The Wiener Process

Recap

A Wiener Process or Standard Brownian Motion can be defined in two ways:

① A process $\{W(t) : t \geq 0\}$ having
   i) Continuous paths
   ii) Stationary, independent increments
   iii) $W(t) \sim N(0,t) \ \forall\, t \geq 0$

or

② A Gaussian process with continuous paths, mean zero and correlation function:

$$\langle W(t_1)\ W(t_2)\rangle_c = \min(t_1, t_2)$$

Some properties of the Wiener Process

Thm: Suppose W is a Wiener process and let $c > 0$. Then the process

$$X(t) = c^{-1/2}\, W(ct)$$

is a Wiener process.

Pf: Follows immediately from ②: $X(t)$ is clearly Gaussian.

$$\langle X(t_1)\ X(t_2)\rangle_c = c^{-1}\langle W(ct_1)\ W(ct_2)\rangle_c$$

$$= c^{-1}\min(ct_1, ct_2) = \min(t_1, t_2) \qquad \blacksquare$$

Thm: The process $X(t) = W(c+t) - W(c)$ is a Wiener process independent of $W(t \leq c)$, for any $c > 0$.

Pf: Follows immediately from ① ii and ②. $\blacksquare$

This is the sense in which Brownian motion is Markovian and stationary - it is constantly restarting.

Thm: The process $X(t) = t\,W(1/t)$ is a Wiener process.

Pf: $X(t)$ is Gaussian and continuous (ignore technicalities at $t=0$). Covariance function:

$$\langle t_1 W(1/t_1)\ t_2 W(1/t_2)\rangle_c = t_1 t_2 \min(1/t_1, 1/t_2) = \frac{t_1 t_2}{\max(t_1, t_2)} = \min(t_1, t_2).$$

General Brownian motions

We may generalize the wiener process by adding a drift and a scale to $W(t)$.

A process $X(t)$ defined as

$$X(t) = X(0) + \mu t + \sigma W(t)$$

is called a (μ, σ^2)-Brownian motion (though this terminology is rarely used in physics).

I bring this up because of an important fact:

Thm: If a stochastic process X has continuous paths and stationary independent increments, then X is a Brownian motion as defined above.

The Gaussianity of increments thus comes "for free". This can be intuited from the central limit theorem: If increments are stationary and independent, then any one increment can be partitioned as the sum of many smaller increments that are i.i.d.

Some pathologies of Brownian Paths

1) For any $\varepsilon > 0$, $W(t)$ crosses zero infinitely many times on the interval $(0, \varepsilon)$.

One way to understand this is to use that $t\,W(1/t)$ is a Wiener process. It is believable that a wiener process crosses 0 infinitely many times on the interval $(1/\varepsilon, \infty)$ (we may come back to this when we talk about recurrence of random walks). But if $s\,W(1/s) = 0$ for infinitely many $s \in (1/\varepsilon, \infty)$, then $W(t) = 0$ for infinitely many $t \in (0, \varepsilon)$.

2) $W(t)$ is nowhere differentiable, with prob 1.

This may be understood from the independence of increments: $W(t+\delta) - W(t)$ and $W(t-\delta) - W(t)$ are independent $\forall\, t, \delta > 0$.

Another way: consider $\Delta W = W(t+\delta) - W(t)$.

This has variance $\langle \Delta W^2 \rangle = \delta$, hence $\lim_{\delta \to 0} \frac{\Delta W}{\delta} \sim \frac{\sqrt{\delta}}{\delta}$

Computing things: First passage times

Let $\{W(t)\}$ be a Wiener process. Define the first passage time for some $b>0$:

$$\tau_b = \inf\{t : W_t \geq b\}$$
$$= \inf\{t : W_t = b\}.$$

Let us compute the cumulative distribution function $F(t) = \mathbb{P}[\tau_b \leq t]$.

Then
$$\mathbb{P}[\tau_b \leq t] = \mathbb{P}[\tau_b \leq t, W_t < b] + \mathbb{P}[\tau_b \leq t, W_t > b]$$
$$= \mathbb{P}[W_t < b \mid \tau_b \leq t]\,\mathbb{P}[\tau_b < t] + \mathbb{P}[W_t > b]$$

The second term simply follows from $W_t \sim \mathcal{N}(0,t)$:

$$\mathbb{P}[W_t > b] = \int_b^\infty dx \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} = \frac{1}{2} \mathrm{Erfc}\left(b/\sqrt{2t}\right).$$

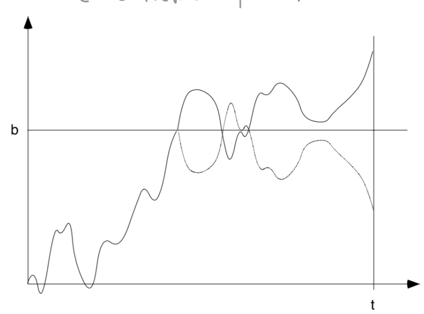
How about the first term? First convince yourself that, by symmetry,

$$\mathbb{P}[W_t < b \mid \tau_b \leq t] = 1/2 \qquad \text{"Reflection Principle"}$$

i.e. once you condition on hitting b at some time $\tau_b \leq t$, $W(t)$ is equally likely to be above or below b at subsequent times. Thus:

$$\boxed{\mathbb{P}[\tau_b < t] = 2\,\mathbb{P}[W_t > b] = \mathrm{erfc}\left(b/\sqrt{2t}\right).}$$

The reflection principle:



$(\mathrm{erfc} \equiv 1 - \mathrm{erf})$. The corresponding probability density is

$$\boxed{P_{\tau_b}(t) = \frac{b e^{-b^2/2t}}{\sqrt{2\pi t^3}}} \qquad (\text{supported on } \mathbb{R}^+).$$

This is called a Levy distribution. It has diverging mean:

$$\langle \tau_b \rangle = \infty$$

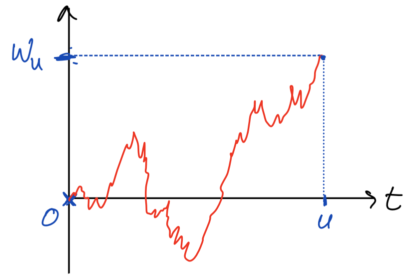
But τ_b is finite with Prob 1: $\lim_{t \to \infty} \mathbb{P}[\tau_b < t] = \mathrm{erfc}(0) = 1$.

We say that a Brownian motion in 1D is "null-recurrent": if it crosses zero, it comes back almost surely, but the time to so is infinite on average.

Conditional distributions

Consider a Wiener Process $W(t)$. Suppose I give you $W(u)$ for some $u>0$. What is the conditional distribution of $(W(t\leq u)\mid W(u))$?

The following fact is useful:



Lemma: $W_t-(t/u)W_u$ is independent of W_u for $t<u$.

Pf: Clearly the quantitity is Gaussian, so it suffices to show that the covariance is zero.

$$\langle (W_t - t/u\, W_u)\, W_u\rangle_c = \min(t,u) - \frac{t}{u}\langle W_u^2\rangle_c$$
$$= t - \frac{t}{u}u = 0. \quad \blacksquare$$

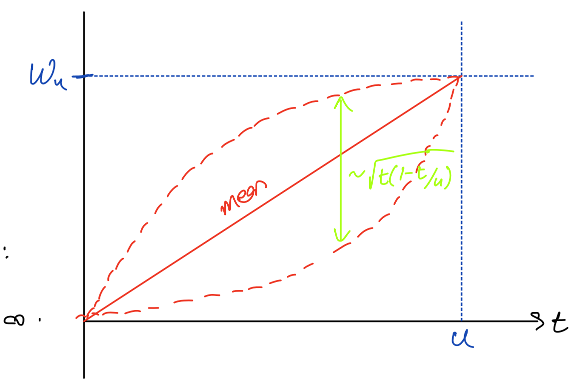
This makes life easy:

$$\langle W_t \mid W_u\rangle = \langle W_t - (t/u)W_u \mid W_u\rangle + \frac{t}{u}W_u$$
$$= \langle W_t - \frac{t}{u}W_u\rangle + t/u\, W_u$$
$$= \boxed{t/u\, W_u}$$

This makes sense: the mean is just obtained by linearly interpolating btw the two points. The variance is:

$$\langle W_t^2 \mid W_u\rangle_c = \langle (W_t - \langle W_t \mid W_u\rangle)^2 \mid W_u\rangle$$
$$= \langle (W_t - \frac{t}{u}W_u)^2 \mid W_u\rangle$$
$$\overset{\text{by lemma}}{=} \langle W_t - \frac{t}{u}W_u\rangle^2$$
$$= t - 2\frac{t}{u}\min(t,u) + \frac{t^2}{u^2}u$$
$$= \boxed{t(1-t/u)}$$

As expected, variance is smallest near t or u, and maximized at $t=u/2$. Strangely, this is independent of W_u! As W_u is made larger, the relative width vanishes:

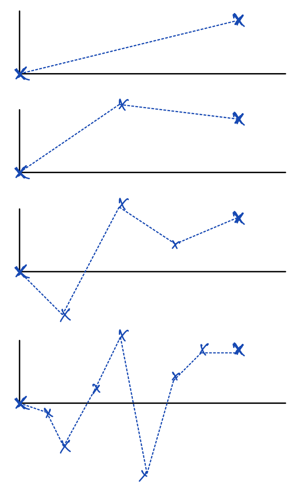


$$\frac{\sqrt{\langle W_t^2 \mid W_u\rangle_c}}{\langle W_t \mid W_u\rangle} \lesssim \frac{u/2}{t/u\, W_u} \to 0 \text{ as } W_u\to\infty.$$

(Maximum variance is always $u/4$)

Note that the result above gives a procedure for sampling a Wiener process on the interval $(0, u)$

1) Set $W(0) = 0$.
2) Set $W(u)$ by drawing a $\mathcal{N}(0, u)$ number.
3) Set $W(u/2)$ by drawing a $\mathcal{N}\left(\frac{1}{2}W(u), u/4\right)$
4) Repeat for the intervals $(0, u/2)$ and $(u/2, u)$, continue recursively.



The conditional correlation function is, for $s < t < u$

$$\langle W(s) W(t) \mid W(u) \rangle_c = \frac{s(u-t)}{u}$$

I leave the proof as an exercise.

The Brownian Bridge

In the special case $u = 1$ and $W_u = 0$, the resulting conditional distribution is called a Brownian Bridge. This can be uniquely defined as follows

Def: A standard Brownian Bridge is a Gaussian process $\{X(t) : t \in [0, 1]\}$ with continuous paths, mean zero, and $\langle X(s) X(t) \rangle_c = s(1-t)$ for $0 \leq s \leq t \leq 1$.

You can verify that if W is a Wiener process, the following give Brownian bridges:

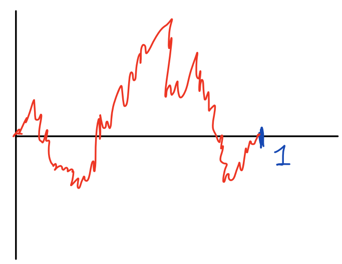
$$X(t) = W(t) - t\,W(1)$$

$$Y(t) = (1-t)\,W\left(\frac{t}{1-t}\right)$$

You can interpret this as a Wiener process "pinned" at $W(1) = 0$.

You may verify that a Brownian bridge has the following Fourier representation:

$$B_t = \sum_{k=1}^{\infty} \frac{Z_k}{k} \frac{\sqrt{2}\,\sin(k\pi t)}{\pi}$$



where the $\{Z_k\}$ are i.i.d $\mathcal{N}(0, 1)$.

The spectral density thus falls as $1/k^2 \Rightarrow$ this is like a simple Gaussian field theory with Hamiltonian $\mathcal{H}[\psi] = \int dx (\nabla\psi)^2$ (don't worry if this is meaningless to you).