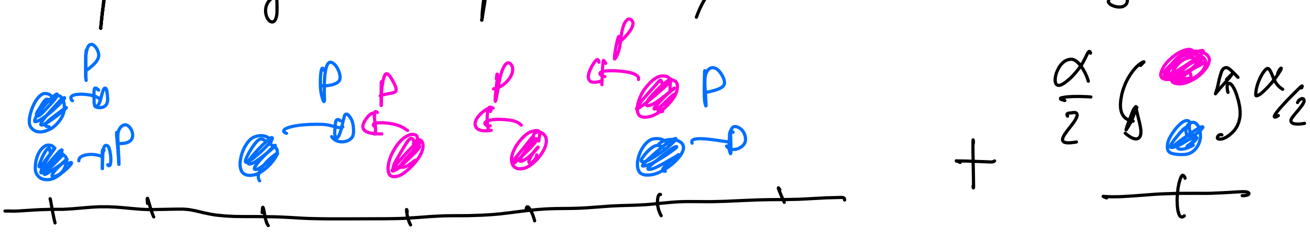
Conclusion: if $\frac{v'}{v} \leq -\frac{1}{\rho}$ (or $(\rho v)' < 0$), a homogeneous phase is linearly unstable to MIPS

More general results on the phase-separation, surface tension and how to compute the phase diagram in off-lattice models can be found in: [Solon et al, New Journal of Physics 20, 075001 (2018)]

7.7.2) RTPs on lattice in 1D

Here, we focus on the simpler case of on-lattice RTPs in 1d. (inspired by [Thompson et al, J. Stat. Mech 2011].



Lattice gas model of non-interacting run-and-tumble bacteria.

Configurations $\mathcal{C} \Leftrightarrow$ occupancies (n_i^+, n_i^-) which describe the number of particles hopping to the right or to the left on site i.

Add excluded volume interaction: partial exclusion

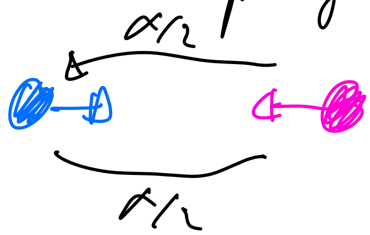
Hopping $i \to i+1$ for a right going particle $p\left(1 - \frac{n_{i+1}}{n_M}\right)$

$i \to i-1$ —— left —— $p\left(1 - \frac{n_{i-1}}{n_M}\right)$

$n_i = n_i^+ + n_i^-$ is the total occupancy.

n_M is the maximal occupancy allowed on a lattice site

Tumbling rate



Idea: Describe the dynamics of $\rho_i^\pm = \langle n_i^\pm(t) \rangle$ and show that homogeneous profiles are unstable.

Master equation:

$$\partial_t P(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} W(\mathcal{C}' \to \mathcal{C})\, P(\mathcal{C}') - W(\mathcal{C} \to \mathcal{C}')\, P(\mathcal{C})$$

Here $\mathcal{C} = \{n_i^+, n_i^-\} = \{n_1^+, n_1^-, n_2^+, n_2^-, \dots, n_L^+, n_L^-\}$

(1) $\mathcal{C}'$ such that $W(\mathcal{C}' \to \mathcal{C}) \neq 0$

* $\mathcal{C}' = \{n_1^+, n_1^-, \dots, n_{i-1}^+ + 1, n_{i-1}^-, n_i^+ - 1, n_i^-, \dots, n_L^+, n_L^-\} \equiv \{n_{i-1}^+ + 1, n_i^+ - 1\}$

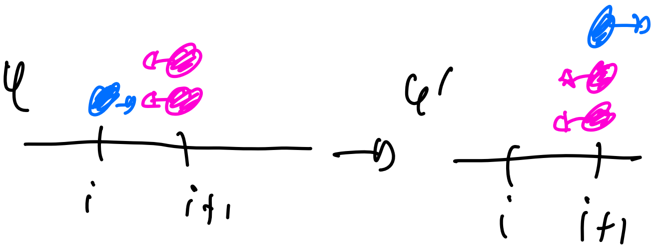
$\mathcal{C}' = n_{i-1}^+ + 1,\ n_i^+ - 1 \to \mathcal{C}$ at rate $W = p\,(n_{i-1}^+ + 1)\left(1 - \frac{n_i - 1}{n_M}\right)$

and $n_{j \neq i}^{+-}$ same as in $\mathcal{C}$

* $\mathcal{C}'$ as $\mathcal{C}$ but with $n_{i+1}^- + 1,\ n_i^- - 1 \to \mathcal{C}$ at rate $W(\mathcal{C}' \to \mathcal{C}) = p\,(n_{i+1}^- + 1)\left(1 - \frac{n_i - 1}{n_M}\right)$

* φ' as φ but with $n_i^+ + 1, n_i^- - 1$; then $W(\varphi' \to \varphi) = \frac{\alpha}{2}(n_i^+ + 1)$
* φ' as φ but with $n_i^- + 1, n_i^+ - 1$; then $W(\varphi' \to \varphi) = \frac{\alpha}{2}(n_i^- + 1)$

(2) φ' such that $W(\varphi \to \varphi') \neq 0$, e.g.



* φ' as φ but with $\{n_i^+ - 1, n_{i+1}^+ + 1\}$, $W(\varphi \to \varphi') = p\, n_i^+ \left(1 - \frac{n_{i+1}}{n_M}\right)$
* ———— $\{n_{i+1}^- - 1, n_i^- + 1\}$, $W(\varphi \to \varphi') = p\, n_{i+1}^- \left(1 - \frac{n_i}{n_M}\right)$
* ———— $\{n_i^+ - 1, n_i^- + 1\}$, $W(\varphi \to \varphi') = \frac{\alpha}{2} n_i^+$
* ———— $\{n_i^- - 1, n_i^+ + 1\}$, $W(\varphi \to \varphi') = \frac{\alpha}{2} n_i^-$

Master eq:

$$
\begin{aligned}
\partial_t P(\{n_j^+, n_j^-\}) = \sum_i \; & p(n_{i-1}^+ + 1)\left(1 - \frac{n_i - 1}{n_M}\right) P(\{n_{i-1}^+ + 1, n_i^+ - 1\}) \\
& + p(n_{i+1}^- + 1)\left(1 - \frac{n_i - 1}{n_M}\right) P(\{n_{i+1}^- + 1, n_i^- - 1\}) \\
& + \frac{\alpha}{2}(n_i^+ + 1) P(\{n_i^+ + 1, n_i^- - 1\}) + \frac{\alpha}{2}(n_i^- + 1) P(\{n_i^- + 1, n_i^+ - 1\}) \\
& - \left[p\, n_i^+ \left(1 - \frac{n_{i+1}}{n_M}\right) + p\, n_i^- \left(1 - \frac{n_{i-1}}{n_M}\right) + \frac{\alpha}{2} n_i^+ + \frac{\alpha}{2} n_i^- \right] P(\{n_i^+, n_i^-\})
\end{aligned}
$$

where $P(\{n_i^\pm + 1, n_j^\pm - 1\}) = P(\varphi' = \varphi$ with $n_i^\pm$ replaced by $n_i^\pm + 1$, $n_j^\pm$ ———— $n_j^\pm - 1)$

From there, use $\langle n_i^+ \rangle = \sum_{\{n_j\}} n_i^+ P(\{n_j\})$ to get $\partial_t \langle n_i^+ \rangle$ from the master equation → good luck!

Evolution of observables: $\langle O(\varphi) \rangle = \sum_\varphi O(\varphi) P(\varphi, t) = \langle O \rangle_\varphi$

$$\partial_t \langle O \rangle_\varphi = \sum_\varphi O(\varphi)\, \partial_t P(\varphi, t)$$ → Master equation

$$= \sum_{\varphi, \varphi'} O(\varphi) \left[W(\varphi' \to \varphi) P(\varphi') - W(\varphi \to \varphi') P(\varphi) \right]$$

$\varphi \leftrightarrow \varphi'$

$$= \sum_{\varphi, \varphi'} O(\varphi') W(\varphi \to \varphi') P(\varphi) - O(\varphi) W(\varphi \to \varphi') P(\varphi)$$

$$= \sum_\varphi \left[\sum_{\varphi'} (O(\varphi') - O(\varphi)) W(\varphi \to \varphi') \right] P(\varphi)$$

$$\boxed{\partial_t \langle O \rangle_\varphi = \left\langle \sum_{\varphi'} (O(\varphi') - O(\varphi)) W(\varphi \to \varphi') \right\rangle_\varphi}$$

$\Delta O(\varphi \to \varphi')$: change in O from φ to φ'

$W(\varphi \to \varphi')$: rate at which this change occurs.

Dynamics of the average occupancies: $\partial_t \langle n_i^+ \rangle = ?$

Consider φ' that can be reached from $\varphi = \{n_i^+, n_i^-\}$ and compute $n_i^+(\varphi') - n_i^+(\varphi) = \Delta n_i^+$

Moves that impact n_i^+ are:
- hops into or out of site i
- tumble at site i

$$\partial_t \langle n_i^+ \rangle = \langle \overbrace{(+1)}^{\Delta n_i^+} \times \overbrace{p\, n_{i-1}^+ \left(1 - \tfrac{n_i}{n_M}\right)}^{W} \rangle + \langle \overbrace{(-1)}^{\Delta n_i^+} \times \overbrace{p\, n_i^+ \left(1 - \tfrac{n_{i+1}}{n_M}\right)}^{W} \rangle$$

$i-1 \to i$; $i \to i+1$

$$+ \langle \overbrace{(-1)}^{\Delta n_i^+} \times \overbrace{\tfrac{\alpha}{2} n_i^+}^{W} \rangle + \langle \overbrace{(+1)}^{\Delta n_i^+} \times \overbrace{\tfrac{\alpha}{2} n_i^-}^{W} \rangle$$

tumble $+ \to -$; tumble $- \to +$

$$\partial_t \langle n_i^+ \rangle = p \langle n_{i-1}^+ \left(1 - \tfrac{n_i}{n_M}\right) \rangle - p \langle n_i^+ \left(1 - \tfrac{n_{i+1}}{n_M}\right) \rangle - \tfrac{\alpha}{2} \langle n_i^+ \rangle + \tfrac{\alpha}{2} \langle n_i^- \rangle$$

$$\partial_t \langle n_i^- \rangle = p \langle n_{i+1}^- \left(1 - \tfrac{n_i}{n_M}\right) \rangle - p \langle n_i^- \left(1 - \tfrac{n_{i-1}}{n_M}\right) \rangle + \tfrac{\alpha}{2} \langle n_i^+ \rangle - \tfrac{\alpha}{2} \langle n_i^- \rangle$$

Comment: These equations involve $\langle n_{i-1}^+ n_i \rangle$, $\langle n_i^+ n_{i+1} \rangle$, etc

$\Rightarrow$ not closed equations for $\langle n_i^\pm \rangle$.

In practice N-points functions will involve N+1 point functions in their dynamics. We need approximations to get a solvable system.

Mean-field approximation: $\langle n_i^+ n_j^- \rangle \simeq \langle n_i^+ \rangle \langle n_j^- \rangle$

In general, this is quantitatively wrong, but often qualitatively right.

$\rho_i^\pm = \langle n_i^\pm \rangle$; $\rho_i = \langle n_i \rangle$; $\rho_m = n_M$

Dynamics:

$$\begin{cases} \partial_t \rho_i^+ = p\left[\rho_{i-1}^+ \left(1 - \tfrac{\rho_i}{\rho_m}\right) - \rho_i^+ \left(1 - \tfrac{\rho_{i+1}}{\rho_m}\right)\right] - \tfrac{\alpha}{2} \rho_i^+ + \tfrac{\alpha}{2} \rho_i^- & (1) \\ \partial_t \rho_i^- = p\left[\rho_{i+1}^- \left(1 - \tfrac{\rho_i}{\rho_m}\right) - \rho_i^- \left(1 - \tfrac{\rho_{i-1}}{\rho_m}\right) + \tfrac{\alpha}{2} \rho_i^+ - \tfrac{\alpha}{2} \rho_i^-\right] & (2) \end{cases}$$

Typical microscopic length $l_p = \frac{p}{\alpha}$ $\Rightarrow$ expect variations of $\rho_i^\pm$ on this scale. If $\frac{p}{\alpha} \gg 1$, little differences between $\rho_i^\pm$ and $\rho_{i+1}^\pm$

$\Rightarrow$ Taylor expand $O_{i\pm1} \simeq O(x) \pm \frac{1}{L} \nabla O(x) + \frac{1}{2L} \Delta O(x)$; $x = \frac{i}{L} \in [0,1]$

Carrying out this expansion in (1) & (2) leads to $\nabla = \partial_x$; $\Delta = \partial_{xx}$ (6)

$$\begin{cases} \partial_t \rho^+(x) = -\frac{p}{L}(\partial_x \rho^+)\left(1-\frac{\rho}{\rho_m}\right) + \frac{p}{L}\rho^+ \frac{\nabla \rho}{\rho_m} + \frac{p}{2L^2}\Delta\rho^+\left(1-\frac{\rho}{\rho_m}\right) + \frac{p\rho^+}{2L^2}\frac{\Delta\rho}{\rho_M} - \frac{\alpha}{2}\rho^+ + \frac{\alpha}{2}\rho^- & (3) \\ \partial_t \rho^-(x) = \frac{p}{L}(\partial_x \rho^-)\left(1-\frac{\rho}{\rho_m}\right) - \frac{p}{L}\rho^- \frac{\partial_x\rho}{\rho_m} + \frac{p}{2L^2}(\partial_{xx}\rho^-)\left(1-\frac{\rho}{\rho_m}\right) + \frac{p}{2L^2}\rho^- \frac{\partial_{xx}\rho}{\rho_M} + \frac{\alpha}{2}\rho^+ - \frac{\alpha}{2}\rho^- & (4) \end{cases}$$

$\rho(x) = \rho^+(x) + \rho^-(x)$; $m(x) = \rho^+(x) - \rho^-(x)$

$$(3)+(4) \Rightarrow \partial_t \rho(x) = -\partial_x\left[\frac{p}{L} m \left(1-\frac{\rho}{\rho_m}\right) - \frac{p}{2L^2}\partial_x \rho\right] \qquad (5)$$

$$(3)-(4) \quad \partial_t m(x) = -\alpha m(x) - \partial_x\left[\frac{p}{L}\rho\left(1-\frac{\rho}{\rho_m}\right)\right] + \frac{p}{2L^2}(\partial_{xx} m)\left(1-\frac{\rho}{\rho_m}\right) + \frac{p}{2L^2} m\left(\frac{\partial_{xx}\rho}{\rho_m}\right) \qquad (6)$$

Analysis of (5) - (6) in the large L limit:

$L \to \infty$ (5): $\partial_t \rho = 0 \Rightarrow$ Nothing happens in a time of order 1 for a conserved field in the large size limit.

Speed up time $t = L\tau$

$$(5) \Rightarrow \frac{d\rho}{d\tau} = -\partial_x\left[p\, m(x)\left(1-\frac{\rho(x)}{\rho_m}\right) - \frac{p}{2L}\partial_x \rho\right] \qquad (5)'$$

$$(6) \quad \dot m(x) = -\alpha L m - p\,\partial_x\left[\rho\left(1-\frac{\rho}{\rho_m}\right)\right] + \frac{p}{2L}(\partial_{xx} m)\left(1-\frac{\rho}{\rho_m}\right) + \frac{p}{2L} m\, \partial_{xx}\rho$$

If $\partial_{xx}\rho$ & $\partial_{xx} m$ are finite, then the last two terms are negligible.

$$\dot m(x) = \underbrace{-\alpha L m(x)}_{\text{exponential relaxation in a time scale } \tau \sim \frac{1}{\alpha L}} \underbrace{- p\,\partial_x\left[\rho\left(1-\frac{\rho}{\rho_m}\right)\right]}_{\text{external source}}$$

$\Longrightarrow m(x) \simeq -\frac{p}{\alpha L}\partial_x\left[\rho\left(1-\frac{\rho}{\rho_m}\right)\right]$ for $\tau \sim O(1)$

Injecting $m(x)$ back into (5)' then leads to

$$\dot{\rho} = \partial_x \left[\frac{p^2}{\alpha L}\left(1 - \frac{\rho(x)}{\rho_M}\right) \partial_x \left[\rho\left(1 - \frac{\rho}{\rho_M}\right)\right] + \frac{p}{2L}\partial_x \rho \right]$$

Again $\dot{\rho} \sim 0$ for $\tau \sim O(1) \Rightarrow \tau = L\tilde{\tau}$

$$\frac{d\rho(x)}{d\tilde{\tau}} = \partial_x \left[\left(\frac{p}{2} + \frac{p^2}{\alpha}\left(1 - \frac{\rho}{\rho_M}\right)\left(1 - \frac{2\rho}{\rho_M}\right)\right) \partial_x \rho \right]$$

$$\equiv \partial_x \left[D_{eff}(\rho(x))\, \partial_x(\rho) \right]$$

At the macroscopic $x = \frac{i}{L}$; $\tilde{\tau} = \frac{t}{L^2}$; we obtain an effective dynamics for the density field which is a non-linear diffusion equation.

If $D_{eff}(\rho_0) > 0 \Rightarrow$ small fluctuations around ρ_0 relax to 0.

— $D_{eff}(\rho_0) < 0 \Rightarrow$ ———————— get amplified

$$\left(\dot{\rho} = D\Delta\rho \xrightarrow{t \to -t} \dot{\rho} = (-D)\Delta\rho \right)$$

$\Rightarrow$ linear instability

$$D_{eff} < 0 \Leftrightarrow \left(\frac{2\rho}{\rho_M} - 1\right)\left(1 - \frac{\rho}{\rho_M}\right) \geq \frac{\alpha}{2p} = \frac{1}{2\ell_p}$$ $\to$ persistence length

$\ell_p \gg 1 \Rightarrow$ instability for $\rho \geq \frac{\rho_M}{2}$

Comments: hopping rate $\Leftrightarrow v(\rho) = p\left(1 - \frac{\rho}{\rho_M}\right)$

$$\frac{v'}{v} \leq -\frac{1}{\rho} \Leftrightarrow \rho \geq \frac{\rho_M}{2}$$

} the result of our computations is consistent with the hand-waving argument.

What remains is to characterize the phase coexistence emerging from this instability $\Rightarrow$ Solon et al., NJP 2018.

Diffusive limit: $\alpha_R = \alpha_L = \alpha$; $v_R = v_L = v$; $t \gg 1/\alpha$

$\rho = R + L$ local probability density

$$\boxed{\partial_t \rho = (1)+(2) = -\partial_x[J_R - J_L] = -\partial_x[v(R-L)] = -\partial_x \cdot J} \; ; \; J = vR - vL \quad (*)$$

what is the value of J?

$$v(1) - v(2) = \partial_t J$$

$$= -v\partial_x[v(R+L)] - \alpha v R + \alpha v L$$

$$\boxed{\partial_t J = -v\,\partial_x[\rho \cdot v] - \alpha J} \quad (**)$$

$$(**) \Rightarrow \partial_t J_H = -\alpha J_H \Rightarrow J_H = J_H^0 e^{-\alpha t}$$

$$J_p = J_0 e^{-\alpha t} + e^{-\alpha t}\int_0^t ds\, e^{\alpha s}\left(-v\partial_x(\rho v)\right)$$

ρ is a conserved field $\Rightarrow$ on time-scale $\sim \frac{1}{\alpha}$ it barely evolves

$$J = J_0 e^{-\alpha t} + \int_0^t ds\, \underbrace{e^{-\alpha(t-s)}}_{\simeq 0 \text{ if } t-s \gg 1/2}\left(-v\partial_x(\rho v)\right)$$

$$\simeq \underbrace{J_0 e^{-\alpha t}}_{\simeq 0 \; t \gg 1/\alpha} - [v\,\partial_x(\rho v)]\underbrace{\int_0^t ds\, e^{-\alpha(t-s)}}_{\simeq 1/\alpha \; t \gg 1/\alpha}$$

$$\Rightarrow J \simeq -\frac{v}{\alpha}\partial_x(\rho v)$$

Injecting this in $(*)$ leads to $\boxed{\partial_t \rho = \partial_x\left[\frac{v}{\alpha}\partial_x(\rho v)\right]}$ $(***)$

* If v, α is a constant $\Rightarrow \partial_t \rho = \frac{v^2}{\alpha} \partial_{xx} \rho \Rightarrow D = \frac{v^2}{\alpha}$ is the ~~large-scale~~ large-scale diffusivity

* If $v(x)$ non-constant $\Rightarrow \rho = \frac{K}{v(x)}$ is a steady-state solution if $\int_{\mathbb{R}} dx \frac{1}{v(x)} < \infty$

(***) is a diffusion-drift approximation of the microscopic run and tumble dynamics.

The same in 2D: Cates, Tailleur, EPL (2013).