What did we learn in 8.08/8.S308?

This is an attempt at summarising the main results, concepts and methods that we encountered in 8.08/8.S308. I strongly encourage you to build your own summary of the class! This is **not** a list of things you should know by heart; I've tried to cover all the results that we derived that correspond to the core of the class. There are other interesting things (e.g. the Carnot cycle) that I'd be happy for you to remember, but that are further away from the core topics, and thus not listed here.

When you look at this list, it's probably a good idea to know items that are single-line results (e.g. what is an overdamped Langevin equation? What is the corresponding Fokker-Planck equation?) and to be comfortable redoing the derivations of more complex results. Some of the things that we did are really hard and I'm of course not expecting you to be able to rederive these results without guidance.

1 Concepts & Models

Here you can ask yourself: am I familiar with these notions? Do I know how to write the corresponding equations?

- Fluctuating forces and stochastic equations
- Underdamped and overdamped Langevin equation
- White noise and colored noise
- Markovian and non-Markovian dynamics
- The mobility of a colloid. Its relation to friction.
- Fokker-Planck equation and Fokker-Planck operator
- Probability current
- Steady-state distribution (and its requirements, boundary conditions, normalizability)
- The Boltzmann weight
- The Fokker-Planck equation for combination of stochastic & deterministic variables and its application to underdamped Brownian motion: The Kramers equation.
- The notion of time-reversal symmetry in the steady state. The notion of an arrow of time (e.g. page 5, lecture 5)
- The notion of detailed balance
- Two-point correlation functions C_{BA}
- Response functions R_{BA}
- Defining heat and work starting from an underdamped dynamics
- The notion of escape rate
- Run-and-tumble bacteria/particles. How we model their dynamics and what their master equation is.
- Active Brownian particles. What kind of system they model. How we model their dynamics in two dimensions. What their Master equation is.

- Persistence length & persistence time. How they are related to model parameters.

2 Methods

Here you can ask yourself: do I know how to use this?

- Rescaling time to go from an underdamped to an overdamped Langevin equation in the presence of white noise (basic). To derive the diffusive approximation to run-and-tumble dynamics (advanced).
- Formal solution of a Langevin equation
- How to time discretize a Langevin equation
- Ito Formula in all possible forms & how to use it
- Using Itō formula in the presence of a combination of deterministic and stochastic equations
- What is the probability of a realization $\eta(t)$ of a Gaussian white noise?
- How to get the probability of a trajectory of a Langevin equation starting from that of $\eta(t)$ (hard)
- How to derive a Fokker-Planck equation starting from $P(x) =< \delta(x - x_0) >$ using Itō formula
- How to compute a steady-state distribution from a Master/Fokker-Planck equation
- How to compute the steady-state distribution of an equilibrium system (the Boltzmann weight)
- How to relate the time relaxation of a probability distribution to the spectrum of the evolution operator.
- How to use the bra-ket notations, in particular to express propagators.
- How detailed balance is encoded in relation between the evolution operator (H_{FP} or the Markov matrix) and the steady-state distribution P_S
- How detailed balance is encoded for Markov processes as a relation between transition rates and steady-state distribuion
- How one studies the fluctuations of energy starting from the Langevin equation.
- How to use path-integral formalism to compare the probability of forward and backward trajectories (hard)
- How to write Master equations for Langevin dynamics + jump processes (e.g. molecular motors, run-and-tumble bacteria)
- How to write Master equation for Markov chains (discrete states, continuous time, like for molecular motors or MIPS on lattice)
- How to write the evolution equation for the average of an observable starting from the master equation (as for <i> in the chapter on molecular motors)
- How to write the evolution equation for an average observable directly, as we did for the average occupancies for MIPS on lattice.
- How to build the path probability of a markov process (hard)

3 Results

Here you can ask yourself: do I know this? Where it is coming from? What is the way to derive this result?

- Interactions between tracer and baths can be split between an average force and fluctuations
- For an equilibrium bath the friction kernel and the noise temporal correlations are related through an Einstein relation
- General expression of a Langevin equation with inertia and memory
- The characteristic function of a Gaussian is a Gaussian
- A linear combination of Gaussian random variables is a Gaussian random variables
- The scaling of the noise term $d\eta$ with dt
- Expression of Fokker-Planck equation for additive and multiplicative noise, in the Ito discretization, in all dimensions.
- Time-reversal symmetry implies that $C_{BA}(t) = C_{AB}(t)$
- The fluctuation dissipation theorem $R_{BA}(t) = -\frac{1}{k_B T}\frac{d}{dt}C_{BA}(t)$
- Steady 'ratchet' currents require breaking TRS and parity symmetry
- There is no steady-state spatial current in an equilibrium system
- For Markov processes and overdamped Langevin equation, detailed balance implies the existence of a basis where the Markov matrix and H_{FP} become Hermitian
- The time until the next configuration change occurs in a Markov process is exponentially distributed.
- The probability to go from configuration C to configuration C' given that a configuration change is happening is $W(\mathcal{C} \to \mathcal{C}')/r(\mathcal{C})$.
- How the mean-square displacement of an active particle looks like. What the large-scale effective diffusion an active particle is.
- The steady-state distribution of active particles generically differs from that of passive ones. Typical example of accumulation of active particles at boundaries.
- Existence of an equilibrium limit when $\tau \to 0$, $\ell_p \to \infty$, keeping $D_{\text{eff}} = \ell_p^2/[d\tau]$ constant.
- Active particles undergo a motility-induced phase separation when their motility decreases sufficiently rapidly as their density increases.
- What the steady-state of an active particle with non-uniform self-propulsion speed v(r) is.
- MIPS arises from a feedback loop between the tendancy of active particles to accumulate where they go slower and interactions that make they go slower at high density.
- How to build the evolution of $\langle O(\mathcal{C}\rangle$ out of the master equation
- How to compute the transition rate between configurations by summing individual transitions that lead to the same effect (e.g. in the evolution of the distance between two motors, in MIPS on lattice)