Recitation 5: Numerical Simulation of SDEs

Recap: A stochastic differential equation
$$\dot{x} = f(x) + \sigma(x)\,\eta(t)$$
is a shorthand for the integral equation
$$x(t) = x_0 + \int_0^t ds\, f(x(s)) + \int_0^t dW(s)\, \sigma(x(s))$$
we defined Ito integrals and computed explicitly an example
$$\int_0^t W(s)\, dW(s) = \frac{1}{2}\left(W_t^2 - t\right)$$

1 The Stochastic Ito-Taylor expansion

(1.1) Warm up: Deterministic case

Let us first consider deterministic ODEs of the form:
$$\dot{x} = f(x)$$
We write this in integral form
$$x(t) = x(0) + \int_0^t f(x(s))\, ds \qquad (2)$$
Now note that for any function $g(x)$, we have $\dot{g}(x) = g'(x)\dot{x} = f(x)g'(x)$, so that
$$g(x(t)) = g(x(0)) + \int_0^t ds\, f(x(s))\, g'(x(s))$$
$$\equiv g(x(0)) + \int_0^t ds\, \mathcal{L} g(x(s)), \qquad \text{where } \mathcal{L} \equiv f(x)\,\partial_x$$
Let us now use this to rewrite the integral in (2) by setting $g = f$:
$$x(t) = x(0) + f(x(0)) \int_0^t ds + \int_0^t ds \int_0^s du\, \mathcal{L} f(x(u))$$
$$= x(0) + f(x(0))\, t + R_2,$$
where $R_2 \equiv \int_0^t ds \int_0^s du\, \mathcal{L} f(x(u)) \in \mathcal{O}(t^2)$. For small t, we may thus neglect R_2.
This gives the Euler scheme $x(t) = x(0) + f(x(0))t + R_2$.
To do better, we continue the expansion, setting $g = \mathcal{L} f$:
$$x(t) = x(0) + f(x(0))\, t + \mathcal{L} f(x(0)) \int_0^t ds \int_0^s du + \underbrace{\int_0^t ds \int_0^s du \int_0^u dv\, \mathcal{L}^2 g(x(v))}$$
$$\equiv x(0) + f(x(0))\, t + \frac{1}{2} \mathcal{L} f(x(0))\, t^2 + R_3.$$

We may in general work up to any order:

$$x(t) = x(0) + \sum_{m=1}^{r} \frac{t^m}{m!} \mathcal{L}^{m-1} f(x(0)) + R_{r+1}$$

where $R_{r+1} \in \mathcal{O}(t^{r+1})$ is

$$R_{r+1} = \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_r} ds_{r+1}\, \mathcal{L}^r f(x(s_{r+1}))$$

This is effectively a Taylor expansion in integral form.

(1.2) Stochastic case and the Milstein scheme

In moving to SDEs, the main difference is that we must use Ito's formula in place of the chain rule. For an SDE

$$\dot{x} = f(x) + \sigma(x)\, \eta(t),$$

we have

$$\begin{aligned}\dot{g}(x) &= g'(x)\,\dot{x} + \tfrac{1}{2}\sigma(x)\, g''(x) \\ &= g'(x) f(x) + \tfrac{1}{2}\sigma(x) g''(x) + g'(x)\sigma(x)\,\eta(t)\end{aligned}$$

Integrating:

$$g(x(t)) = g(x(0)) + \int_0^t ds\, \mathcal{L}_0\, g(x(s)) + \int_0^t \mathcal{L}_1\, g(x(s))\, dW(s).$$

where

$$\mathcal{L}_0 \equiv f(x)\,\partial_x + \tfrac{1}{2}\sigma(x)\,\partial_x'', \qquad \mathcal{L}_1 \equiv \sigma(x)\,\partial_x$$

we now apply this to

$$x(t) = x(0) + \int_0^t ds\, f(x(s)) + \int_0^t \sigma(x(s))\, dW(s).$$

Setting $g = f$ and $g = \sigma$ in the above:

$$x(t) = x(0) + f(x(0)) \underbrace{\int_0^t ds}_{t} + \sigma(x(s)) \underbrace{\int_0^t dW(s)}_{W(t)} + R_1$$

where

$$R_1 \equiv \underbrace{\int_0^t ds \int_0^s du\, \mathcal{L}_0 f(x(u))}_{\mathcal{O}(t^2)} + \overbrace{\int_0^t ds \int_0^s dW(u)\, \mathcal{L}_1 f(x(s))}^{\mathcal{O}(t^{3/2})}$$

$$+ \overbrace{\int_0^t dW(s) \int_0^s du\, \mathcal{L}_0 \sigma(x(u))}^{\mathcal{O}(t^{3/2})} + \overbrace{\int_0^t dW(s) \int_0^s dW(u)\, \mathcal{L}_1 \sigma(x)}^{\mathcal{O}(t)}$$

If we set $R_1 = 0$, we get the EM scheme. This is <u>accurate only to $\mathcal{O}(\sqrt{t})$</u> because we have neglected a term of order t in R_1. To get an algorithm that's accurate to $\mathcal{O}(t)$, we must further expand this term:

$$\int_0^t dW(s)\int_0^s dW(u)\ \mathcal{L}_1\sigma(x) = \mathcal{L}_1\sigma(x(0))\int_0^t dW(s)\int_0^s dW(u) + o(t)$$
$$= \mathcal{L}_1\sigma(x_0)\int_0^t W(s)\,dW(s) \longleftarrow \text{we calculated this above}$$
$$= \frac{\sigma(x_0)\sigma'(x_0)}{2}(W_t^2 - t)$$

To conclude, we have

$$x(t) = x_0 + f(x_0)t + \sigma(x_0)W(t) + \tfrac{1}{2}\sigma(x_0)\sigma'(x_0)(W_t^2 - t) + R_2$$

where $R_2 \in O(t^{3/2})$. Setting $R_2 = 0$ gives the Milstein scheme, accurate to $O(\Delta t)$:

Def Milstein Scheme: to integrate an SDE $\dot{x} = f(x) + \sigma(x)\eta(t)$, discretize time $t_n = n\Delta t$, and compute $x(t_n) \equiv x_n$ recursively as

$$x_{n+1} = x_n + f(x_n)\Delta t + \sigma(x_n)\sqrt{\Delta t}\,Z_n + \tfrac{1}{2}\sigma(x_n)\sigma'(x_n)(\Delta t\, Z_n - t)$$

where $\{Z_n\}$ are i.i.d. $\mathcal{N}(0,1)$.

Note that for additive noise $\sigma(x) = \sigma$, the Milstein scheme reduces to the EM scheme, implying that EM is also accurate to $O(\Delta t)$ in the additive case.

Q: How can this be extended to higher dimensions? Trying the above for a 2D system, we would encounter integrals like

$$\int_0^t\int_0^s dW_u^{(1)}\,dW_s^{(2)}$$

where $W^{(1)}$ and $W^{(2)}$ are two independent Wiener processes. Such integrals have no known analytical expressions (though they can be simulated).

For vectorial problems it is often best to look for a method specialized to the particular SDE at hand.

2 Strong and Weak Convergence

How can we judge the quality of a numerical scheme?

In general, there are two kinds of convergence results that one can look for

1) Strong convergence (pathwise convergence): For a given realization of $\{W_t\}$, how well does the discrete algorithm reproduce the continuum limit?

2) <u>Weak convergence</u>: suppose I measure some observable $g(x)$. How well is $\langle g(x)\rangle$ approximated by the algorithm?

We now give explicit definitions. Let $x(t)$ denote the true (stochastic) solution to an SDE on the interval $[0,T]$. Suppose I run a numerical integrator with time step Δt, and that it produces a discrete approximation $(y_0^{\Delta t}, y_1^{\Delta t}, \dots, y_N^{\Delta t}) \equiv y$ with $N = T/\Delta t$.

<u>Def</u> $y^{\Delta t}$ converges strongly to X at time T with order α if $\exists\, C>0,\ \delta>0$ s.t.

$$\langle |y_N^{\Delta t} - x(T)| \rangle \le C(\Delta t)^{\alpha} \qquad \forall \Delta t < \delta$$

<u>Def</u> $y^{\Delta t}$ converges weakly to X at T with order β if for any smooth test function f, $\exists\, C_f>0,\ \delta>0$, independent of Δt, s.t.

$$|\langle f(y_N^{\Delta t})\rangle - \langle f(x(T))\rangle| \le C_f(\Delta t)^{\beta} \qquad \forall \Delta t < \delta$$

You can check that for f that don't grow too fast, $\beta \ge \alpha$, so that strong $\Rightarrow$ weak. One can show

	Strong order	Weak order
EM	1/2	1
Milstein	1	1

Proving the strong or weak order of an algorithm is difficult for generic f and σ. But it can be checked numerically: if you have an exact solution, you can estimate the above by running many simulations for different Δt. In the strong case, use the same Wiener process sampled at successively higher resolution (recall Lec 3 on conditional distributions), compute $|y^{\Delta t} - x|$ for different Δt, and then repeat for many different Wiener realizations to get $\langle |y-x| \rangle$.

In the weak case, use a different $\{W(t)\}$ on each run.

What if you don't have an exact solution? Compare to a very high res simulation, or look at the difference as you double the resolution:

$$\frac{|y^{\Delta t} - y^{\Delta t/2}|}{|y^{\Delta t/2} - y^{\Delta t/4}|}$$

For sufficiently small Δt, $y^{\Delta t} = x + C(\Delta t)^{\alpha}$, so that

$$\frac{|y^{\Delta t} - y^{\Delta t/2}|}{|y^{\Delta t/2} - y^{\Delta t/4}|} = \frac{|C(\Delta t)^{\alpha} - C(\Delta t/2)^{\alpha}|}{|C(\Delta t/2)^{\alpha} - C(\Delta t/4)^{\alpha}|} = \frac{2^{\alpha} - 1}{1 - 2^{-\alpha}} = 2^{\alpha}$$