Recitation 6: Manipulating Ito SDEs

Consider an Ito SDE:

$$\dot{x} = \mu(x) + \sigma(x)\,\eta(t), \qquad \langle \eta(t)\,\eta(t')\rangle = \delta(t-t')$$

"Multiplying by dt", we get the math notation

$$dx = \mu(x)\,dt + \sigma(x)\,dW \qquad (*)$$

The above is intended to denote the corresponding integral. We can develop heuristic rules for manipulating infinitesimals like the above. Here are the rules:

Rules for eqns like $(*)$: (1) Set $dW^2 = dt$

(2) Neglect terms of order higher that dt.

e.g. $dt^2 = 0$, $dt\,dW = 0$ $\left(O(t^{3/2})\right)$

What is meant by the above is that, for an adapted process $\{M(t)\}$, the following hold given a partitioning of $[0,t]$ into $0<t_1<\cdots<t_{n-1}<t$, with $\Delta t \equiv t/n$:

$$\lim_{n\to\infty} \sum_i M_{i-1}(\Delta W_i)^2 = \int_0^t ds\, M(s) \qquad (1)$$

$$\lim_{n\to\infty} \sum_i M_{i-1}\,\Delta t^2 = 0, \qquad \lim_{n\to\infty} \sum_i M_{i-1}\, dW^n = 0 \quad \text{for } n>2, \qquad \text{etc}$$

Let's check (1). Define $I_n = \sum_{i=1}^n \underbrace{M_{i-1}\,(\Delta W_i^2 - \Delta t)}_{\text{indep}}$

Notice

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \sum_{i=1}^n M_{i-1}\,\Delta W_i^2 - \int_0^t ds\, M(s)$$

So to check equation 1, it suffices to show that I_n converges to zero. We will show this in the mean square sense: i.e. we will show

$$\lim_{n\to\infty} \langle I_n\rangle = 0, \qquad \lim_{n\to\infty} \langle I_n^2\rangle = 0$$

Indeed:

$$\langle I_n\rangle = \sum_i \langle M_{i-1}\rangle \langle \Delta W_i^2 - \Delta t\rangle = 0$$

$$\langle I_n^2\rangle = \left\langle \left[\sum_{i=1}^n M_{i-1}(\Delta W_i^2 - \Delta t)\right]^2\right\rangle$$

$$= \sum_{i=1}^n \langle \underbrace{M_{i-1}^2\,(\Delta W_i^2 - \Delta t)^2}_{\text{indep}}\rangle + \sum_{i>j} 2\langle \underbrace{M_{i-1}M_{j-1}(\Delta W_j^2-\Delta t)(\Delta W_i^2 - \Delta t)}_{\text{indep},\ \langle \Delta W_i^2 - \Delta t\rangle = 0}\rangle$$

$$= \sum_{i=1}^n \langle M_{i-1}^2\rangle \langle (\Delta W_i^2 - \Delta t)^2\rangle$$

Now use $\langle (\Delta W_i^2 - \Delta t)^2 \rangle = \langle \Delta W_i^4 \rangle - 2\langle \Delta W_i^2 \rangle \langle \Delta t \rangle + \Delta t^2 = 3\Delta t^2 - 2\Delta t^2 + \Delta t^2 = 2\Delta t^2$

Thus: $$\langle I_n^2 \rangle = 2 \sum_{i=1}^{n} \langle M_{i-1}^2 \rangle \frac{t^2}{n^2} \xrightarrow{n\to\infty} 0 \qquad \blacksquare$$

The other rules can be proven similarly.

Using these rules, we can manipulate infinitesimals like dW and dt and Taylor expand with respect to them, knowing when to terminate the series.

Example: Ito's lemma

Let $$dx = \mu(x)\,dt + \sigma(x)\,dW$$

Consider $f(x)$ for some smooth f. Then

$$df = f'(x)\,dx + f''(x)\frac{dx^2}{2} + \frac{f'''(x)\,dx^3}{6} + \cdots$$

$$= f'(x)\,dx + \frac{f''}{2}\left(\mu(x)\,dt + \sigma(x)\,dW\right)^2 + \cdots$$

$$= f'(x)\,dx + \frac{f''}{2}\left(\mu(x)^2 dt^2 + 2\mu(x)\,dt\,dW + \sigma^2 dt\right)$$

Dividing by dt:

$$\dot{f} = f'(x)\,\dot{x} + \frac{\sigma^2}{2} f''$$

we have recovered Ito's formula.

Example: From Stratonovich to Ito

Suppose I have a stratonovich SDE $\quad dx = \mu(x)\,dt + \sigma(x) \circ dW$, (stratonovich notation)

corresponding to the integral

$$x(t) = x_0 + \int_0^t ds\, \mu(x(s)) + \int_0^t dW(s) \circ \sigma(x(s))$$

where $\int dW \circ$ denotes the stratonovich integral, defined by

$$\int_0^t dW(s) \circ \sigma(x(s)) \equiv \lim_{n\to\infty} \sum_{i=1}^{n} \sigma\left(\frac{x_i + x_{i-1}}{2}\right) \Delta W_i$$

I wish to rewrite this as an Ito SDE

$$dx = \tilde{\mu}(x)\,dt + \tilde{\sigma}(x)\,dW$$

where $\tilde{\mu}, \tilde{\sigma}$ are to be given in terms of μ, σ. Let write $x_i = x_{i+1} + \Delta x_i$ and expand the stratonovich integral w.r.t Δx_i:

$$\sum_{i=1}^{n} \sigma\left(\frac{x_i + x_{i-1}}{2}\right) \Delta W_i = \sum_{i=1}^{n} \left[\sigma(x_i) + \sigma'(x_i)\frac{\Delta x_i}{2} + \sigma''(x_i)\frac{\Delta x_i^2}{4} + \cdots\right] \Delta W_i$$

$$\sum_{i=1}^{n} \sigma\left(\frac{x_i + x_{i-1}}{2}\right) \Delta W_i = \underbrace{\sum_{i=1}^{n} \sigma(x_i) \Delta W_i}_{\text{Ito}} + \sum_{i=1}^{n} \frac{\sigma'(x_i)}{2} \left(\tilde{\mu}(x_i) \Delta t + \tilde{\sigma}(x_i) \Delta W_i\right) \Delta W_i$$

Taking $n \to \infty$, the term $\Delta t \Delta W_i \sim O(\Delta t^{3/2})$ is subleading. ΔW_i^2 becomes $dW^2 = dt$, so that $\lim_{n\to\infty} \sum_{i=1}^{n} \sigma'(x_i)\, \tilde{\sigma}(x_i)\, \Delta W_i^2 = \int_0^t ds\, \sigma'(x(s))\, \tilde{\sigma}(x(s))$

Thus

$$\int_0^t dW(s) \circ \sigma(x(s)) = \int dW(s)\, \sigma(x(s)) + \frac{1}{2} \int ds\, \sigma'(x(s))\, \tilde{\sigma}(x(s))$$

Altogether, the Ito expression for $x(t)$ is

$$x(t) = x_0 + \int_0^t ds \left[\mu(x(s)) + \frac{1}{2} \tilde{\sigma}(x(s))\, \partial_x \sigma(x(s))\right] + \int_0^t dW(s)\, \sigma(x(s))$$

Matching terms, we see $\tilde{\sigma} = \sigma$

$$\tilde{\mu} = \mu + \frac{1}{2} \sigma \partial_x \sigma .$$

In summary:

The Stratonovich SDE $\quad dx = \mu(x)\, dt + \sigma(x) \circ dW \quad$ (a)

is the same as the Ito SDE $\quad dx = \left(\mu(x) + \frac{1}{2} \sigma \sigma'(x)\right) dt + \sigma(x)\, dW . \quad$ (b)

Notice that for additive noise $\sigma'(x) = 0$, these are the same.

We may take the above as the definition of the Stratonovich SDE, and study its properties.

For example, we can prove that Stratonovich SDEs obey the standard chain rule:

Let $f(x)$ be some smooth function, with x obeying (a). Apply Ito's lemma to (b):

$$\dot{f} = f' \dot{x} + \frac{1}{2} \sigma^2 f''$$

$$\begin{aligned} df &= f' \mu\, dt + \frac{1}{2} f' \sigma \sigma'\, dt + f' \sigma\, dW + \frac{1}{2} \sigma^2 f''\, dt \\ &= \left[f' \mu + \frac{1}{2}\left(f' \sigma \sigma' + \sigma^2 f''\right)\right] dt + f' \sigma\, dW \qquad (*) \end{aligned}$$

This last result is an Ito equation. To convert to Strato, we use

Ito SDE: $\quad df = \mu_f(f)\, dt + \sigma_f(f)\, dW$

equivalent Strato: $\quad df = \left(\mu_f(f) - \frac{1}{2} \sigma_f \sigma_f'(f)\right) dt + \sigma_f(f) \circ dW$

so that $(*)$ above has the strato form

$$df = \left[f' \mu + \frac{1}{2}\left(f' \sigma \sigma' + \sigma^2 f''\right) - \frac{1}{2} f' \sigma\, \partial_f (f' \sigma)\right] dt + f' \sigma(x) \circ dW \qquad \boxed{**}$$

Notice:

$$\frac{d}{df}\left[f'(x)\, \sigma(x)\right] = \frac{d}{df}\left[f'(f^{-1}(f))\, \sigma(f^{-1}(f))\right]$$

$$= \sigma f'' \frac{df^{-1}(f)}{df} + f'\sigma' \frac{df^{-1}(f)}{df}$$

$$= \frac{1}{f'}\left(\sigma f'' + f'\sigma'\right)$$

Plugging into ** above, we get

$$df = f'\mu\, dt + f'\sigma \circ dW \Rightarrow \dot{f} = f'\dot{x}$$

Thus the stratonovich equation obeys the standard chain rule.