Recitation 3 – Amer AL-Hiyasat

Topics:

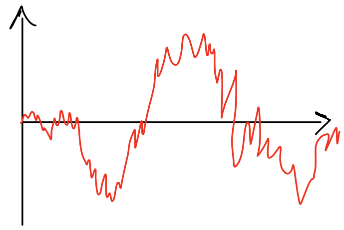
1) Conditional distributions + constructive definition of the Wiener process
2) General Brownian motions
3) Stochastic Differential Equations

Recap: Wiener process $W(t)$ is defined by:

i) Continuous paths

ii) Stationary, independent increments

iii) $W(t) \sim \mathcal{N}(0,t)$



An equivalent definition is:

$W(t)$ is a Gaussian process with $\langle W(t)\rangle = 0$ and $\langle W(t_1) W(t_2)\rangle = \min(t_1, t_2)$.

The physics definition of $W(t)$ is that it is the solution to the Langevin eqn:

$$\dot{x} = \eta(t), \qquad \text{where } \langle \eta(t)\eta(t')\rangle = \delta(t_1 - t_2)$$

Thus

$$W(t) = \int_0^t ds\, \eta(s)$$

so that we may write

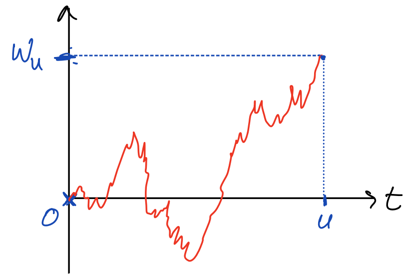
$$\text{“}\frac{dW}{dt} = \eta(t)\text{”}$$

where the quotes emphasize that $W(t)$ is not differentiable in the rigorous sense.

Conditional distributions

Consider a Wiener Process $W(t)$. Suppose I give you $W(u)$ for some $u>0$. What is the conditional distribution of $(W(t\leq u) \mid W(u))$?

The following fact is useful:



Claim: $W_t - (t/u)\,W_u$ is independent of W_u for $t<u$.

Pf: Clearly the quantity is Gaussian, so it suffices to show that the covariance is zero.

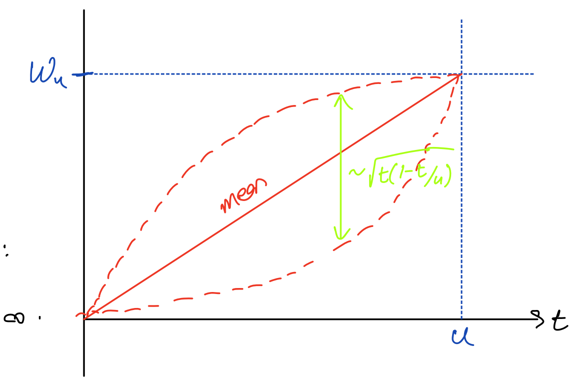
$$\left\langle \left(W_t - t/u\, W_u\right) W_u \right\rangle_c = \min(t,u) - \frac{t}{u}\langle W_u^2\rangle_c$$
$$= t - \frac{t}{u}u = 0. \quad ■$$

This makes life easy:

$$\langle W_t \mid W_u\rangle = \langle W_t - (t/u)\,W_u \mid W_u\rangle + \frac{t}{u}W_u$$
$$= \langle W_t - \frac{t}{u}W_u\rangle + t/u\, W_u$$
$$= \boxed{t/u\; W_u}$$

This makes sense: the mean is just obtained by linearly interpolating btw the two points. The variance is:

$$\langle W_t^2 \mid W_u\rangle_c = \left\langle \left(W_t - \langle W_t \mid W_u\rangle\right)^2 \mid W_u\right\rangle$$
$$= \left\langle \left(W_t - \frac{t}{u}W_u\right)^2 \mid W_u\right\rangle$$
$$\overset{\text{by claim}}{=} \left\langle \left(W_t - \frac{t}{u}W_u\right)^2\right\rangle$$
$$= t - 2\frac{t}{u}\min(t,u) + \frac{t^2}{u^2}u$$
$$= \boxed{t(1-t/u)}$$



As expected, variance is smallest near t or u, and maximized at $t=u/2$.

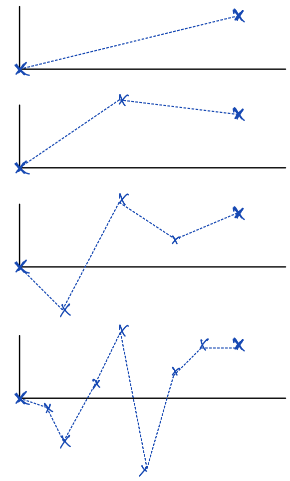
Strangely, this is independent of W_u!

As W_u is made larger, the relative width vanishes:

$$\frac{\sqrt{\langle W_t^2 \mid W_u\rangle_c}}{\langle W_t \mid W_u\rangle} \leq \frac{u/2}{t/u\, W_u} \to 0 \text{ as } W_u \to \infty.$$

(Maximum variance is always $u/4$)

Note that the result above gives a procedure for sampling a Wiener process on the interval $(0,u)$



1) Set $W(0) = 0$.
2) Set $W(u)$ by drawing a $\mathcal{N}(0,u)$ number.
3) Set $W(u/2)$ by drawing a $\mathcal{N}\left(\frac{1}{2}W(u), u/4\right)$
4) Repeat for the intervals $(0,u/2)$ and $(u/2,u)$, continue recursively.

The conditional correlation function is, for $s<t<u$

$$\langle W(s)\,W(t)\,|\,W(u)\rangle_c = \frac{s(u-t)}{u}$$

I leave the proof as an exercise.

The Brownian Bridge

In the special case $u=1$ and $W_u = 0$, the resulting conditional distribution is called a Brownian Bridge. This can be uniquely defined as follows

Def: A standard Brownian Bridge is a Gaussian process $\{X(t): t\in[0,1]\}$ with continuous paths, mean zero, and $\langle X(s)\,X(t)\rangle_c = s(1-t)$ for $0\le s\le t\le 1$.

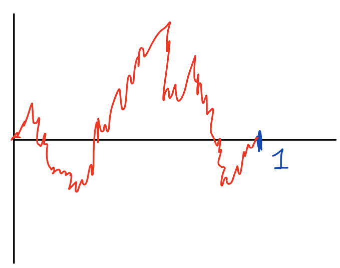
You can verify that if W is a Wiener process, the following give Brownian bridges:

$$X(t) = W(t) - t\,W(1)$$
$$Y(t) = (1-t)\,W\!\left(\frac{t}{1-t}\right)$$

You can interpret this as a Wiener process "pinned" at $W(1)=0$.

You may verify that a Brownian bridge has the following Fourier representation:

$$B_t = \sum_{k=1}^{\infty} \frac{Z_k}{k}\,\frac{\sqrt{2}\,\sin(k\pi t)}{\pi}$$



where the $\{Z_k\}$ are i.i.d $\mathcal{N}(0,1)$.

The spectral density thus falls as $1/k^2 \Rightarrow$ this is like a simple Gaussian field theory with Hamiltonian $\mathcal{H}[\psi] = \int dx\,(\nabla\psi)^2$ (don't worry if this is meaningless to you).

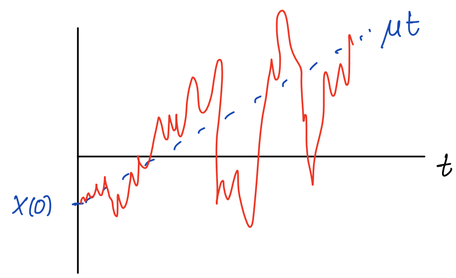
General Brownian motions (with drift)

We may generalize the Wiener process by adding a drift and a scale to $W(t)$.

A process $X(t)$ defined as

$$X(t) = X(0) + \mu t + \sigma W(t)$$

is called a (μ, σ^2)-Brownian motion (though this terminology is rarely used in physics).



I bring this up because of an important fact:

Thm: If a stochastic process X has continuous paths and stationary independent increments, then X is a Brownian motion as defined above.

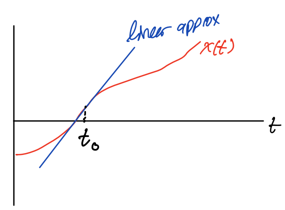
The Gaussianity of increments thus comes "for free". This can be intuited from the central limit theorem: If increments are stationary and independent, then any one increment can be partitioned as the sum of many smaller increments that are i.i.d.

Stochastic differential equations

The solution to a (deterministic) ODE of the form

$$\dot{x} = f(x)$$

is a smooth function $x(t)$ (so long as f is smooth).



Such a solution can everywhere be locally approximated by a linear function:

$$x(t) \simeq x(t_0) + f(x(t_0))\,(t - t_0) \qquad \text{as } t \to t_0$$

A stochastic differential equation

$$\dot{x} = f(x) + \sigma(x)\,\eta(t)$$

has solutions $x(t)$ which can be locally approximated by a general Brownian motion:

$$x(t) \simeq x(t_0) + f(x(t_0))\,(t - t_0) + \sigma(x(t_0))\,W(t - t_0) \qquad \text{as } t \to t_0$$

Stochastic Integrals

A stochastic differential equation is a shorthand for the corresponding stochastic integral:

$$x(t) = \int_0^t ds\, f(x(s)) + \int_0^t ds\, \sigma(x(s))\, \eta(s)$$

In math, integrals of the form $\int_{t_0}^t ds\, \eta(s)\, h(s)$ are typically denoted,

$$\int_{t_0}^t ds\, \eta(s)\, h(s) = \int_{t_0}^t dW(s)\, h(s) \qquad (1)$$

i.e. we replace $ds\, \eta(s) \longrightarrow dW(s)$.

The corresponding notation for an SDE is

$$dx = f(x,t)\, dt + \sigma(x,t)\, dW$$

We define (1) as a stochastic version of the Riemann-Stieltjes integral:

Partition the interval $[t_0, t]$ into n subintervals,

$$t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t$$

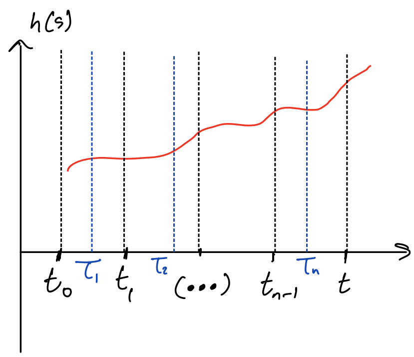
Pick times τ_i within these subintervals:

$$t_{i-1} \leq \tau_i \leq t_i$$

The stochastic integral is defined as the limit $\int_{t_0}^t h(s)\, dW(s) = \lim_{n\to\infty} S_n$, where

$$S_n \equiv \sum_{i=1}^n h(\tau_i)\left[W(t_i) - W(t_{i-1})\right]$$

where W is a standard Wiener process.



Note that the value of this integral depends on the choice of τ_i. For example, consider $h(s) = W(s)$. Then:

$$\langle S_n \rangle = \sum_{i=1}^n \left\langle W(\tau_i)\left[W(t_i) - W(t_{i-1})\right]\right\rangle$$

$$= \sum_{i=1}^n \left[\min(\tau_i, t_i) - \min(\tau_i, t_{i-1})\right]$$

$$= \sum_{i=1}^n (\tau_i - t_{i-1})$$

If we use an "α-discretization" as mentioned in lecture:

$$\tau_i \equiv \alpha t_i + (1-\alpha) t_{i-1}, \qquad 0 \le \alpha \le 1,$$

Then

$$\langle S_n \rangle = \sum_{i=1}^{n} \alpha (t_i - t_{i-1}) = \alpha (t - t_0)$$

This ranges anywhere between 0 and $t - t_0$.

In the Itō perscription, we make the choice

$$\alpha = 0$$

In the Stratonovich perscription, we use

$$\alpha = 1/2.$$

For arbitrary $h(s)$, there is no general correspondence between the two integrals, but in the important special case where $h(s) = h(x(s))$ and $x(s)$ is the solution to an SDE, there is in fact a general formula relating the two perscriptions. We will come back to this in a future recitation.