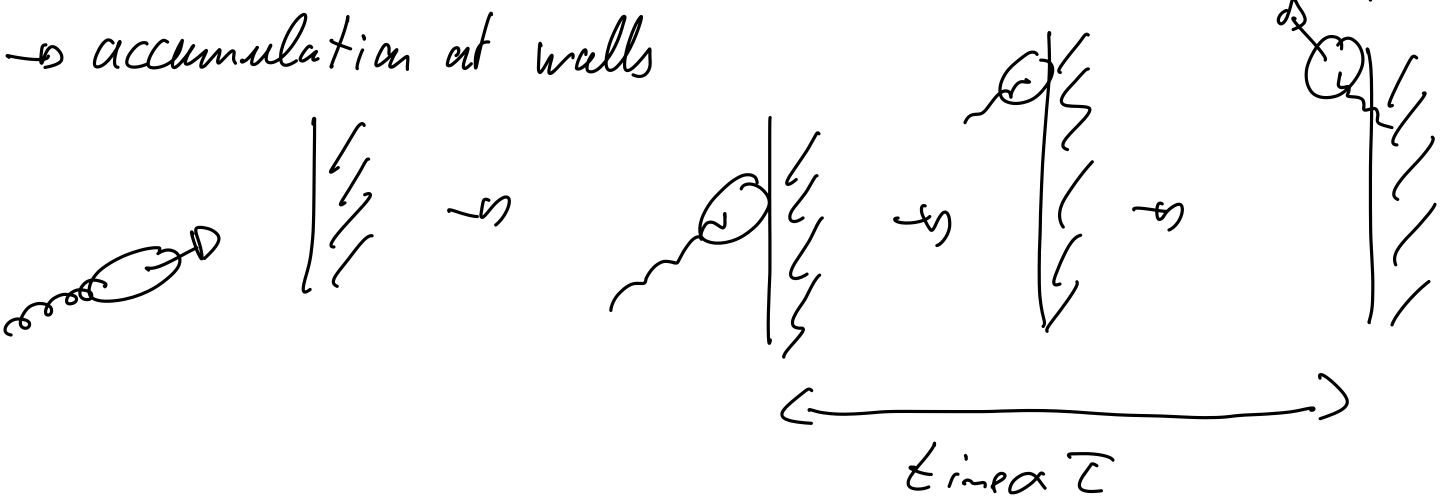
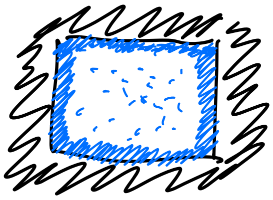
c) active particles and walls

This result is much more general than the case of harmonic traps.

$\rightarrow$ accumulation at walls



The persistence of active particles make them accumulate at walls



$\underline{v_p} \tau \gg L \Rightarrow$ all particles at the wall

$P(x)$ is then very different from $e^{-\beta V(x)}$

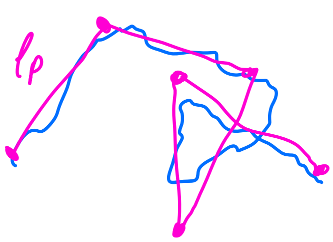
[Elgeti, Gompper, EPL 2009, 2013]

d) the effective equilibrium regime

Active particles are characterized by a persistence length ℓ_p and a persistence time τ.

Naive picture: Make steps in a random direction every $1/\tau$ time units.

$\Rightarrow$ passive random walk picture!



Exercise: write down the master equation of the particle position $\Rightarrow$ compute the diffusivity.

$[D] = L^2 . T^{-1} \Rightarrow D \propto \frac{\ell_p^2}{\tau} \times \frac{1}{d}$

$$\boxed{\text{diffusivity} : \lim_{t\to\infty} \frac{1}{t} \frac{\langle r^2(t)\rangle}{2d} \qquad \langle r^2 \rangle \sim 2dDt}$$

e.g. ABP, $\tau = \frac{1}{D_r}$, $\ell_p = \frac{v}{D_r} \Rightarrow D = \frac{v^2}{dD_r}$ ✓

RTP, $\tau = \frac{1}{\alpha}$, $\ell_p = \frac{v}{\alpha} \Rightarrow D = \frac{v^2}{d\alpha}$ ✓

⚠ This works because there are no other sources of fluctuations. If v_0 fluctuates, the relation between ℓ_p, τ & D is not universal.

The equilibrium limit

In the limit $\tau \to 0$, D constant, active dynamics become fully equivalent to colloidal dynamics at an effective temperature $kT_{eff} = \frac{D}{\mu}$ where D is the large-scale diffusivity of the active particle and μ is their mobility.

* This explains the $\tau \to 0$ result for the harmonic well above.

Take home message

① Universal equilibrium regime at $\tau = 0$, D constant

effective temperature : $kT_{eff} = \frac{D}{\mu}$

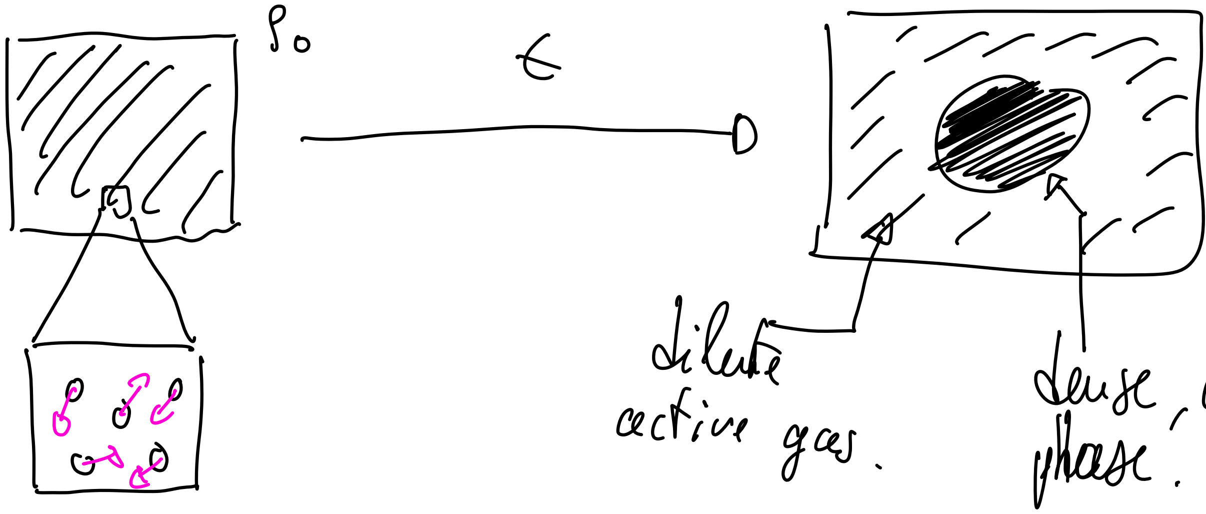
where D is the large scale diffusivity of the particle.

② Large $\tau \Rightarrow$ strong non-equilibrium effects

- accumulation at walls
- gravitational collapse
- non-monotonous $\rho(x)$ in traps.

7.7) Motility-induced phase separation

Phase transition through which an active system undergoes a phase separation resembling liquid-gas phase separation in the absence of attractive forces.



Very general in active matter:

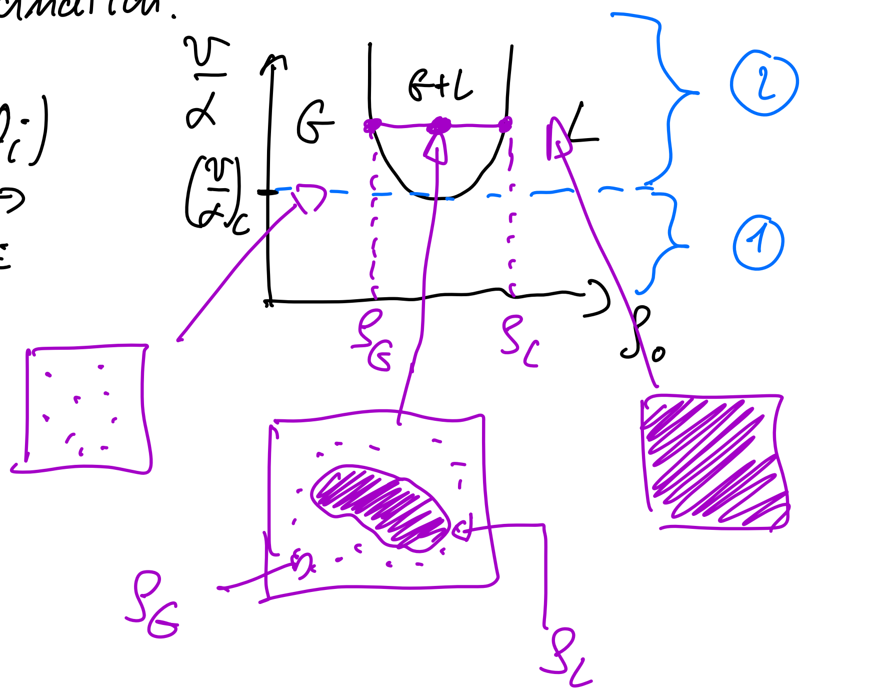
* RTPs [Tailleur, Cates, PRL 101, 218103, (2008)]

on lattice [Thompson et al., J. Stat. Mech. P02029, (2011)]

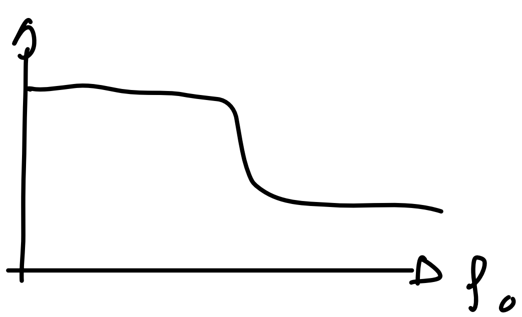
⇒ quorum-sensing interaction.

$$\dot{\vec{r}}_i = v(\vec{r}_i, [\rho])\,\vec{u}(\theta_i) + \sqrt{2D_t}\,\vec{\eta}_i$$

$$\theta_i \xrightarrow{\alpha} \theta_i'$$



$v(\vec{r}, \rho_0)$



In region (1), the translational noise is strong enough to enforce a homogeneous system.

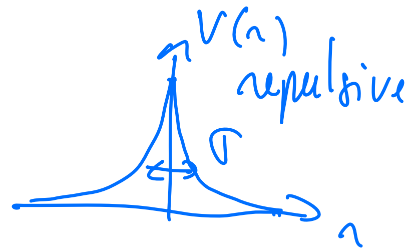
In region (2), Phase-separation is observed.

If $D_t \to 0$, $\left(\frac{v}{\alpha}\right)_c \to 0$; MIPS is seen everywhere.

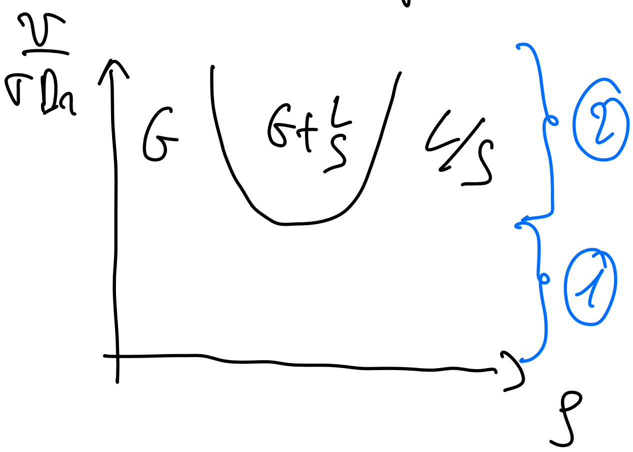
* ABPs: [Fily, Marchetti, PRL 108, 235702 (2012)]
[Redner, Hagan, Baskaran, PRL 110, 055701 (2013)]

→ pairwise forces

$$\begin{cases}\dot{\vec{r}}_i = v_0\, \vec{u}(\theta_i) - \sum\limits_{j\neq i} \vec{\nabla}_{\vec{r}_i} V(\vec{r}_j - \vec{r}_i)\\ \dot{\theta}_i = \sqrt{2D_r}\,\xi_i\end{cases}$$



Phase diagram



In region ①, as in equilibrium, pairwise repulsive forces leads to a homogeneous fluid.

In region ②, phase-separation occurs.

* In experiments:

→ Enhanced tendency for clustering: [Theurkauff et al, PRL 108, 268303 (2012); Palacci et al., Science 339, 936 (2013)]

→ Phase separation: [Buttinoni et al., PRL 2013, 110, 238301]

→ self-propelled Janus colloids

→ Bacterial colonies: [Liu et al, PRL 122, 248102, (2019)]

→ Quincke rollers: [Geyer et al, PRX 9, 031043 (2019)]
Coexistence between ordered polar active liquid & an arrested solid.

7.7.1) A linear stability analysis

ABP's or RTPs interacting via quorum-sensing interaction $v(\rho(r))$

① Non-uniform speed $v(\vec{r})$ without interactions in 2D with PBC

$$\partial_t P(\vec{r},\theta) = -\vec{\nabla}_{\vec{r}}\cdot\left[v(r)\,\vec{u}(\theta)\,P(\vec{r},\theta,t)\right] + \text{isotropic reorientation term } \textcircled{H}\cdot P$$

ABPs: $\textcircled{H} P = D_r\,\partial_{\theta\theta} P(\vec{r},\theta)$

RTPs: $\textcircled{H} P = -\alpha P(\vec{r},\theta) + \alpha \int \frac{d\theta'}{2\pi} P(\vec{r},\theta')$

Steady-state $\partial_t P(\vec{r},\theta) = 0$

⑥

(α) $P(\vec{r},\theta) = f(\vec{r})$ then (H) $P = 0$

(β) $\vec{\nabla}_{\vec{r}} \cdot \left[v(\vec{r})\, \vec{u}(\theta)\, f(\vec{r}) \right] = 0 \Rightarrow f(\vec{r}) = \dfrac{K}{v(\vec{r})}$

$$\vec{\nabla}_{\vec{r}} \cdot \Big[\underbrace{K\, \vec{u}(\theta)}_{\text{no } \vec{r}\text{-dependency}} \Big] = 0$$

$\Rightarrow$ Steady-state $\boxed{P_{ss}(\vec{r},\theta) = \dfrac{K}{v(\vec{r})}}$

Active particles spend more time when they go slowly

$\Rightarrow$ accumulation in slow regions

(1) Repulsive force / crowding: model $v(\rho(r))$ with $v'(\rho) < 0$

$\Rightarrow$ particles that go slower where are denser.

(1) model a lack of food $\to$ quorum-sensing [Curatolo et al, Nat. Phys. 2020]

(2) pairwise forces

$$\dot{\vec{r}}_i = v_0\, \vec{u}(\theta_i) + \vec{F}_i \,; \quad \vec{F}_i = -\sum_j \vec{\nabla}_{\vec{r}_i} V(\vec{r}_i - \vec{r}_j)$$

$$= \underbrace{\left[v_0 + \vec{F}_i \cdot \vec{u}(\theta_i) \right] \vec{u}(\theta_i)} + \left(1 - \vec{u}(\theta_i)\, \vec{u}(\theta_i)\right) \vec{F}_i$$

$$= \text{“} v(\rho) \text{”}\ \vec{u}(\theta_i) + 0$$

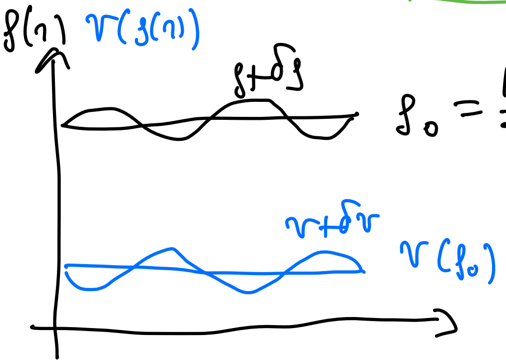
MF + neglect 2nd term.

Repulsive forces lead to a decrease of $v(\rho)$ as

$$v(\rho) \simeq v_0 \left(1 - \frac{\rho}{\rho^*}\right)$$ [Fily et al, PRL 2012]

ρ^*: crowding density

Feed back loop leading to MIPS:



$\rho_0 = \frac{K}{v_0}$

Perturbation $\rho(r) = \rho_0 + \delta\rho(r)$

$\delta\rho(r) \ll \rho_0$

Since $v(\rho(r)) = v(\rho_0) + \delta v(r)$

$= v(\rho_0) + v'(\rho_0)\,\delta\rho$

Q: where does $\rho(\vec{r})$ wants to relax?

$\rho(\vec{r})$ is a conserved field $\Rightarrow$ its evolution on large scales is slow

Locally particles want to relax towards $\rho(\vec{r}) \propto \frac{K}{v(\vec{r})}$

$$\frac{K}{v(\rho(r))} = \frac{K}{v(\rho_0)+v'(\rho_0)\,\delta\rho} = \frac{K}{v(\rho_0)} \cdot \frac{1}{1+\frac{v'}{v}\delta\rho} = \underbrace{\frac{K}{v(\rho_0)}}_{\rho_0}\left(1 - \frac{v'(\rho_0)}{v(\rho_0)}\,\delta\rho\right)$$

Start from $\rho_0 + \delta\rho$; relax to $\rho_0 - \frac{v'(\rho_0)}{v(\rho_0)}\,\rho_0\,\delta\rho$

if $|\delta\rho| < \left|-\frac{v'}{v}\rho_0\,\delta\rho\right| \Rightarrow$ perturbation is amplified $\Rightarrow$ instability

linear instability criterion: $\boxed{\frac{v'(\rho_0)}{v(\rho_0)} < -\frac{1}{\rho_0}}$

Conclusion: When $v'(\rho_0)$ is sufficiently negative, this hand-waving argument predicts a linear instability.

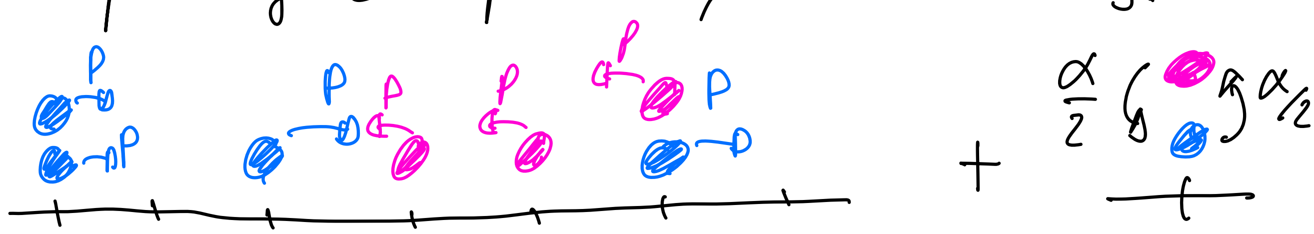
Q: Can we derive this quantitatively in a specific model?

Comment: This argument can be made more quantitative and can be generalized to non-vanishing translational diffusion [Cates, Tailleur, Ann. Rev. Cond. Mat. Physics 6, 219-244 (2015)]

7.7.2) RTPs on lattice in 1D

More general results on the phase-separation, surface tension and how to compute the phase diagram in off-lattice models can be found in: [Solon et al, New Journal of Physics 20, 075001 (2018)]

Here, we focus on the simpler case of on-lattice RTPs in 1d. (inspired by [Thompson et al, J. Stat. Mech 2011].



Lattice gas model of non-interacting run-and-tumble bacteria.

Configurations $\varphi \Leftrightarrow$ occupancies (n_i^+, n_i^-) which describe the number of particles hopping to the right or to the left on site i.

Add excluded volume interaction: partial exclusion

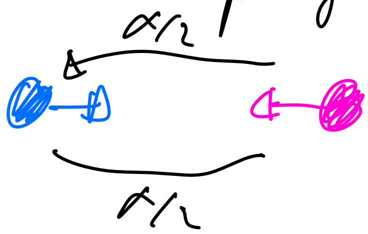
Hopping $i \to i+1$ for a right going particle $p\left(1-\frac{n_{i+1}}{n_M}\right)$

$i \to i-1$ —— left —— $p\left(1-\frac{n_{i-1}}{n_M}\right)$

$n_i = n_i^+ + n_i^-$ is the total occupancy.

n_M is the maximal occupancy allowed on a lattice site

Tumbling rate



Idea: Describe the dynamics of $\rho_i^\pm = \langle n_i^\pm(x)\rangle$ and show that homogeneous profiles are unstable.

Master equation:

$$\partial_t P(\varphi) = \sum_{\varphi' \neq \varphi} W(\varphi' \to \varphi) P(\varphi') - W(\varphi \to \varphi') P(\varphi)$$

Here $\varphi = \{n_i^+, n_i^-\} = \{n_1^+, n_1^-, n_2^+, n_2^-, \dots, n_L^+, n_L^-\}$

(1) φ' such that $W(\varphi' \to \varphi) \neq 0$

$* \ \varphi' = \{n_1^+, n_1^-, \dots, n_{i-1}^+ + 1, n_{i-1}^-, n_i^+ - 1, n_i^-, \dots, n_L^+, n_L^-\} \equiv \{n_{i-1}^+ + 1, n_i^+ - 1\}$

$\varphi' = n_{i-1}^+ + 1,\ n_i^+ - 1 \longrightarrow \varphi$ at rate $\boxed{W = p\,(n_{i-1}^+ + 1)\left(1 - \frac{n_i - 1}{n_M}\right)}$

and $n_{j \neq i}^{+-}$ same as in φ