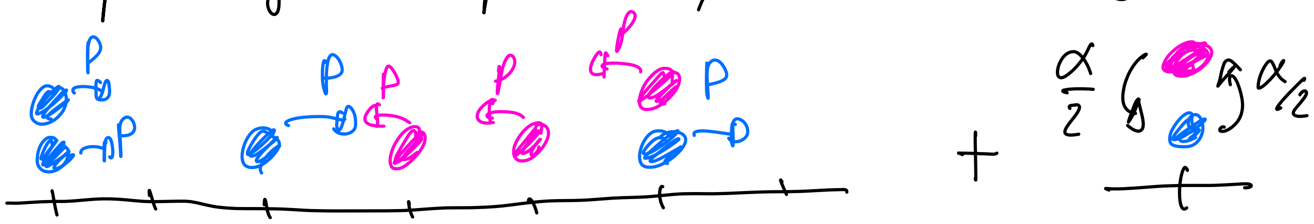
IV) MIPS on lattice

More general results on the phase-separation, surface tension and how to compute the phase diagram in off-lattice models can be found in: [Solon et al, New Journal of Physics 20, 075001 (2018)]

Here, we focus on the simpler case of on-lattice RTPs in 1d. (inspired by [Thompson et al, J. Stat. Mech 2011].



Lattice gas model of non-interacting run-and-tumble bacteria.

Configurations $\varphi \Leftrightarrow$ occupancies (n_i^+, n_i^-) which describe the number of particles hopping to the right or to the left on site i.

Add excluded volume interaction: partial exclusion

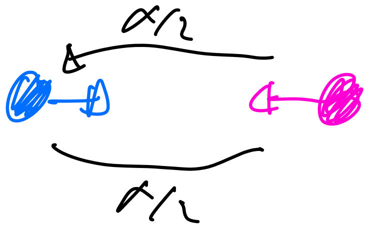
Hopping $i \to i+1$ for a right going particle $p\left(1 - \frac{n_{i+1}}{n_M}\right)$

$i \to i-1$ —— left —— $p\left(1 - \frac{n_{i-1}}{n_M}\right)$

$n_i = n_i^+ + n_i^-$ is the total occupancy.

n_M is the maximal occupancy allowed on a lattice site

Tumbling rate



Idea: Describe the dynamics of $\rho_i^{\pm} = \langle n_i^{\pm}(x)\rangle$ and show that homogeneous profiles are unstable.

Master equation: $\partial_t P(\varphi) = \sum_{\varphi' \neq \varphi} W(\varphi' \to \varphi) P(\varphi') - W(\varphi \to \varphi') P(\varphi)$

Here $\varphi = \{n_i^+, n_i^-\} = \{n_1^+, n_1^-, n_2^+, n_2^-, \dots, n_L^+, n_L^-\}$

① φ' such that $W(\varphi' \to \varphi) \neq 0$

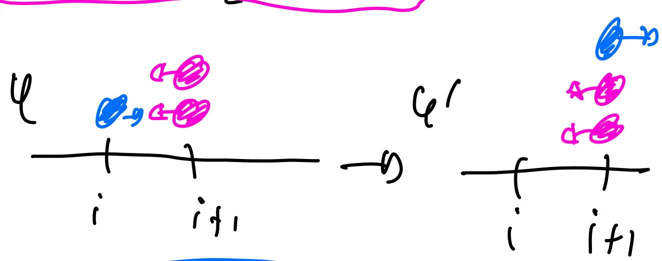
* $\varphi' = \{n_1^+, n_1^-, \dots, n_{i-1}^+ + 1, n_{i-1}^-, n_i^+ - 1, n_i^-, \dots, n_L^+, n_L^-\} \equiv \{n_{i-1}^+ + 1, n_i^+ - 1\}$

$\varphi' = n_{i-1}^+ + 1,\ n_i^+ - 1 \longrightarrow \varphi$ at rate $\boxed{W = p\,(n_{i-1}^+ + 1)\left(1 - \frac{n_i - 1}{n_M}\right)}$

and $n_{j \neq i}^{+-}$ same as in φ

* φ' as φ but with $n_{i+1}^- + 1, n_i^- - 1 \to \varphi$ at rate $\boxed{W(\varphi' \to \varphi) = p\,(n_{i+1}^- + 1)\left(1 - \frac{n_i - 1}{n_M}\right)}$
* φ' as φ but with $n_i^+ + 1, n_i^- - 1$; then $\boxed{W(\varphi' \to \varphi) = \frac{\alpha}{2}(n_i^+ + 1)}$
* φ' as φ but with $n_i^- + 1, n_i^+ - 1$; then $\boxed{W(\varphi' \to \varphi) = \frac{\alpha}{2}(n_i^- + 1)}$

② φ' such that $W(\varphi \to \varphi') \neq 0$, e.g.



* φ' as φ but with $\{n_i^+ - 1, n_{i+1}^+ + 1\}$, $\boxed{W(\varphi \to \varphi') = p\, n_i^+ \left(1 - \frac{n_{i+1}}{n_M}\right)}$
* ———— $\{n_{i+1}^- - 1, n_i^- + 1\}$, $\boxed{W(\varphi \to \varphi') = p\, n_{i+1}^- \left(1 - \frac{n_i}{n_M}\right)}$

$* \text{———} \quad \{n_i^+ - 1, n_i^- + 1\}, \quad \boxed{W(\varphi \to \varphi') = \frac{\alpha}{2} n_i^+}$

$* \text{———} \quad \{n_i^- - 1, n_i^+ + 1\}, \quad \boxed{W(\varphi \to \varphi') = \frac{\alpha}{2} n_i^-}$

Master eq:

$$\partial_t P(\{n_j^+, n_j^-\}) = \sum_i p(n_{i-1}^+ + 1)\left(1 - \frac{n_i - 1}{n_M}\right) P(\{n_{i-1}^+ + 1, n_i^+ - 1\})$$

$$+ p(n_{i+1}^- + 1)\left(1 - \frac{n_i - 1}{n_M}\right) P(\{n_{i+1}^- + 1, n_i^- - 1\})$$

$$+ \frac{\alpha}{2}(n_i^+ + 1) P(\{n_i^+ + 1, n_i^- - 1\}) + \frac{\alpha}{2}(n_i^- + 1) P(\{n_i^- + 1, n_i^+ - 1\})$$

$$-\left[p n_i^+ \left(1 - \frac{n_{i+1}}{n_M}\right) + p n_i^- \left(1 - \frac{n_{i-1}}{n_M}\right) + \frac{\alpha}{2} n_i^+ + \frac{\alpha}{2} n_i^-\right] P(\{n_i^+, n_i^-\})$$

where $P(\{n_i^\pm + 1, n_j^\mp - 1\}) = P(\varphi' = \varphi$ with $n_i^\pm$ replaced by $n_i^\pm + 1$
$n_j^\mp$ ——— $n_j^\mp - 1$

From there, use $\langle n_i^\pm \rangle = \sum_{\{n_j\}} n_i^\pm P(\{n_j\})$ to get $\partial_t \langle n_i^\pm \rangle$ from the master equation $\Rightarrow$ good luck!

Evolution of observables: $\langle O(\varphi) \rangle = \sum_\varphi O(\varphi) P(\varphi, t) = \langle O \rangle_\varphi$

$$\partial_t \langle O \rangle_\varphi = \sum_\varphi O(\varphi)\, \partial_t P(\varphi, t)$$ Master equation

$$= \sum_{\varphi, \varphi'} O(\varphi)\left[W(\varphi' \to \varphi) P(\varphi') - W(\varphi \to \varphi') P(\varphi)\right]$$

$\varphi \Leftrightarrow \varphi'$

$$= \sum_{\varphi, \varphi'} O(\varphi') W(\varphi \to \varphi') P(\varphi) - O(\varphi) W(\varphi \to \varphi') P(\varphi)$$

$$= \sum_{\varphi} \left[\sum_{\varphi'} \left(\mathcal{O}(\varphi') - \mathcal{O}(\varphi) \right) W(\varphi \to \varphi') \right] P(\varphi)$$

$$\boxed{\partial_t \langle \mathcal{O} \rangle_\varphi = \Big\langle \sum_{\varphi'} \underbrace{\left(\mathcal{O}(\varphi') - \mathcal{O}(\varphi) \right)}_{\Delta \mathcal{O}(\varphi \to \varphi')} \underbrace{W(\varphi \to \varphi')}_{W(\varphi \to \varphi')} \Big\rangle_\varphi}$$

$\Delta \mathcal{O}(\varphi \to \varphi')$: change in $\mathcal{O}$ from φ to φ'

$W(\varphi \to \varphi')$: rate at which this change occurs.

Dynamics of the average occupancies: $\partial_t \langle n_i^+ \rangle = ?$

Consider φ' that can be reached from $\varphi = \{ n_i^+, n_i^- \}$

and compute $n_i^+(\varphi') - n_i^+(\varphi) = \Delta n_i^+$

Moves that impact n_i^+ are: . hops into or out of site i

. tumble at site i

$$\partial_t \langle n_i^+ \rangle = \Big\langle \underbrace{(+1)}_{\Delta n_i^+} \times \underbrace{p\, n_{i-1}^+ \left(1 - \tfrac{n_i}{n_M}\right)}_{W} \Big\rangle + \Big\langle \underbrace{(-1)}_{\Delta n_i^+} \times \underbrace{p\, n_i^+ \left(1 - \tfrac{n_{i+1}}{n_M}\right)}_{W} \Big\rangle$$

$i-1 \to i$ $\qquad$ $i \to i+1$

$$+ \Big\langle \underbrace{(-1)}_{\Delta n_i^+} \times \underbrace{\tfrac{\alpha}{2} n_i^+}_{W} \Big\rangle + \Big\langle \underbrace{(+1)}_{\Delta n_i^+} \times \underbrace{\tfrac{\alpha}{2} n_i^-}_{W} \Big\rangle$$

tumble $+ \to -$ $\qquad$ tumble $- \to +$

$$\boxed{\begin{aligned} \partial_t \langle n_i^+ \rangle &= p \Big\langle n_{i-1}^+ \left(1 - \tfrac{n_i}{n_M}\right) \Big\rangle - p \Big\langle n_i^+ \left(1 - \tfrac{n_{i+1}}{n_M}\right) \Big\rangle - \tfrac{\alpha}{2} \langle n_i^+ \rangle + \tfrac{\alpha}{2} \langle n_i^- \rangle \\ \partial_t \langle n_i^- \rangle &= p \Big\langle n_{i+1}^- \left(1 - \tfrac{n_i}{n_M}\right) \Big\rangle - p \Big\langle n_i^- \left(1 - \tfrac{n_{i-1}}{n_M}\right) \Big\rangle + \tfrac{\alpha}{2} \langle n_i^+ \rangle - \tfrac{\alpha}{2} \langle n_i^- \rangle \end{aligned}}$$

Comment: These equations involve $\langle n_{i-1}^+ n_i \rangle$, $\langle n_i^+ n_{i+1} \rangle$, etc

$\Rightarrow$ not closed equations for $\langle n_i^+ \rangle$.

In practice N-points functions will involve $N+1$ point functions in their dynamics. We need approximations to get a solvable system. SS

Mean-field approximation: $\langle n_i^{\pm} n_j^{\pm} \rangle \simeq \langle n_i^{\pm} \rangle \langle n_j^{\pm} \rangle$

In general, this is quantitatively wrong, but often qualitatively right.

$\rho_i^{\pm} = \langle n_i^{\pm} \rangle$; $\rho_i = \langle n_i \rangle$; $\rho_M = n_M$

Dynamics:
$$\begin{cases} \partial_t \rho_i^+ = p\left[\rho_{i-1}^+ \left(1 - \frac{\rho_i}{\rho_M}\right) - \rho_i^+ \left(1 - \frac{\rho_{i+1}}{\rho_M}\right)\right] - \frac{\alpha}{2}\rho_i^+ + \frac{\alpha}{2}\rho_i^- & (1) \\ \partial_t \rho_i^- = p\left[\rho_{i+1}^- \left(1 - \frac{\rho_i}{\rho_M}\right) - \rho_i^- \left(1 - \frac{\rho_{i-1}}{\rho_M}\right) + \frac{\alpha}{2}\rho_i^+ - \frac{\alpha}{2}\rho_i^-\right] & (2) \end{cases}$$

Typical microscopic length $l_p = \frac{p}{\alpha}$ $\Rightarrow$ expect variations of $\rho_i^{\pm}$ on this scale. If $\frac{p}{\alpha} \gg 1$, little differences between $\rho_i^{\pm}$ and $\rho_{i+1}^{\pm}$

$\Rightarrow$ Taylor expand $\quad O_{i\pm1} \simeq O(x) \pm \frac{1}{L}\nabla O(x) + \frac{1}{2L^2}\Delta O(x)$; $x = \frac{i}{L} \in [0,1]$

Carrying out this expansion in (1) & (2) leads to $\quad \nabla = \partial_x$; $\Delta = \partial_{xx}$

$$\begin{cases} \partial_t \rho^+(x) = -\frac{p}{L}(\partial_x \rho^+)\left(1 - \frac{\rho}{\rho_M}\right) + \frac{p}{L}\rho^+ \frac{\nabla\rho}{\rho_M} + \frac{p}{2L^2}\Delta\rho^+\left(1 - \frac{\rho}{\rho_M}\right) + \frac{p\rho^+}{2L^2}\frac{\Delta\rho}{\rho_M} - \frac{\alpha}{2}\rho^+ + \frac{\alpha}{2}\rho^- & (3) \\ \partial_t \rho^-(x) = \frac{p}{L}(\partial_x \rho^-)\left(1 - \frac{\rho}{\rho_M}\right) - \frac{p}{L}\rho^- \frac{\partial_x\rho}{\rho_M} + \frac{p}{2L^2}(\partial_{xx}\rho^-)\left(1 - \frac{\rho}{\rho_M}\right) + \frac{p}{2L^2}\rho^- \frac{\partial_{xx}\rho}{\rho_M} + \frac{\alpha}{2}\rho^+ - \frac{\alpha}{2}\rho^- & (4) \end{cases}$$

$\rho(x) = \rho^+(x) + \rho^-(x)$; $m(x) = \rho^+(x) - \rho^-(x)$

$(3)+(4) \Rightarrow \quad \partial_t \rho(x) = -\partial_x\left[\frac{p}{L} m \left(1 - \frac{\rho}{\rho_M}\right) - \frac{p}{2L^2}\partial_x \rho\right] \qquad (5)$

$(3)-(4)$
$$\partial_t m(x) = -\alpha m(x) - \partial_x\left[\frac{p}{L}\rho\left(1 - \frac{\rho}{\rho_M}\right)\right] + \frac{p}{2L^2}(\partial_{xx} m)\left(1 - \frac{\rho}{\rho_M}\right) + \frac{p}{2L^2} m \left(\frac{\partial_{xx}\rho}{\rho_M}\right) \qquad (6)$$

Analysis of (5) - (6) in the large L limit:

$L \to \infty$ (5): $\partial_t \rho = 0 \Rightarrow$ Nothing happens in a time of order 1 for a conserved field in the large size limit.

Speed up time $t = L\tau$

$$(5) \Rightarrow \frac{d\rho}{d\tau} = -\partial_x\left[p\, m(x)\left(1 - \frac{\rho(x)}{\rho_M}\right) - \frac{p}{2L}\partial_x \rho\right] \qquad (5)'$$

$$(6) \quad \dot m(x) = -\alpha L m - p\,\partial_x\left[\rho\left(1 - \frac{\rho}{\rho_M}\right)\right] + \frac{p}{2L}(\partial_{xx} m)\left(1 - \frac{\rho}{\rho_M}\right) + \frac{p}{2L} m\, \partial_{xx}\rho$$

If $\partial_{xx}\rho$ & $\partial_{xx} m$ are finite, then the last two terms are negligible.

$$\dot m(x) = \underbrace{-\alpha L m(x)}_{\text{exponential relaxation in a time scale } \tau \sim \frac{1}{\alpha L}} \underbrace{- p\,\partial_x\left[\rho\left(1 - \frac{\rho}{\rho_M}\right)\right]}_{\text{external source}}$$

$$\Rightarrow \quad m(x) = -\frac{p}{\alpha L}\partial_x\left[\rho\left(1 - \frac{\rho}{\rho_M}\right)\right]$$

for $\tau \sim O(1)$

Injecting $m(x)$ back into (5)' then leads to

$$\dot\rho = \partial_x\left[\frac{p^2}{\alpha L}\left(1 - \frac{\rho(x)}{\rho_M}\right)\partial_x\left[\rho\left(1 - \frac{\rho}{\rho_M}\right)\right] + \frac{p}{2L}\partial_x \rho\right]$$

Again $\dot\rho \sim 0$ for $\tau \sim O(1) \Rightarrow \tau = L\tilde\tau \Rightarrow t = L^2\tilde\tau$ (diffusive scaling)

$$\frac{d\rho(x)}{d\tilde\tau} = \partial_x\left[\left(\frac{p}{2} + \frac{p^2}{\alpha}\left(1 - \frac{\rho}{\rho_M}\right)\left(1 - \frac{2\rho}{\rho_M}\right)\right)\partial_x \rho\right]$$

$$\equiv \partial_x\left[D_{\text{eff}}(\rho(x))\,\partial_x(\rho)\right]$$

At the macroscopic $x = \frac{i}{L}$; $\tau = \frac{t}{L^2}$; we obtain an effective dynamics for the density field which is a non-linear diffusion equation.

If $D_{eff}(\rho_0) > 0 \Rightarrow$ small fluctuations around ρ_0 relax to 0.

— $D_{eff}(\rho_0) < 0 \Rightarrow$ ______ get amplified

$$\left(\dot{\rho} = D \Delta \rho \xrightarrow{t \to -t} \dot{\rho} = (-D) \Delta \rho \right)$$

$\Rightarrow$ linear instability

$$D_{eff} < 0 \Leftrightarrow \left(\frac{2\rho}{\rho_m} - 1\right)\left(1 - \frac{\rho}{\rho_m}\right) \geq \frac{\alpha}{2p} = \frac{1}{2\ell_p}$$ $\to$ persistence length

$\ell_p \gg 1 \Rightarrow$ instability for $\rho \geq \frac{\rho_m}{2}$

Comments: hopping rate $\Leftrightarrow v(\rho) = p\left(1 - \frac{\rho}{\rho_m}\right)$

$$\frac{v'}{v} \leq -\frac{1}{\rho} \Longleftrightarrow \rho \geq \frac{\rho_m}{2}$$

the result of our computations is consistent with the hand-waving argument.

What remains is to characterize the phase coexistence emerging from this instability $\Rightarrow$ Derive the noise terms & look for effective free energy.

IV Population dynamics & pattern formation

Time scales:

① R&T dynamics $\sim 1\,s$

② diffusive approximation ~ 1 min

③ bacterial division ~ 1 hour (20' for e.coli in good conditions)

To describe experiments on time scale of days:

→ approximate ① by ②

→ model ③

IV.1) The logistic growth

Model: bacteria divide at rate $\tilde{b}(\rho) \simeq \tilde{b}_0 \underbrace{- \tilde{b}_1 \rho}_{\text{competition [for food]}} + O(\rho^2)$

bacteria "die" at rate $d(\rho) \simeq d_0 + d_1 \rho$

Comment: death is a very complicated process for bacteria: stop growing, stop running, cell lysis ⇒ many different time-scales.

Here ⇒ assume $\rho(\vec{r},t)$ track "healty" bacteria

Mean-field dynamics: $\frac{d}{dt}\rho = \nabla[\dots] + \rho b(\rho) - \rho d(\rho)$; $b = \tilde{b} \ln 2$ $\rho(t) = 2^{\tilde{b} t} \rho_0 = e^{\tilde{b} \ln 2\, t} \rho_0$

$$= \nabla[\dots] + \underbrace{\mu \rho \left(1 - \frac{\rho}{\rho_0}\right)}_{\text{logistic term}}$$

with $\mu = b_0 - d_0$ and $\rho_0 = \frac{\mu}{b_1 + d_1}$

homogeneous dynamics: $\partial_t \rho = \mu \rho \left(1 - \frac{\rho}{\rho_0}\right)$

* two fixed points $\rho = 0$ & $\rho = \rho_0$ ⇒ stability?

$\rho = 0 + \delta\rho \Rightarrow \frac{d}{dt}\delta\rho = \mu\,\delta\rho \Rightarrow \delta\rho \sim e^{\mu t} \Rightarrow \rho = 0$ unstable

$\rho = \rho_0 + \delta\rho \Rightarrow \frac{d}{dt}\delta\rho = -\mu\,\delta\rho \Rightarrow \delta\rho \sim e^{-\mu t} \Rightarrow \rho = \rho_0$ stable

* How does $\rho(t)$ relax towards $\rho = \rho_0$?

$\Rightarrow$ Solve through separation of variables $\quad \dfrac{d\rho}{\rho\left(1 - \frac{\rho}{\rho_0}\right)} = \mu\, dt$

$$\phi = \frac{\rho}{\rho_0} \Rightarrow \underbrace{\frac{d\phi}{\phi} + \frac{d\phi}{1-\phi}}_{d\ln\phi \,-\, d\ln(1-\phi)} = \mu\, dt \Rightarrow \ln\frac{\phi}{1-\phi} = \tilde{k} + \mu t$$

$$\Rightarrow \frac{\phi}{1-\phi} = k e^{\mu t} = \frac{\rho(t)}{\rho_0 - \rho(t)}$$

$$\Rightarrow \rho(t) = \frac{\rho_0 k e^{\mu t}}{1 + k e^{\mu t}}$$

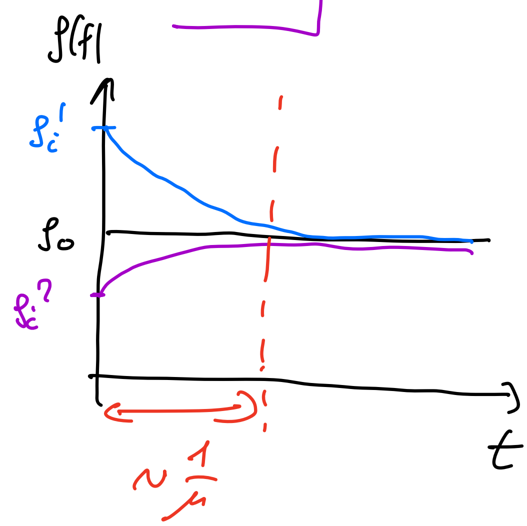
$$t = 0 \quad \rho(t) = \rho_i = \frac{\rho_0 k}{1+k} \Rightarrow k = \frac{\rho_i/\rho_0}{1 - \rho_i/\rho_0}$$

$$\Rightarrow \rho(t) = \frac{\rho_i}{1 - \frac{\rho_i}{\rho_0}} \cdot \frac{1}{e^{-\mu t} + \frac{\rho_i/\rho_0}{1 - \frac{\rho_i}{\rho_0}}} = \frac{\rho_i}{1 - \frac{\rho_i}{\rho_0}} \cdot \frac{1 - \frac{\rho_i}{\rho_0}}{\frac{\rho_i}{\rho_0} + e^{-\mu t}\left(1 - \frac{\rho_i}{\rho_0}\right)}$$

$$\rho(t) = \frac{\rho_i\, \rho_0}{\rho_i + (\rho_0 - \rho_i)\, e^{-\mu t}}$$

$t = 0 \quad \rho(t) = \rho_i$

$t = +\infty \quad \rho(t) = \rho_0$



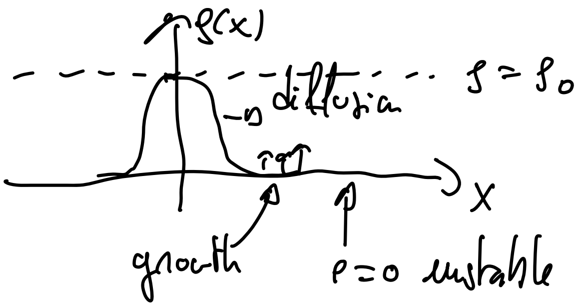
$\Rightarrow$ exponential relaxation towards $\rho(t) = \rho_0$

IV.2) Colony spreading & Fisher waves

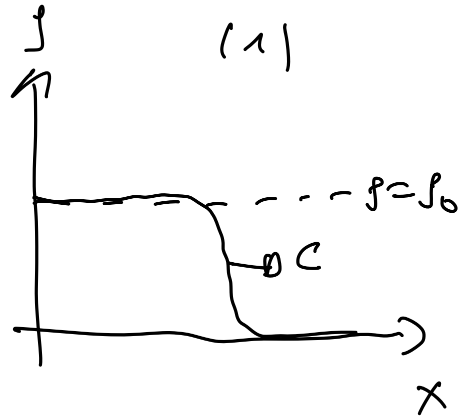
diffusion + logistic growth

$$\dot{\rho} = D\Delta\rho + \alpha\rho\left(1-\frac{\rho}{\rho_0}\right) \qquad (1)$$

Initial condition



diffusion + growth $\Rightarrow$ steady wave



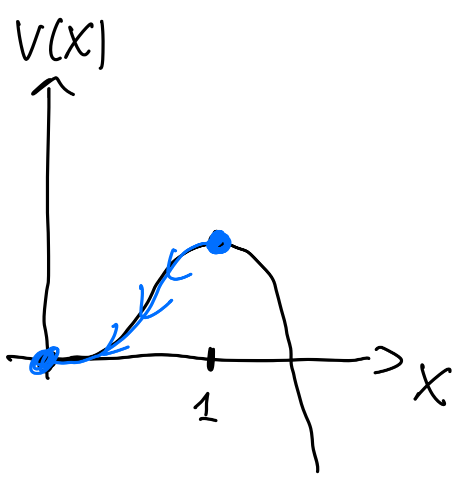
existence of travelling wave

$\frac{\rho}{\rho_0}(x,t) \equiv X(x-ct)$ for some function $X(\tau)$ when $\tau = x-ct$

(1) $\Leftrightarrow$ $D X'' + \alpha X(1-X) = -cX'$

$$\Leftrightarrow D\frac{d^2}{d\tau^2}X(\tau) = -c\frac{dX}{d\tau} - \frac{d}{dX}V(X) \; ; \; V(X) = -\alpha\frac{X^3}{3} + \alpha\frac{X^2}{2}$$

(1) $\Longleftrightarrow$



No oscillation in $X<0$ region!

(1) $\Leftrightarrow$ overdamped relaxation in the potential well

linear stab of: $\begin{cases} \dot{X} = v \\ D\dot{v} = -cv - \alpha X(1-X) \end{cases}$ close to $(0,0)$

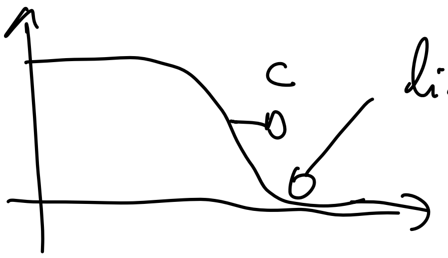
$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta v\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{\alpha}{D} & -\frac{c}{D}\end{pmatrix}\begin{pmatrix}\delta x\\ \delta v\end{pmatrix}$$

$$M = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$\lambda_\pm = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a-d)^2+4bc}$$

$$\lambda^\pm = -\frac{c}{2D} \pm \frac{1}{2}\sqrt{\frac{c^2}{D^2} - \frac{4\alpha}{D}} \Rightarrow \boxed{c^2 > 4\alpha D}$$

Selection principle: pulled wave



linearize for small ρ

$$\partial_t \rho = D\Delta\rho + \alpha\rho \Rightarrow \partial_t \hat{\rho}(q) = (\alpha - Dq^2)\,\hat{\rho}(q)$$

$$\Rightarrow \hat{\rho}(q) = \hat{\rho}_i(q)\, e^{(\alpha - Dq^2)t}$$

$$\Rightarrow \rho(\underbrace{x-ct}_{z}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dq\; \hat{\rho}_i(q)\, e^{(\alpha-Dq^2)t}\, e^{iq(\overbrace{x-ct}^{z}+ct)}$$

$$\rho(z) = \int_{-\infty}^{+\infty} \frac{dq}{2\pi}\, e^{iqz + t[\alpha - Dq^2 + iqc]}$$

large $t \Rightarrow$ saddle point $\frac{d}{dq}[\alpha - Dq^2 + iqc] = 0 \Rightarrow \boxed{c = -2iq^*D}$

$$q^* = q_r + iq_i \Rightarrow q_r = 0 \;\&\; \boxed{c = 2Dq_i}$$

Steady solution $\Rightarrow \mathrm{Re}\left[\alpha - Dq^{*2} + iq^*c\right] = 0 = \alpha + Dq_i^2 - cq_i = 0$

$$\Rightarrow \alpha = Dq_i^2 \Rightarrow q_i = \sqrt{\frac{\alpha}{D}} \Rightarrow \boxed{c = 2\sqrt{D\alpha}}$$

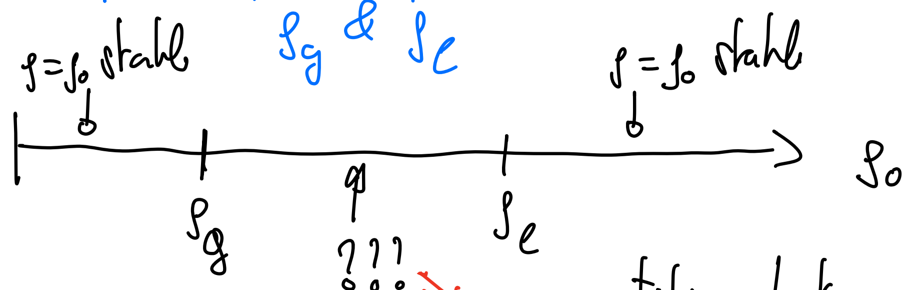
The slowest speed is selected.

IV.3 Mobility-induced pattern formation

IV.3.1 From MIPS to finite-size patterns

Long-time dynamics

$$\partial_t \rho = \underbrace{\partial_x\left\{\rho D\, \partial_x\left[\ln\rho + \ln v(\rho) + \frac{\sigma^2 v'}{v}\,\partial_{xx}\rho\right]\right\}}_{\text{promotes phase-separation between } \rho_g \,\&\, \rho_e} + \underbrace{\alpha\,\rho\left(1-\frac{\rho}{\rho_0}\right)}_{\text{promote } \rho \simeq \rho_0}$$



$\rightsquigarrow$ competition between MIPS & population dynamics

$$\rho(x) = \rho_0 + \delta\rho(x)$$

$$\Rightarrow \partial_t\,\delta\rho(x) = \partial_x\left\{\rho_0 D(\rho_0)\,\partial_x\left[\frac{\delta\rho}{\rho_0} + \frac{v'(\rho_0)}{v(\rho_0)}\,\delta\rho + \frac{\sigma^2 v'(\rho_0)}{v(\rho_0)}\,\partial_{xx}\delta\rho\right]\right\} - \alpha\,\delta\rho$$

Fourier transform $\delta\rho(x) \to \delta\rho(q)$ $\qquad \partial_x \to iq$

$$\partial_t\,\delta\rho_q(t) = -q^2\rho_0 D(\rho_0)\left[\frac{1}{\rho_0} + \frac{v'(\rho_0)}{v(\rho_0)}\right]\delta\rho_q + \frac{\sigma^2 v'(\rho_0)}{v(\rho_0)}\,q^4\,\delta\rho_q - \alpha\,\delta\rho_q$$

$$= \Lambda_q\,\delta\rho_q$$

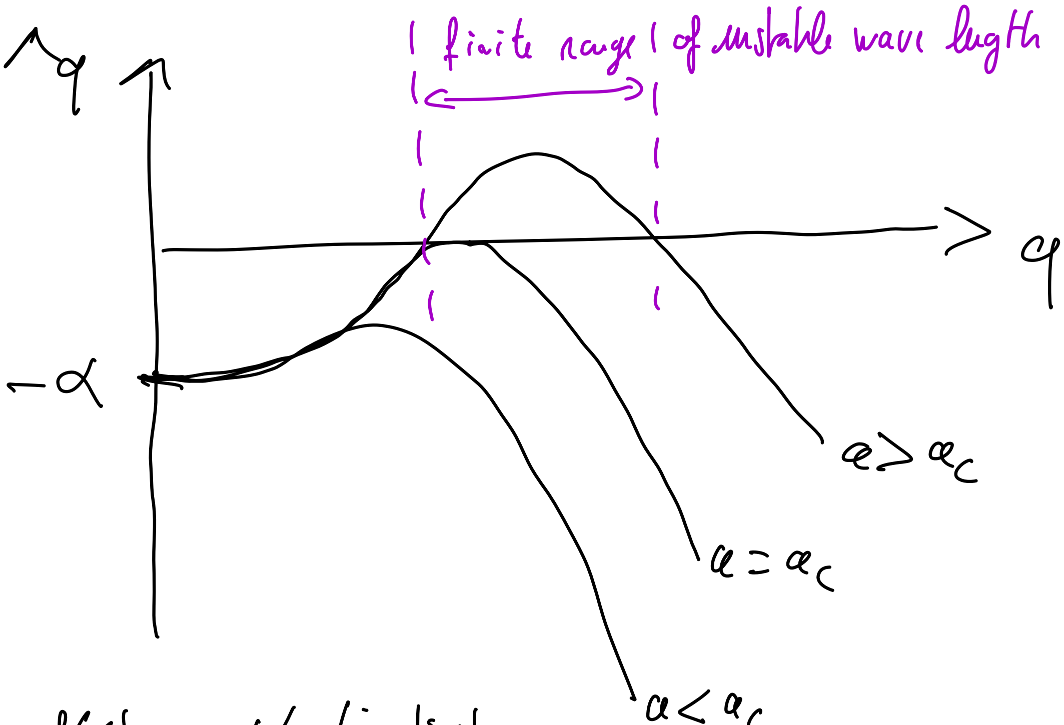
$$\Lambda_q = -\alpha - q^2\rho_0 D(\rho_0)\left[\frac{1}{\rho_0} + \frac{v'(\rho_0)}{v(\rho_0)}\right] + \sigma^2\frac{v'(\rho_0)}{v(\rho_0)}\,q^4$$

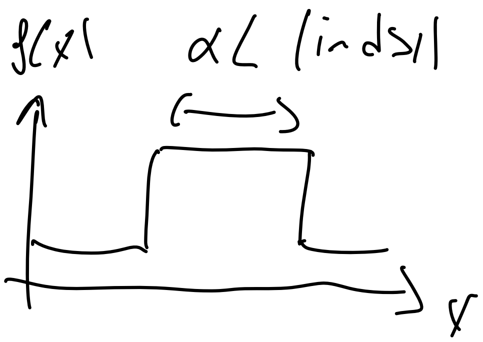
MIPS: $\dfrac{v'(\rho_0)}{v(\rho_0)} < 0 \Rightarrow \Lambda_q = -\alpha + a q^2 - b q^4 \quad ; a,b>0$

$$\Lambda_q = 0 \iff a^2 - 4\alpha b > 0$$

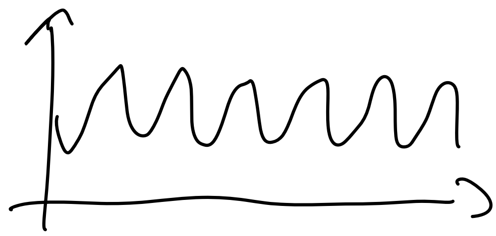
$$a_c = 2\sqrt{\alpha b}$$

finite range of unstable wave length ⇒ patterns of fixed size



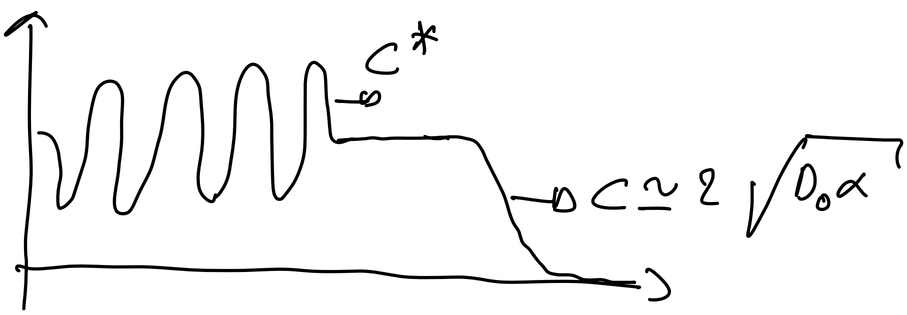


MIPS

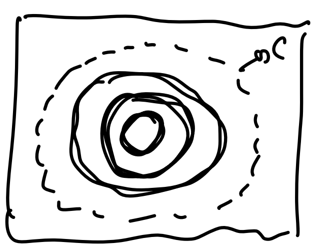


MIPF

(IV.3.2) Colony growth & MIPF



$D_0 = D(\rho \simeq 0)$



c^* can also be computed and can be larger or smaller than c.

This can be extended to active mixtures and complex ecosystems.

Refs: Cates et al, PNAS 107, 11715 - 11720, (2010)

Liu et al, Science 334, 238 (2011)

Curatolo et al, Nat. Phys. 16, 1152 (2020)