r}, t) = \frac{\delta\theta_0}{(4\pi D t)^{d/2}} e^{-\frac{(\vec{r} - \vec{r}_0)^2}{4Dt}}$

1. $\delta\theta(\vec{r})$ spreads slowly. In a time t, average over $N(t)$ spins where $N(t) \sim (\sqrt{t})^d = t^{d/2}$
2. Maximum decreases as $\frac{1}{t^{d/2}}$
3. During this time, each of the $N(t)$ spins makes $O(1)$ new errors because of η

$\Rightarrow$ Total # errors $\sim t \, N(t)$

uncorrelated $\Rightarrow$ typical scale $\sqrt{t N(t)}$ for total magnetisation

error per spin $\delta\theta \sim \frac{\sqrt{t N(t)}}{N(t)} = \sqrt{\frac{t}{N(t)}} \sim t^{\frac{1}{2} - \frac{d}{4}}$ (*)

If $d > 2$ $\delta\theta(t) \to 0 \Rightarrow$ error relax to $0 \Rightarrow$ LRO

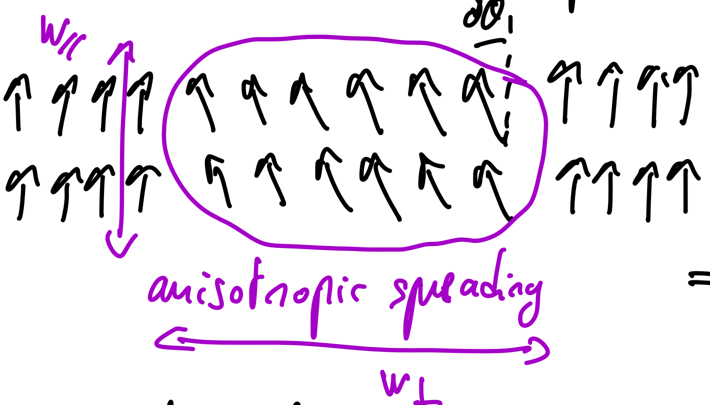
$d = 2$ $t^0 \sim \ln t$, as usual $\Rightarrow$ no LRO

This is because $N(t)$ too small!

2.2.2) Why do we point better when we walk?

$\vec{S}(\vec{n})$ is now the speed $\quad \vec{v}(n) = \vec{v}_0 + \delta\vec{v} \qquad \vec{v}_0 = v_0\,\vec{u}_\parallel$

$\theta = \theta_0 + \delta\theta$



$$\vec{u}(\theta_0+\delta\theta) \simeq \vec{u}(\theta_0) + \underbrace{\partial_{\theta_0}\vec{u}(\theta_0)}_{\vec{u}_\perp(\theta_0)}\cdot\delta\theta + \frac{1}{2}\underbrace{\partial^2_{\theta_0}\vec{u}(\theta_0)}_{-\vec{u}(\theta_0)}\,\delta\theta^2$$

$$\simeq \vec{u}(\theta_0)(1-\delta\theta^2/2) + \vec{u}_\perp(\theta_0)\,\delta\theta$$

$$\Rightarrow \delta\vec{v} = \delta v_\parallel\,\vec{u}_\parallel + \delta v_\perp\,\vec{u}_\perp$$

$$\delta v_\parallel \sim \delta\theta^2 \,;\; \delta v_\perp \sim \delta\theta$$

How does the error spread? Two possible mechanisms:

① diffusive scaling as before

② ballistic spreading due to δv: after a time t $\delta r \sim \delta v . t$

Self consistent assumption: since $\delta\theta^2 \ll \delta\theta$, we assume that $\delta v_\parallel . t \ll \sqrt{t}$

$\Rightarrow w_\parallel \sim t^{1/2}$

What about $w_\perp$?

① assume diffusive scaling of $w_\perp$: as before, $(*) \Rightarrow \delta\theta \sim t^{1/2 - d/4}$. In turns, advection leads to $\delta w_\perp^{adv} = v_0\,\delta\theta\, t \sim t^{3/2 - d/4}$

$$\frac{\delta w_\perp^{adv}}{\delta w_\perp^{diffusive}} \sim \frac{t^{3/2-d/4}}{t^{1/2}} \sim t^{1-\frac{d}{4}}$$ if $d<4$, advection faster than diffusion.

$\Rightarrow w_\perp$ given by advective spreading $\Rightarrow \delta\theta$ unknown $\Rightarrow$ self consistency

② Again, after a time t, error spread over $N(t)$ spins $\simeq w_\parallel . w_\perp \sim t^{1/2}\, w_\perp(t)^{d-1}$

$$\Rightarrow \text{Typical } \delta\theta \sim \sqrt{\frac{t}{N(t)}} \sim \frac{t^{1/2}}{t^{1/4}\, w_\perp(t)^{\frac{d-1}{2}}} \simeq t^{1/4}\, w_\perp(t)^{\frac{1-d}{2}}$$

but $w_\perp(t) \sim v_0\,\delta\theta . t \Rightarrow \delta\theta \sim t^{1/4}\,\delta\theta^{\frac{1-d}{2}} . t^{\frac{1-d}{2}}$

$$\Longleftrightarrow \delta\theta^{\frac{1+d}{2}} \sim t^{\frac{3-2d}{4}} \Longleftrightarrow \delta\theta \sim t^{\frac{3-2d}{2+2d}} \xrightarrow[d=2]{} 0$$

The advection in transverse direction make the ever needs!

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From there:

dynamical exponent z

time t for the information to spread over w along $\vec{e}_\perp$

$t(w) \sim w^z$; $z = \frac{2d+2}{5}$

$w \sim t\,\delta\theta \sim t^{\frac{3-2d+2+2d}{2+2d}}$

$\Rightarrow t \sim w^{\frac{2+2d}{5}}$

Asymmetry exponent

When the correlation along $\vec{e}_\perp$ reach a size w, along $\vec{e}_\parallel$ they scale as $w_\parallel = w^\chi$.

diffusif scaling $\Rightarrow w_\parallel = \sqrt{t(w)} \Rightarrow \chi = \frac{z}{2} = \frac{d+1}{5}$

Comment: These exponents are the same as those predicted by John Toner using RG.

Giant number fluctuations: Take a box with $\langle n \rangle$ particles

Q: how does $\langle n^2 \rangle_c$ scale?

Gaussian scaling (SRC) $\Rightarrow \langle n^2 \rangle_c \sim \langle n \rangle^1$

TT : long-range correlation $\Rightarrow \langle n^2 \rangle_c \sim \langle n \rangle^\alpha$; $\alpha = 1.6$

⚠ Giant = correlated over large scales not of giant amplitudes :-)

Numerics: B. Mahault $\chi \simeq 0.95$ TT $(z = 0.60)$

$z \simeq 1.33$ TT $(z = 1.20)$

$\alpha \simeq 1.67$ TT $(\alpha \simeq 1.60)$

$\Rightarrow$ still some mystery!

But at least it is clear why moving helps $\Rightarrow N(t)$ grows faster with $t \Rightarrow$ better averaging of the noise.

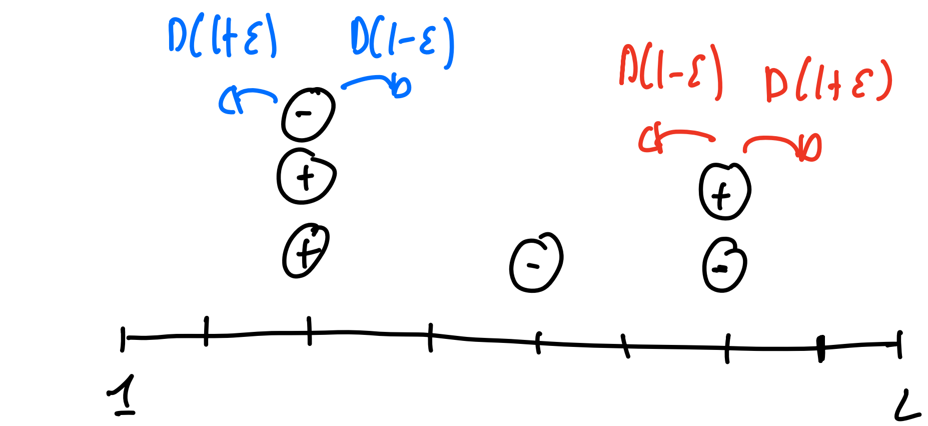
3) The transition to collective motion

15 years after the introduction of the VM $\to$ Nature of transition still unclear

$\Rightarrow$ simpler model: Heisenberg spins $\Rightarrow$ Ising spins

3.1) The active Ising Model

① Be wise discretize ② Scalar is easier



- L^d sites
- periodic boundary conditions

- Symmetric diffusion in $(d-1)$ dimensions
- Diffusion biased by spins along $\vec{e}_x$.

$\Rightarrow$ Self propulsion

- Aligning rates $s = \pm 1$; $W_f(s \to -s) = \Gamma e^{-\beta s \frac{m_i}{\rho_i}}$