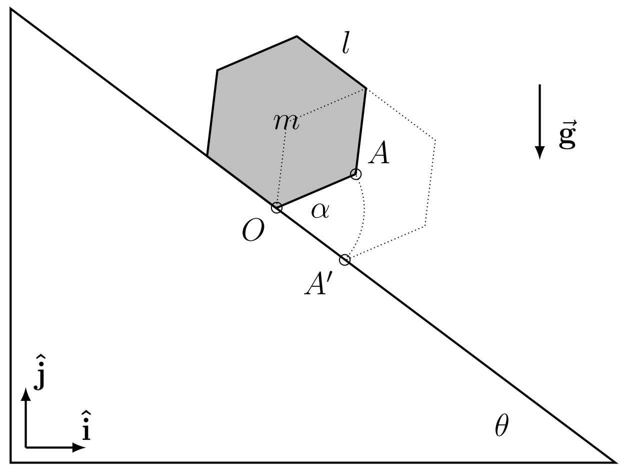
2.003J Spring 2011: Dynamics and Control I	**Recitation 01**
Massachusetts Institute of Technology	**February 3rd, 2011**
Department of Mechanical Engineering	**February 4th, 2011**

Problem 1



A pencil with regular hexagonal cross-section, mass m, and side length l is placed on a no-slip inclined surface. Define I as the moment of inertia of the pencil about an edge. Given that $\theta > 30°$, it begins to rotate due to gravity.

(a) Find the velocity $\vec{\mathbf{v}}_A$ of point A just before it hits the incline.

(b) Derive a differential equation that describes the motion of point A as a function of α.

Solution

(a) The pencil will rotate about point O traversing a circular path. Because of this geometry, we know that the direction of the velocity $\vec{\mathbf{v}}_A$ when A reaches the incline must be perpendicular to the inclined surface, or $\vec{\mathbf{v}}_A/|\vec{\mathbf{v}}_A| = (-\sin\theta, -\cos\theta)$. To fine the magnitude of the velocity, we use energy conservation:

$$E_i = E_f \implies \Delta PE = KE_f \implies mgh = \frac{1}{2}I\omega^2$$

From the initial to final positions, the center of mass moves in the direction of the incline by a distance l, thus taking the vertical component yields $h = l\sin\theta$. Also, since point A is rotating at a radius $r = l$, its velocity is related to the angular velocity by $l\omega = |\vec{\mathbf{v}}_A|$. Combining equations yields:

$$\boxed{\vec{\mathbf{v}}_A = |\vec{\mathbf{v}}_A|(-\sin\theta, -\cos\theta) = \sqrt{\frac{2mgl^3\sin\theta}{I}}(-\sin\theta, -\cos\theta)}$$

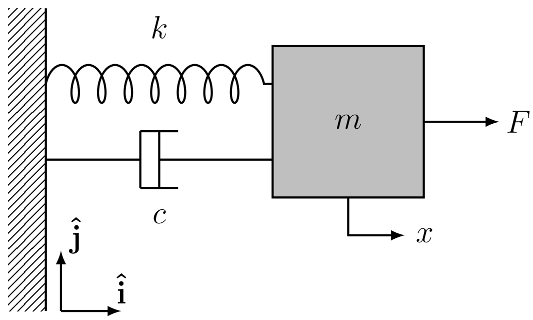
(b) Gravity drives the movement of the pencil by exerting a torque τ on the pencil's center of mass. The magnitude of this torque increases while the pencil rotates as the lever arm r increases. The length of this lever arm is equal to $r = l\cos\phi$ where ϕ is the angle that the line from the center of mass to O makes with the horizontal. A little geometry gives us that $\phi = 60^\circ + \alpha - \theta$. We can use Newton's Second law for a rotational system to derive the equation of motion:

$$I\ddot{\alpha} = \tau = mgr = mglcos(60^\circ + \alpha - \theta)$$

$$\boxed{I\ddot{\alpha} - mgl\cos(60^\circ + \alpha - \theta) = 0}$$

This is a very complicated non-linear differential equation. Solving differential equations like this is beyond the scope of this course. Even so, it is worth noting that we were still able to derive information about the system by exploiting energy conservation in part (a).

Problem 2



Mass m is connected to a fixed wall by a spring k and dashpot c in the absence of gravity. Initially the system is at rest with $x(0) = 0$ and $\dot{x}(0) = 0$. At time $t = 0$, a constant force $F\,\hat{\mathbf{i}}$ is exerted on the mass.

(a) Derive a differential equation that describes the motion of mass m.

(b) Solve for the position of mass m as a function of time.

Solution

(a) The spring will impart a force kx proportional to displacement x. The dashpot will impart a force $c\dot{x}$ proportional to velocity $\dot{x}$. We can use Newton's Second law directly to derive the equation of motion:

$$m\ddot{x} = \sum F = F - kx - c\dot{x}$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = F}$$

(b) $x(t)$ will be a combination of a homogeneous solution and a particular solution, $x(t) = x_h(t) + x_p(t)$ evaluated using the given initial conditions. The particular solution for a constant force should also be a constant. $\ddot{x}$ and $\dot{x}$ terms will go to zero, thus:

$$x_p(t) = \frac{F}{k}$$

This particular solution represents the steady state behavior of the system. As $t \to \infty$, $x(t)$ will look like $\frac{F}{k}$. The homogeneous solution represents the transient, or time dependent behavior of the system. The homogeneous solution to a second order damped system will be a family of exponential solutions of the form Ae^{st}. Substituting into the homogeneous equation, we arrive at the differential equation's characteristic equation:

$$m\frac{d^2}{dt^2}Ae^{st} + c\frac{d}{dt}Ae^{st} + kAe^{st} = 0 \qquad ms^2 + cs + k = 0 \qquad s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

To better understand the parameters of the system, we can make a change of variables to the system's natural frequency ω and a non-dimensional damping coefficient ξ given by:

$$s = -\xi\omega \pm \omega\sqrt{\xi^2 - 1} \qquad \omega = \sqrt{\frac{k}{m}} \qquad \xi = \frac{c}{2\sqrt{m\omega}}$$

We have solved for the exponent of our proposed solution of the form Ae^{st}. There are two parts to this solution. The first part has the form $e^{-\xi\omega}$. This negative exponential represents the system's damping behavior, with the transient solution decaying to zero as $t \to \infty$. The second part has the form $e^{\pm\omega\sqrt{\xi^2-1}}$. This part has generally two regimes. If $\xi^2 - 1 > 0$, this part represents additional damping to the system, and we refer to the system as "over-damped". If $\xi^2 - 1 < 0$, this part represents an oscillatory response with the presence of a complex exponential, and we refer to the system as "under-damped". The boundary between these two regimes is when $\xi^2 - 1 = 0$, and this term vanishes. We call a system on this boundary "critically damped". The solution will be different with respect to each regime.

Over-damped An over-damped system be a linear combination of the two characteristic equation solutions and will have a homogeneous solution of the form:

$$x_h(t) = Ae^{s_1 t} + Be^{s_2 t} \qquad s_1 = -\xi\omega + \omega\sqrt{\xi^2 - 1} \qquad s_2 = -\xi\omega - \omega\sqrt{\xi^2 - 1}$$

Combining with the particular solution, the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ become:

$$0 = A + B + \frac{F}{k} \qquad 0 = As_1 + Bs_2 \qquad | \qquad A = \frac{F}{k}\frac{s_2}{s_1 - s_2} \qquad B = -\frac{F}{k}\frac{s_1}{s_1 - s_2}$$

Yielding the solution:

$$\boxed{x(t) = \frac{F}{k} + \frac{F}{k}\frac{s_2 e^{s_1 t} - s_1 e^{s_2 t}}{s_1 - s_2}}$$

Critically Damped A critically damped system has repeated roots so its homogeneous equation will have the form:

$$x_h(t) = (A + Bt)e^{st} \qquad s_1 = -\xi\omega$$

Combining with the particular solution, the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ become:

$$0 = A + \frac{F}{k} \qquad 0 = As + B \qquad | \qquad A = -\frac{F}{k} \qquad B = \frac{Fs}{k}$$

Yielding the solution:

$$\boxed{x(t) = \frac{F}{k} + \frac{F}{k}(st - 1)e^{st}}$$

Under-damped An under-damped system has complex roots and its homogeneous equation will have the form:

$$x_h(t) = e^{-\xi\omega t}\left(Ae^{i\omega_d t} + Be^{-i\omega_d t}\right) \qquad \omega_d = \omega\sqrt{1-\xi^2}$$

Remembering a complex exponential identity that $\cos x = \frac{1}{2}(e^{ix}+e^{-ix})$ and adjusting our unknown constant variables, we can rewrite this solution as:

$$x_h(t) = e^{-\xi\omega t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$

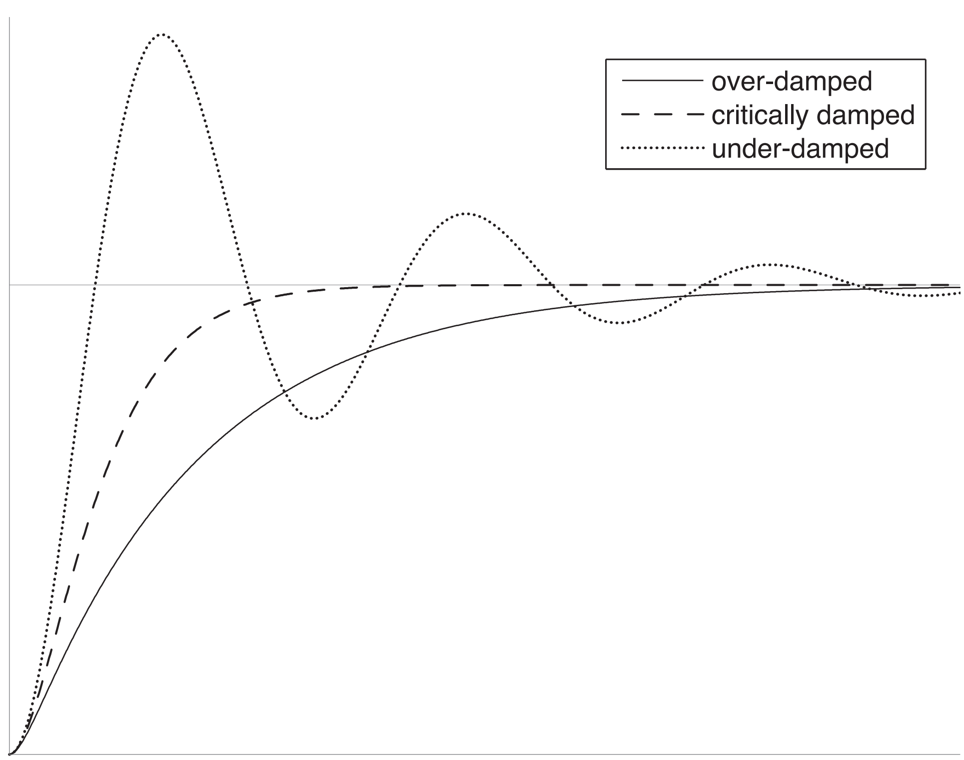
Combining with the particular solution, the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ become:

$$0 = B + \frac{F}{k} \qquad 0 = A\omega_d - B\xi\omega \qquad | \qquad A = -\frac{F}{k}\frac{\omega}{\omega_d}\xi \qquad B = -\frac{F}{k}$$

Yielding the solution:

$$\boxed{x(t) = \frac{F}{k} - \frac{F}{k}e^{-\xi\omega t}\left(\frac{\omega}{\omega_d}\xi\sin\omega_d t - \cos\omega_d t\right)}$$

Below is a graph of all three solutions for three different values of ξ. Over-damped is shown as a solid line with $\xi = 2$. Critically damped is shown as a dashed line with $\xi = 1$. Under-damped is shown as a dotted line with $\xi = 0.2$. A line at $x = \frac{F}{k}$ is shown in gray marking the steady state solution. Note that $x(0) = 0$ and $\dot{x}(0) = 0$ as was specified.

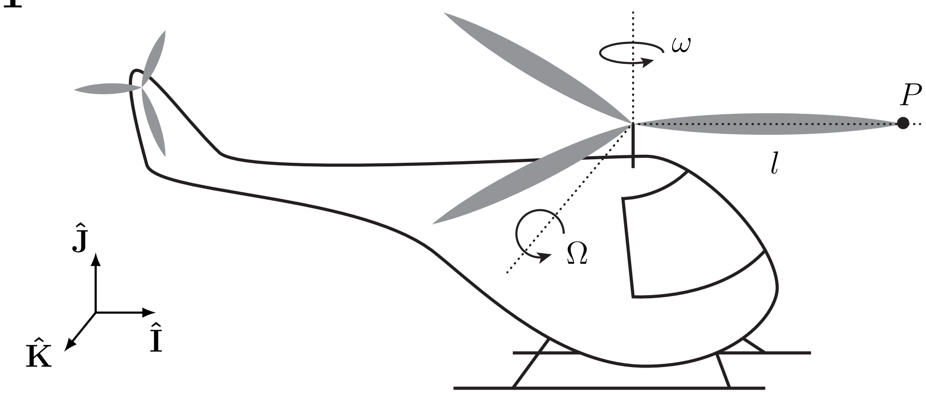


Here is the MATLAB code used to generate the above graph.

```
% define time step and system frequency
t = 0:0.001:1;
omega = 20;

% over-damped damping parameter
xi = 2;
omegad = omega*sqrt(xi^2-1);
s1 = -xi*omega+omegad;
s2 = -xi*omega-omegad;
x1 = (s2*exp(s1*t)-s1*exp(s2*t))/(s1-s2)+1;

% critically damped damping parameter
xi = 1;
s = -xi*omega;
x2 = (s*t-1).*exp(s*t)+1;

% under-damped damping parameter
xi = 0.2;
omegad = omega*sqrt(1-xi^2);
x3 = 1-exp(-xi*omega*t).*(omega/omegad*xi*sin(omega*t)+cos(omegad*t));

% steady state
x4 = t*0+1;

% plot
plot(t,x1,‘k’,t,x2,‘k--’,t,x3,‘k-.’,t,x4,‘k:’)
legend(‘over-damped’,‘critically damped’,‘under-damped’)
axis off; set(gcf,‘Color’,‘w’)
```

2.003J Spring 2011: Dynamics and Control I — **Recitation 02**
Massachusetts Institute of Technology — **February 10th, 2011**
Department of Mechanical Engineering — **February 11th, 2011**

Problem 1



A helicopter is hovering in the air. The blades of its main rotor have length l and are spinning with angular velocity ω with respect to the helicopter. At the time shown, the helicopter is also tilting upward with constant angular velocity Ω with respect to ground. Find the velocity and acceleration of point P at this instant.

Solution

First, define our reference frames. Define point O as the center of the rotor. Define inertial ground reference frame $\hat{\mathrm{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, frame $\hat{\mathrm{H}} = (H, \hat{\mathbf{i}}_H, \hat{\mathbf{j}}_H, \hat{\mathbf{k}}_H)$ attached to the helicopter with $H \equiv O$, and frame $\hat{\mathrm{R}} = (R, \hat{\mathbf{i}}_R, \hat{\mathbf{j}}_R, \hat{\mathbf{k}}_R)$ attached to the rotors also with $R \equiv O$.

Second, write the given variables with respect to these reference frames. We have:

$$^{\mathrm{R}}\boldsymbol{r}_P = l\,\hat{\mathbf{i}}_R \qquad {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} = \Omega\,\hat{\mathbf{K}} \qquad {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}} = \omega\,\hat{\mathbf{j}}_H$$

Note that at this instant, $\hat{\mathbf{i}}_R = \hat{\mathbf{I}}$ and $\hat{\mathbf{j}}_H = \hat{\mathbf{J}}$. These coordinate orientations were chosen for convenience. Recall the general formula for taking the derivative with respect to frame $\hat{\mathrm{O}}$ of a vector $\boldsymbol{r}$ defined in frame $\hat{\mathrm{A}}$. We will use this relation often in our derivation:

$$\frac{^{\mathrm{O}}d}{dt}\left(^{\mathrm{A}}\boldsymbol{r}_p\right) = \frac{^{\mathrm{A}}d}{dt}\left(^{\mathrm{A}}\boldsymbol{r}_p\right) + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_p = {}^{\mathrm{A}}\dot{\boldsymbol{r}}_p + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_p$$

Now, we solve for variable $^{\mathrm{O}}\boldsymbol{v}_P$:

$$^{\mathrm{O}}\boldsymbol{v}_P = \frac{^{\mathrm{O}}d}{dt}\left(^{\mathrm{O}}\boldsymbol{r}_P\right) = \frac{^{\mathrm{O}}d}{dt}\left(\cancelto{0}{^{\mathrm{O}}\boldsymbol{r}_H} + \cancelto{0}{^{\mathrm{H}}\boldsymbol{r}_R} + {}^{\mathrm{R}}\boldsymbol{r}_P\right) = \frac{^{\mathrm{O}}d}{dt}(^{\mathrm{R}}\boldsymbol{r}_P) = \cancelto{0}{^{\mathrm{R}}\dot{\boldsymbol{r}}_p} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{R}} \times {}^{\mathrm{R}}\boldsymbol{r}_p = {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{R}} \times {}^{\mathrm{R}}\boldsymbol{r}_p$$

Here, we note that the rotation of the rotors with respect to the inertial frame, $^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{R}} = {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} + {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}$. Thus:

$$= (^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} + {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}) \times {}^{\mathrm{R}}\boldsymbol{r}_p = (\omega\,\hat{\mathbf{j}}_H + \Omega\,\hat{\mathbf{K}}) \times l\,\hat{\mathbf{i}}_R = \boxed{l(\Omega\hat{\mathbf{J}} - \omega\hat{\mathbf{K}})}$$

Lastly, we solve for ${}^{\mathrm{O}}\boldsymbol{a}_P$:

$$ {}^{\mathrm{O}}\boldsymbol{a}_P = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{v}_P\right) = \frac{{}^{\mathrm{O}}d}{dt}\left(({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} + {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}) \times {}^{\mathrm{R}}\boldsymbol{r}_p\right) = \left(\cancelto{0}{{}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_H} + \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}})\right) \times {}^{\mathrm{R}}\boldsymbol{r}_p + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} + {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}) \times \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{R}}\boldsymbol{r}_P) $$

$$ = \left(\cancelto{0}{{}^{\mathrm{H}}\dot{\boldsymbol{\omega}}_{\mathrm{R}}} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} \times {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}\right) \times {}^{\mathrm{R}}\boldsymbol{r}_p + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} + {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}) \times \left(\cancelto{0}{{}^{\mathrm{R}}\dot{\boldsymbol{r}}_p} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{R}} \times {}^{\mathrm{R}}\boldsymbol{r}_p\right) $$

$$ = \left({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} \times {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}\right) \times {}^{\mathrm{R}}\boldsymbol{r}_p + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} + {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}) \times \left(({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} + {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}) \times {}^{\mathrm{R}}\boldsymbol{r}_p\right) $$

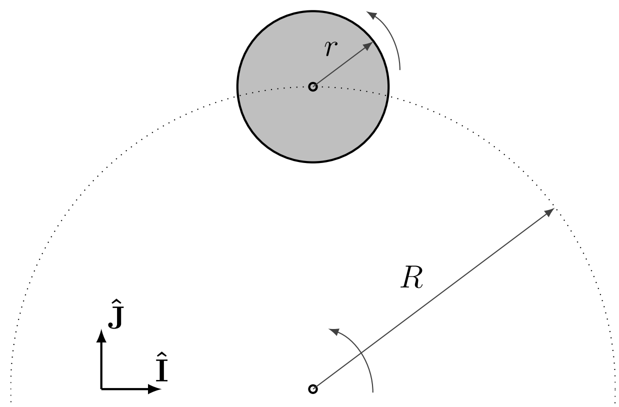
$$ = \cancelto{0}{(\Omega\,\hat{\mathbf{K}} \times \omega\,\hat{\mathbf{j}}_H) \times l\,\hat{\mathbf{i}}_R} + (\Omega\,\hat{\mathbf{K}} + \omega\,\hat{\mathbf{j}}_H) \times \left((\Omega\,\hat{\mathbf{K}} + \omega\,\hat{\mathbf{j}}_H) \times l\,\hat{\mathbf{i}}_R\right) = \boxed{-l(\Omega^2 + \omega^2)\hat{\mathbf{I}}} $$

Note that the eulerian term $\left({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{H}} \times {}^{\mathrm{H}}\boldsymbol{\omega}_{\mathrm{R}}\right) \times {}^{\mathrm{R}}\boldsymbol{r}_p$ happens to be zero at this particular location in the reference frame of the blades, but this is not the case in general. If we had instead asked for the the acceleration of a point P' with ${}^{\mathrm{R}}\boldsymbol{r}_{P'} = l\,\hat{\mathbf{k}}_R$, with $\hat{\mathbf{k}}_R = \hat{\mathbf{K}}$ at this instant, we would instead have:

$$ = (\Omega\,\hat{\mathbf{K}} \times \omega\,\hat{\mathbf{j}}_H) \times l\,\hat{\mathbf{k}}_R + (\Omega\,\hat{\mathbf{K}} + \omega\,\hat{\mathbf{j}}_H) \times \left((\Omega\,\hat{\mathbf{K}} + \omega\,\hat{\mathbf{j}}_H) \times l\,\hat{\mathbf{k}}_R\right) = -\Omega\omega l\hat{\mathbf{J}} - l(\Omega^2 + \omega^2)\hat{\mathbf{K}} $$

Problem 2

Recall our setup from the Problem Set 01 survey questions, but with the Earth as our planet.



The Earth has an approximate radius $r \approx 6400$ km, and orbits the sun at an approximate radius $R \approx 150$ million km. Given that an earth day is approximately $t \approx 24$ hrs and a year is approximately $T \approx 365$ earth days, what is the maximum velocity experienced by any point on the earth with respect to a stationary frame? What is the maximum acceleration? How does this acceleration compare to the acceleration of gravity on the earth's surface?

Solution

First, define our reference frames. Define point O as the center of rotation of the earth about the sun and point B at the center of the earth. Define inertial ground reference frame $\hat{\mathrm{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, frame $\hat{\mathrm{A}} = (A, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A)$ with origin $A \equiv O$ rotating with the earth about the sun, and frame $\hat{\mathrm{B}} = (B, \hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B, \hat{\mathbf{k}}_B)$ with origin B and fixed to the earth. Since the motion is planar, we will let $\hat{\mathbf{K}} = \hat{\mathbf{k}}_A = \hat{\mathbf{k}}_B$.

Second, write the given variables with respect to these reference frames. We have:

$$^{\mathrm{A}}\boldsymbol{r}_B = R\,\hat{\mathbf{i}}_A \qquad ^{\mathrm{B}}\boldsymbol{r}_P = r\,\hat{\mathbf{i}}_B \qquad ^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} = \frac{2\pi}{T}\hat{\mathbf{K}} \qquad ^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} = \frac{2\pi}{t}\hat{\mathbf{K}}$$

Recall the general formula for taking the derivative with respect to frame $\hat{\mathrm{O}}$ of a vector $\boldsymbol{r}$ defined in frame $\hat{\mathrm{A}}$. We will use this relation often in our derivation:

$$\frac{^{\mathrm{O}}d}{dt}\left(^{\mathrm{A}}\boldsymbol{r}_p\right) = \frac{^{\mathrm{A}}d}{dt}\left(^{\mathrm{A}}\boldsymbol{r}_p\right) + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_p = {}^{\mathrm{A}}\dot{\boldsymbol{r}}_p + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_p$$

Now, we solve for variable ${}^{\mathrm{O}}\boldsymbol{v}_P$:

$$ {}^{\mathrm{O}}\boldsymbol{v}_P = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{r}_P\right) = \frac{{}^{\mathrm{O}}d}{dt}\left(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{r}_A} + {}^{\mathrm{A}}\boldsymbol{r}_B + {}^{\mathrm{B}}\boldsymbol{r}_P\right) = \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{A}}\boldsymbol{r}_B) + \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{B}}\boldsymbol{r}_P) $$

$$ = \cancelto{0}{{}^{\mathrm{A}}\dot{\boldsymbol{r}}_B} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + \cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_P} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_P = {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_P $$

Here, we note that the rotation of the earth with respect to the inertial frame, ${}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} = {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}$. Thus:

$$ {}^{\mathrm{O}}\boldsymbol{v}_P = {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_P $$

$$ = 2\pi\frac{R}{T}(\hat{\mathbf{K}} \times \hat{\mathbf{i}}_A) + 2\pi\left(\frac{r}{T} + \frac{r}{t}\right)(\hat{\mathbf{K}} \times \hat{\mathbf{i}}_B) $$

This velocity will be at its maximum when $\hat{\mathbf{K}} \times \hat{\mathbf{i}}_A = \hat{\mathbf{K}} \times \hat{\mathbf{i}}_B$, or when $\hat{\mathbf{i}}_A = \hat{\mathbf{i}}_B$. Thus, plugging in the given values for R, r, T, and t yields:

$$ \left|{}^{\mathrm{O}}\boldsymbol{v}_P\right|_{max} = 2\pi\left(\frac{R}{T} + \frac{r}{T} + \frac{r}{t}\right) \approx \boxed{30{,}352 \text{ m/s}} $$

Lastly, we solve for ${}^{\mathrm{O}}\boldsymbol{a}_P$:

$$ {}^{\mathrm{O}}\boldsymbol{a}_P = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{v}_P\right) = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_P\right) $$

$$ = \cancelto{0}{{}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{A}}\boldsymbol{r}_B) + \left(\cancelto{0}{{}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}}} + \cancelto{0}{{}^{\mathrm{A}}\dot{\boldsymbol{\omega}}_{\mathrm{B}}}\right) \times {}^{\mathrm{B}}\boldsymbol{r}_P + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{B}}\boldsymbol{r}_P) $$

$$ = {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times \left(\cancelto{0}{{}^{\mathrm{A}}\dot{\boldsymbol{r}}_B} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B\right) + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times \left(\cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_P} + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_P\right) $$

$$ = {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times \left({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B\right) + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times \left(({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_P\right) $$

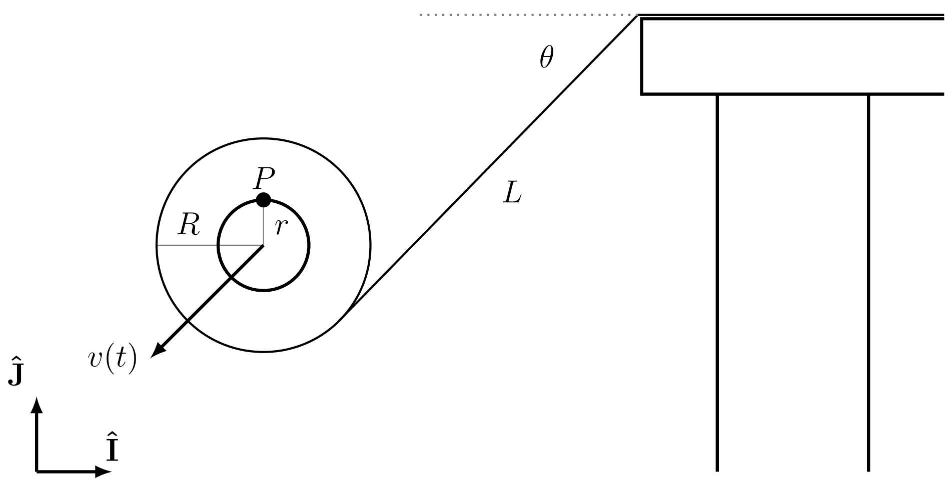
$$ = -4\pi^2\left(\frac{R}{T^2}\hat{\mathbf{i}}_A + \left(\frac{r}{T} + \frac{r}{t}\right)^2 \hat{\mathbf{i}}_B\right) $$

The magnitude of the acceleration will be at its maximum when $\hat{\mathbf{i}}_A = \hat{\mathbf{i}}_B$. Thus, plugging in the given values for R, r, T, and t yields:

$$ \left|{}^{\mathrm{O}}\boldsymbol{a}_P\right|_{max} = 4\pi^2\left(\frac{R}{T^2} + \left(\frac{r}{T} + \frac{r}{t}\right)^2\right) \approx \boxed{0.03999 \text{ m/s}^2} $$

The acceleration of gravity on the earth's surface is given by $\boldsymbol{g} \approx 9.8$ m/s^2. Because $\boldsymbol{g} \approx 245\left|{}^{\mathrm{O}}\boldsymbol{a}_P\right|_{max}$, we conclude that $\boxed{\boldsymbol{g} \gg \left|{}^{\mathrm{O}}\boldsymbol{a}_P\right|_{max}}$

Problem 1



An ant sits inside a roll of toilet paper with inside and outside radius r and R respectively, which itself sits on a table. A cat comes by, steps on the free end of the toilet paper, and hits the roll off the table, sending it unrolling with a variable velocity $v(t)$. After the roll leaves the table, it continues to unroll at velocity $v(t)$, but also begins to rotate about the corner with angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$. Find the velocity and acceleration of the ant located at point P with respect to the ground when the amount the toilet paper that has unrolled off the table is given by L. Assume that at this time, the ant is positioned at the top of the inner tube.

Solution

First, define our reference frames. Define point $O \equiv A$ at the corner of the table and point T at the center of the roll. Define inertial ground reference frame $\hat{\text{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, frame $\hat{\text{A}} = (A = O, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A = \hat{\mathbf{K}})$ rotating with the unrolled tissue about the corner, and frame $\hat{\text{T}} = (T, \hat{\mathbf{i}}_T, \hat{\mathbf{j}}_T, \hat{\mathbf{k}}_A = \hat{\mathbf{K}})$. For convenience, let $\hat{\mathbf{j}}_A$ point in the direction of the unrolled tissue, and at this time instant, let $\hat{\mathbf{j}}_T$ point in the $\hat{\mathbf{J}}$ direction.

Second, write the given variables with respect to these reference frames. We have at this instant:

$$ {}^{\text{O}}\boldsymbol{r}_A = \mathbf{0} \qquad {}^{\text{O}}\dot{\boldsymbol{r}}_A = \mathbf{0} \qquad {}^{\text{A}}\boldsymbol{r}_T = L\hat{\mathbf{j}}_A + R\hat{\mathbf{i}}_A \qquad {}^{\text{A}}\dot{\boldsymbol{r}}_T = v(t)\hat{\mathbf{j}}_A \qquad {}^{\text{T}}\boldsymbol{r}_P = r\hat{\mathbf{j}}_T \qquad {}^{\text{T}}\dot{\boldsymbol{r}}_P = \mathbf{0} $$

We also must specify the angular velocity of each reference frame with respect to one another. ${}^{\text{A}}\boldsymbol{\omega}_{\text{T}}$ should be constant and be related to the speed $v(t)$ at which the roll is unrolling, while ${}^{\text{O}}\boldsymbol{\omega}_{\text{A}}$ should be changing with time according to $\dot{\theta}$:

$$ {}^{\text{O}}\boldsymbol{\omega}_{\text{A}} = \dot{\theta}\hat{\mathbf{K}} \qquad {}^{\text{O}}\dot{\boldsymbol{\omega}}_{\text{A}} = \ddot{\theta}\hat{\mathbf{K}} \qquad {}^{\text{A}}\boldsymbol{\omega}_{\text{T}} = \frac{v(t)}{R}\hat{\mathbf{k}}_A \qquad {}^{\text{A}}\dot{\boldsymbol{\omega}}_{\text{T}} = \frac{\dot{v}(t)}{R}\hat{\mathbf{k}}_A $$

Recall the general formula for taking the derivative with respect to frame $\hat{\text{A}}$ of a vector defined in $\hat{\text{B}}$:

$$\frac{{}^{\text{A}}d}{dt}\left({}^{\text{B}}\boldsymbol{r}_p\right) = \frac{{}^{\text{B}}d}{dt}\left({}^{\text{B}}\boldsymbol{r}_p\right) + {}^{\text{A}}\boldsymbol{\omega}_{\text{B}} \times {}^{\text{B}}\boldsymbol{r}_p = {}^{\text{B}}\dot{\boldsymbol{r}}_p + {}^{\text{A}}\boldsymbol{\omega}_{\text{B}} \times {}^{\text{B}}\boldsymbol{r}_p$$

Now we can, we solve for variable ${}^{\text{O}}\boldsymbol{v}_P$:

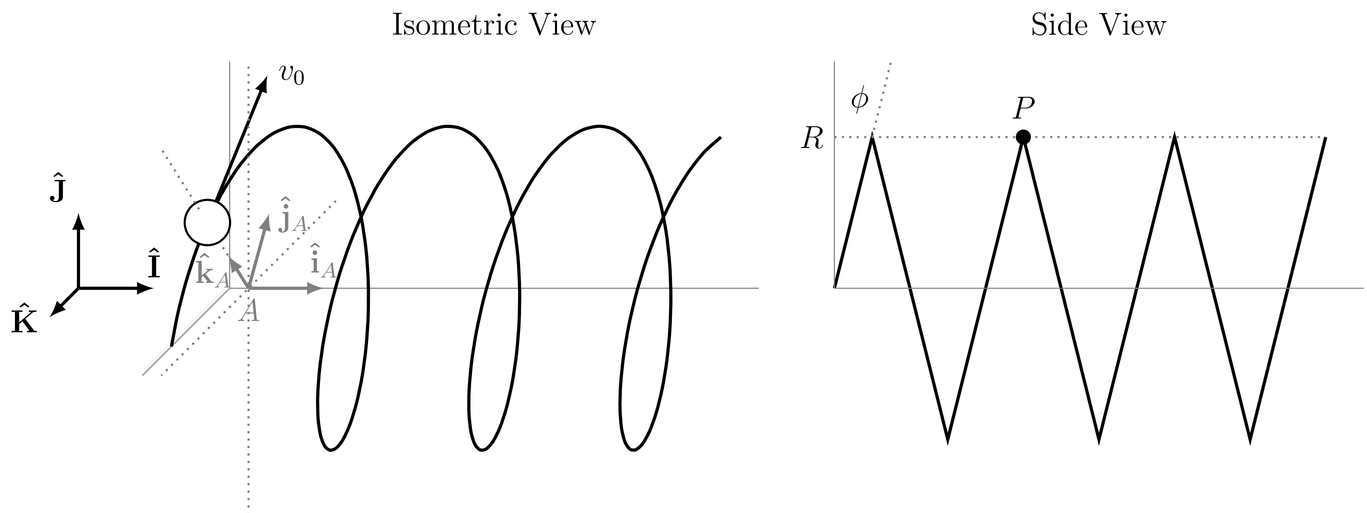
$$\begin{aligned}
{}^{\text{O}}\boldsymbol{v}_P = \frac{{}^{\text{O}}d}{dt}({}^{\text{O}}\boldsymbol{r}_P) &= \frac{{}^{\text{O}}d}{dt}(\cancelto{0}{{}^{\text{O}}\boldsymbol{r}_A} + {}^{\text{A}}\boldsymbol{r}_T + {}^{\text{T}}\boldsymbol{r}_P) \\
&= ({}^{\text{A}}\dot{\boldsymbol{r}}_T + {}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T) + (\cancelto{0}{{}^{\text{T}}\dot{\boldsymbol{r}}_P} + {}^{\text{O}}\boldsymbol{\omega}_{\text{T}} \times {}^{\text{T}}\boldsymbol{r}_P) \\
&= {}^{\text{A}}\dot{\boldsymbol{r}}_T + {}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T + ({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} + {}^{\text{A}}\boldsymbol{\omega}_{\text{T}}) \times {}^{\text{T}}\boldsymbol{r}_P \\
&= v(t)\hat{\mathbf{j}}_A + \dot{\theta}\hat{\mathbf{K}} \times (L\hat{\mathbf{j}}_A + R\hat{\mathbf{i}}_A) + \left(\dot{\theta}\hat{\mathbf{K}} + \frac{v(t)}{R}\hat{\mathbf{k}}_A\right) \times r\hat{\mathbf{j}}_T \\
&= \boxed{-\dot{\theta}L\hat{\mathbf{i}}_A + \left(v(t) + \dot{\theta}R\right)\hat{\mathbf{j}}_A - \left(\dot{\theta}r + \frac{r}{R}v(t)\right)\hat{\mathbf{i}}_T}
\end{aligned}$$

Lastly, we solve for ${}^{\text{O}}\boldsymbol{a}_P$:

$$\begin{aligned}
{}^{\text{O}}\boldsymbol{a}_P &= \frac{{}^{\text{O}}d}{dt}({}^{\text{O}}\boldsymbol{v}_P) = \frac{{}^{\text{O}}d}{dt}\left[{}^{\text{A}}\dot{\boldsymbol{r}}_T + {}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T + ({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} + {}^{\text{A}}\boldsymbol{\omega}_{\text{T}}) \times {}^{\text{T}}\boldsymbol{r}_P\right] \\
&= ({}^{\text{A}}\ddot{\boldsymbol{r}}_T + {}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\dot{\boldsymbol{r}}_T) + ({}^{\text{O}}\dot{\boldsymbol{\omega}}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T + {}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times \frac{{}^{\text{O}}d}{dt}{}^{\text{A}}\boldsymbol{r}_T) + ({}^{\text{O}}\dot{\boldsymbol{\omega}}_{\text{A}} + \frac{{}^{\text{O}}d}{dt}{}^{\text{A}}\boldsymbol{\omega}_{\text{T}}) \times {}^{\text{T}}\boldsymbol{r}_P + ({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} + {}^{\text{A}}\boldsymbol{\omega}_{\text{T}}) \times \frac{{}^{\text{O}}d}{dt}{}^{\text{T}}\boldsymbol{r}_P \\
&= {}^{\text{A}}\ddot{\boldsymbol{r}}_T + 2{}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\dot{\boldsymbol{r}}_T + {}^{\text{O}}\dot{\boldsymbol{\omega}}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T + {}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times \left({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T\right) + ({}^{\text{O}}\dot{\boldsymbol{\omega}}_{\text{A}} + {}^{\text{A}}\dot{\boldsymbol{\omega}}_{\text{T}} + \cancelto{0}{{}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\boldsymbol{\omega}_{\text{T}}}) \times {}^{\text{T}}\boldsymbol{r}_P + ({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} + {}^{\text{A}}\boldsymbol{\omega}_{\text{T}}) \times \frac{{}^{\text{O}}d}{dt}{}^{\text{T}}\boldsymbol{r}_P \\
&= {}^{\text{A}}\ddot{\boldsymbol{r}}_T + 2{}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\dot{\boldsymbol{r}}_T + {}^{\text{O}}\dot{\boldsymbol{\omega}}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T + {}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times \left({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T\right) + ({}^{\text{O}}\dot{\boldsymbol{\omega}}_{\text{A}} + {}^{\text{A}}\dot{\boldsymbol{\omega}}_{\text{T}}) \times {}^{\text{T}}\boldsymbol{r}_P + ({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} + {}^{\text{A}}\boldsymbol{\omega}_{\text{T}}) \times (\cancelto{0}{{}^{\text{T}}\dot{\boldsymbol{r}}_P} + {}^{\text{O}}\boldsymbol{\omega}_{\text{T}} \times {}^{\text{T}}\boldsymbol{r}_P) \\
&= {}^{\text{A}}\ddot{\boldsymbol{r}}_T + 2{}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\dot{\boldsymbol{r}}_T + {}^{\text{O}}\dot{\boldsymbol{\omega}}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T + {}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times \left({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} \times {}^{\text{A}}\boldsymbol{r}_T\right) + ({}^{\text{O}}\dot{\boldsymbol{\omega}}_{\text{A}} + {}^{\text{A}}\dot{\boldsymbol{\omega}}_{\text{T}}) \times {}^{\text{T}}\boldsymbol{r}_P + ({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} + {}^{\text{A}}\boldsymbol{\omega}_{\text{T}}) \times \left[({}^{\text{O}}\boldsymbol{\omega}_{\text{A}} + {}^{\text{A}}\boldsymbol{\omega}_{\text{T}}) \times {}^{\text{T}}\boldsymbol{r}_P\right] \\
&= \dot{v}(t)\hat{\mathbf{j}}_A + 2\dot{\theta}\hat{\mathbf{K}} \times v(t)\hat{\mathbf{j}}_A + \ddot{\theta}\hat{\mathbf{K}} \times (L\hat{\mathbf{j}}_A + R\hat{\mathbf{i}}_A) + \dot{\theta}\hat{\mathbf{K}} \times \left[\dot{\theta}\hat{\mathbf{K}} \times (L\hat{\mathbf{j}}_A + R\hat{\mathbf{i}}_A)\right] \\
&\qquad + \left(\ddot{\theta}\hat{\mathbf{K}} + \frac{\dot{v}(t)}{R}\hat{\mathbf{k}}_A\right) \times r\hat{\mathbf{j}}_T + \left(\dot{\theta}\hat{\mathbf{K}} + \frac{v(t)}{R}\hat{\mathbf{k}}_A\right) \times \left[\left(\dot{\theta}\hat{\mathbf{K}} + \frac{v(t)}{R}\hat{\mathbf{k}}_A\right) \times r\hat{\mathbf{j}}_T\right] \\
&= \dot{v}(t)\hat{\mathbf{j}}_A - 2\dot{\theta}v(t)\hat{\mathbf{i}}_A - \ddot{\theta}L\hat{\mathbf{i}}_A + \ddot{\theta}R\hat{\mathbf{j}}_A + \dot{\theta}\hat{\mathbf{K}} \times \left(-\dot{\theta}L\hat{\mathbf{i}}_A + \dot{\theta}R\hat{\mathbf{j}}_A\right) - \left(\ddot{\theta} + \frac{\dot{v}(t)}{R}\right) r\,\hat{\mathbf{i}}_T + \left(\dot{\theta}\hat{\mathbf{K}} + \frac{v(t)}{R}\hat{\mathbf{k}}_A\right) \times \left[-\left(\dot{\theta}r + v(t)\frac{r}{R}\right)\hat{\mathbf{i}}_T\right] \\
&= \boxed{\dot{v}(t)\hat{\mathbf{j}}_A - 2\dot{\theta}v(t)\hat{\mathbf{i}}_A - \ddot{\theta}(L\hat{\mathbf{i}}_A + R\hat{\mathbf{j}}_A) - \dot{\theta}^2(L\hat{\mathbf{j}}_A + R\hat{\mathbf{i}}_A) - \left(\ddot{\theta} + \frac{\dot{v}(t)}{R}\right) r\,\hat{\mathbf{i}}_T - \left(\dot{\theta} + \frac{v(t)}{R}\right)^2 r\,\hat{\mathbf{j}}_T}
\end{aligned}$$

Here we see six terms. $\dot{v}(t)\hat{\mathbf{j}}_A$ is the translational acceleration of origin A with respect to frame $\hat{\text{O}}$. $2\dot{\theta}v(t)\hat{\mathbf{i}}_A$ is the Coriolis term due to a velocity of origin T relative to the rotating frame $\hat{\text{A}}$. $\ddot{\theta}(L\hat{\mathbf{i}}_A + R\hat{\mathbf{j}}_A)$ is the Eulerian term due to the angular acceleration of frame $\hat{\text{A}}$ with respect to frame $\hat{\text{O}}$. $\dot{\theta}^2(L\hat{\mathbf{j}}_A + R\hat{\mathbf{i}}_A)$ is the centripetal term term from the rotation of frame $\hat{\text{A}}$ with respect to frame $\hat{\text{O}}$. $\left(\ddot{\theta} + \frac{\dot{v}(t)}{R}\right) r\,\hat{\mathbf{i}}_T$ is the Eulerian term due to the angular acceleration of frame $\hat{\text{T}}$ with respect to frame $\hat{\text{O}}$. Lastly, $\left(\dot{\theta} + \frac{v(t)}{R}\right)^2 r\,\hat{\mathbf{j}}_T$ is the centripetal term from the rotation of frame $\hat{\text{T}}$ with respect to frame $\hat{\text{O}}$.

Problem 2



A roller coster car travels with constant tangential velocity v_0, on a helical track with radius R and pitch angle ϕ. What is the velocity and acceleration of the car with respect to the ground when it reaches point P, the second peak of the helical track?

Solution

First, define our reference frames. Define point O at the center of rotation in the plane of the start of the track and point A at the center of rotation of the track traveling along $\hat{\mathbf{I}}$ with the car. Define inertial ground reference frame $\hat{\mathrm{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, and frame $\hat{\mathrm{A}} = (A, \hat{\mathbf{i}}_A = \hat{\mathbf{I}}, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A)$ rotating and translating with the car so the car is fixed in frame $\hat{\mathrm{A}}$. For convenience, let $\hat{\mathbf{k}}_A$ point in the direction of the car.

Second, write the given variables with respect to these reference frames. Since v_0 is the tangential velocity, some component will cause translation and another will cause rotation. We have:

$$^{\mathrm{O}}\boldsymbol{r}_A = t v_0 \sin\phi \hat{\mathbf{I}} \qquad ^{\mathrm{O}}\dot{\boldsymbol{r}}_A = v_0 \sin\phi \hat{\mathbf{I}} \qquad ^{\mathrm{A}}\boldsymbol{r}_P = R\hat{\mathbf{k}}_A \qquad ^{\mathrm{A}}\dot{\boldsymbol{r}}_P = \mathbf{0} \qquad ^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} = -\frac{v_o \cos\phi}{R}\hat{\mathbf{I}}$$

Recall the general formula for taking the derivative with respect to frame $\hat{\mathrm{A}}$ of a vector defined in $\hat{\mathrm{B}}$:

$$\frac{^{\mathrm{A}}d}{dt}\left(^{\mathrm{B}}\boldsymbol{r}_p\right) = \frac{^{\mathrm{B}}d}{dt}\left(^{\mathrm{B}}\boldsymbol{r}_p\right) + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_p = {}^{\mathrm{B}}\dot{\boldsymbol{r}}_p + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_p$$

Now we can, we solve for variable $^{\mathrm{O}}\boldsymbol{v}_P$:

$$^{\mathrm{O}}\boldsymbol{v}_P = \frac{^{\mathrm{O}}d}{dt}(^{\mathrm{O}}\boldsymbol{r}_P) = \frac{^{\mathrm{O}}d}{dt}(^{\mathrm{O}}\boldsymbol{r}_A + {}^{\mathrm{A}}\boldsymbol{r}_P) = {}^{\mathrm{O}}\dot{\boldsymbol{r}}_A + \cancelto{0}{{}^{\mathrm{A}}\dot{\boldsymbol{r}}_P} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_P$$
$$= v_0 \sin\phi \hat{\mathbf{I}} - \frac{v_o \cos\phi}{R}\hat{\mathbf{I}} \times R\hat{\mathbf{k}}_A = \boxed{v_0 \sin\phi \hat{\mathbf{I}} + v_o \cos\phi \hat{\mathbf{j}}_A}$$

Lastly, we solve for ${}^{\mathrm{O}}\boldsymbol{a}_P$:

$$\begin{aligned}
{}^{\mathrm{O}}\boldsymbol{a}_P = \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{O}}\boldsymbol{v}_P) &= \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\dot{\boldsymbol{r}}_A + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_P\right) \\
&= \cancelto{0}{{}^{\mathrm{O}}\ddot{\boldsymbol{r}}_A} + \cancelto{0}{{}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}}} \times {}^{\mathrm{A}}\boldsymbol{r}_P + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times \frac{{}^{\mathrm{O}}d}{dt}{}^{\mathrm{A}}\boldsymbol{r}_P \\
&= {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times (\cancelto{0}{{}^{\mathrm{A}}\dot{\boldsymbol{r}}_P} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_P) \\
&= {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_P) \\
&= -\frac{v_o \cos\phi}{R}\hat{\mathbf{I}} \times \left(-\frac{v_o \cos\phi}{R}\hat{\mathbf{I}} \times R\hat{\mathbf{k}}_A\right) \\
&= \boxed{-\frac{v_o^2 \cos^2\phi}{R}\hat{\mathbf{k}}_A}
\end{aligned}$$

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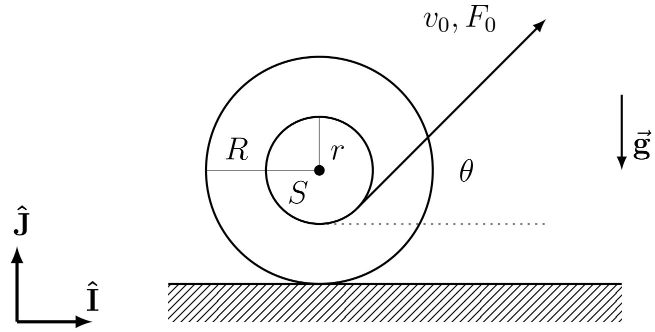
Problem 1



A spool of mass m and moment of inertia about its center I, with inner radius r and outer radius R sits on a table, initially at rest but can roll without slipping. A thread wrapped around the inner radius is pulled at an angle θ from the horizontal. Find the velocity of the center of the spool S with respect to the ground as a function of both time and angle under the following conditions:

(1) **Kinematic constraint:** The thread is pulled with constant velocity v_0

(2) **Force constraint:** The thread is pulled with constant force F_0, with $F_0 < mg$

Solution

Define point O where the spool comes into contact with the ground, and point D where the thread detaches from the spool. Define inertial ground reference frame $\hat{\text{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, and frame $\hat{\text{S}} = (S, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A = \hat{\mathbf{K}})$ attached to and rotating with the spool. For convenience, let the unit vectors of frame $\hat{\text{A}}$ point in the same directions as the unit vectors of frame $\hat{\text{O}}$ at this instant. Also, define another convenient set of ground reference frame unit coordinate vectors $\hat{\text{O}} = (O, \hat{\mathbf{t}}, \hat{\mathbf{r}}, \hat{\mathbf{K}})$ such that $\hat{\mathbf{t}} = \cos\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}}$ and $\hat{\mathbf{r}} = -\sin\theta\,\hat{\mathbf{I}} + \cos\theta\,\hat{\mathbf{J}}$.

Write the given variables with respect to these reference frames. We have:

$$ {}^{\text{O}}\boldsymbol{r}_S = R\,\hat{\mathbf{J}} \qquad {}^{\text{O}}\dot{\boldsymbol{r}}_S = v_s(t)\,\hat{\mathbf{I}} \qquad {}^{\text{S}}\boldsymbol{r}_D = -r\,\hat{\mathbf{r}} \qquad {}^{\text{O}}\boldsymbol{\omega}_S = \omega_s(t)\,\hat{\mathbf{K}} $$

Note that both $v_s(t)$ and ω_s are unknown. We must solve for the magnitude $v_s(t)$ of the velocity ${}^{\text{O}}\dot{\boldsymbol{r}}_S$ given two different constraints. In each case, we will need at least two equations since we have two unknowns.

(1) **Kinematic constraint:** The thread is pulled with constant velocity v_0

This constraint dictates that ${}^{\text{O}}\dot{\boldsymbol{r}}_T \cdot \hat{\mathbf{t}} = v_0$. We know the velocity of point O and point T which will give us two equations. First, the velocity of point O is zero because of the no-slip condition:

$$ 0 = \frac{{}^{\text{O}}d}{dt}{}^{\text{O}}\boldsymbol{r}_O = \frac{{}^{\text{O}}d}{dt}({}^{\text{O}}\boldsymbol{r}_S + {}^{\text{S}}\boldsymbol{r}_O) = {}^{\text{O}}\dot{\boldsymbol{r}}_S + \cancelto{0}{{}^{\text{S}}\dot{\boldsymbol{r}}_O} + {}^{\text{O}}\boldsymbol{\omega}_{\text{S}} \times {}^{\text{S}}\boldsymbol{r}_O = [v_s(t) + \omega_s(t)R]\,\hat{\mathbf{I}} $$

Yielding the familiar result $v_s(t) = -\omega_s(t)R$. Second, is the velocity constraint:

$$v_0 = {}^{\text{O}}\dot{\boldsymbol{r}}_T \cdot \hat{\mathbf{t}} = \frac{{}^{\text{O}}d}{dt}({}^{\text{O}}\boldsymbol{r}_S + {}^{\text{S}}\boldsymbol{r}_T) \cdot \hat{\mathbf{t}} = ({}^{\text{O}}\dot{\boldsymbol{r}}_S + \cancelto{0}{{}^{\text{S}}\dot{\boldsymbol{r}}_T} + {}^{\text{O}}\boldsymbol{\omega}_{\text{S}} \times {}^{\text{S}}\boldsymbol{r}_T) \cdot \hat{\mathbf{t}} = v_s(t)\,\hat{\mathbf{I}} \cdot \hat{\mathbf{t}} + \omega_s(t) r\,\hat{\mathbf{t}} \cdot \hat{\mathbf{t}}$$

Since $\hat{\mathbf{I}} = \cos\theta\,\hat{\mathbf{t}} - \sin\theta\,\hat{\mathbf{r}}$, the component in the $\hat{\mathbf{t}}$ direction yields, $v_0 = v_s(t)\cos\theta + \omega_s(t)r$, and plugging in for $\omega_s(t)$ gives:

$$\boxed{v_s = \frac{v_0}{\cos\theta - r/R}}$$

Note that this velocity is constant in time and diverges when $\cos\theta = r/R$. This happens because the kinematic velocity constraint we have applied is not a reasonable constraint for certain angles under our model that the spool does not lift off the table. Let us examine a force constraint instead.

(2) **Force constraint:** The thread is pulled with constant force F_0, with $F_0 < mg$

This constraint dictates that $\boldsymbol{F}_T = F_0\,\hat{\mathbf{t}}$. As in part (1), the no-slip condition still applies, so $v_s(t) = -\omega_s(t)R$ and $a_s(t) = -\dot{\omega}_s(t)R$. There are four forces acting on the spool: gravity, normal force from the table, frictional force from the table, and tension from the string. We have:

$$\sum \boldsymbol{F} = \boldsymbol{F}_g + \boldsymbol{F}_N + \boldsymbol{F}_f + \boldsymbol{F}_T = \frac{d}{dt}(\boldsymbol{p}) = m\boldsymbol{a}_s \qquad (F_f + F_0\cos\theta)\hat{\mathbf{I}} + (F_N - mg + F_0\sin\theta)\,\hat{\mathbf{J}} = ma_s\,\hat{\mathbf{I}}$$

This adds two unknowns (F_N and F_f) but gives us two more equations. To get our last equation, we balance torques about the center of mass.

$$\sum \boldsymbol{\tau}^S = \cancelto{0}{\boldsymbol{\tau}_g^S} + \cancelto{0}{\boldsymbol{\tau}_N^S} + \boldsymbol{\tau}_f^S + \boldsymbol{\tau}_T^S = \frac{d}{dt}(\boldsymbol{h}) = I\dot{\boldsymbol{\omega}}_s \qquad \text{or} \qquad RF_f + rF_T = I\dot{\omega}_s(t)$$

Substituting in for F_N and F_f yields:

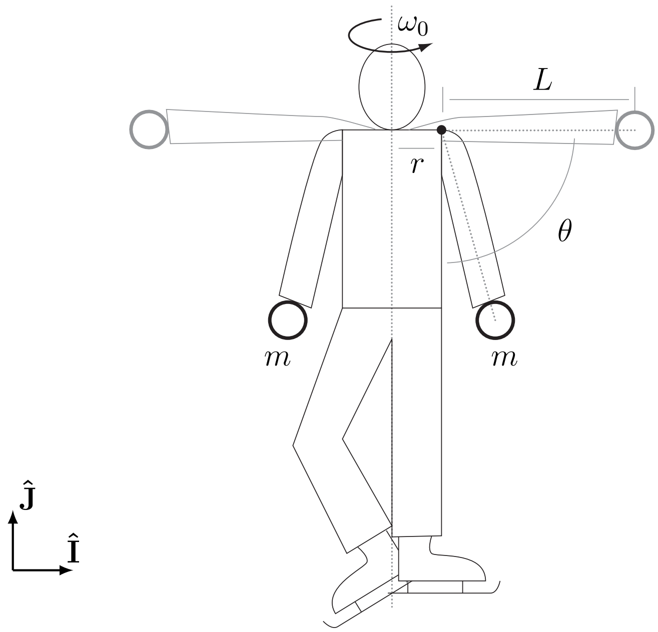
$$I\dot{\omega}_s(t) = R\,[ma_s(t) - F_0\cos\theta] + rF_0$$

And finally, knowing that $a_s(t) = -\dot{\omega}_s(t)R$:

$$a_s = \frac{F_0\,R^2(\cos\theta - r/R)}{I + mR^2} \qquad \text{and since } a_s(t) \text{ is constant in time,} \qquad \boxed{v_s(t) = \frac{F_0\,R^2(\cos\theta - r/R)}{I + mR^2}\,t}$$

Note now when $\cos\theta = r/R$, the acceleration is zero and the spool does not move. This explains the paradox in the kinematic problem. Assuming we don't lift the spool off the ground (i.e. $F_0 < mg$) and friction will keep the spool from sliding, we will be unable to move the spool or string when pulling at this angle.

Problem 2



A figure skater holds two weights of equal mass m at her sides ($\theta_0 = 0$) while spinning in place at constant angular velocity ω_0. At $t = 0$, she lifts her arms outwards at a constant angular velocity $\dot{\theta}$. What torque does the figure skater need to apply with her feet in order to maintain constant angular velocity ω_0 throughout the motion as a function of θ and $\dot{\theta}$ for $0° < \theta < 90°$? Solve first by considering the forces acting on the masses directly, then second, by using the angular momentum formulation.

Solution

Define point $O \equiv A$ between the head and the shoulders on the axis of rotation, point B at the center of the right arm's vertical rotation, point M at the center of the right mass, and point N at the center of the left mass. Define inertial ground reference frame $\hat{\mathrm{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, frame $\hat{\mathrm{A}} = (A, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A = \hat{\mathbf{J}}, \hat{\mathbf{k}}_A)$ attached to and rotating with the skater's body, and frame $\hat{\mathrm{B}} = (B, \hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B, \hat{\mathbf{k}}_B = \hat{\mathbf{k}}_A)$ attached to and rotating with the right arm. For convenience, let the unit vectors of frame $\hat{\mathrm{A}}$ point in the same directions as the unit vectors of frame $\hat{\mathrm{O}}$ at this instant, and choose $\hat{\mathbf{i}}_B$ to point parallel to $\overline{BM}$ such that $\hat{\mathbf{i}}_B = \sin\theta \hat{\mathbf{i}}_A - \cos\theta \hat{\mathbf{j}}_A$.

Write the given variables with respect to these reference frames. We have:

$$ {}^{\mathrm{O}}\boldsymbol{\omega}_A = \omega_0 \, \hat{\mathbf{J}} \qquad {}^{\mathrm{A}}\boldsymbol{\omega}_B = \dot{\theta} \, \hat{\mathbf{k}}_A \qquad {}^{\mathrm{O}}\boldsymbol{r}_A = \mathbf{0} \qquad {}^{\mathrm{A}}\boldsymbol{r}_B = r \, \hat{\mathbf{i}}_A \qquad {}^{\mathrm{B}}\boldsymbol{r}_M = L \, \hat{\mathbf{i}}_B $$

Approach 1

Our first approach is to consider the kinematics of the problem and use Newton's second law to calculate the necessary torque. We are given that ω_0 and $\dot{\theta}$ are constant. To lift the weights, the skater must exert a force on each mass. From Newton's first law, these forces must be equal and opposite to the forces the weights exert on the skater. In order to use Newton's second law to find the magnitude and direction of these forces, we must find the acceleration of each weight. Examine the acceleration of the right mass:

$$
\begin{aligned}
{}^{\mathrm{O}}\boldsymbol{a}_M &= \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}{}^{\mathrm{O}}\boldsymbol{r}_M\right) = \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{r}_A} + {}^{\mathrm{A}}\boldsymbol{r}_B + {}^{\mathrm{B}}\boldsymbol{r}_M)\right) \\
&= \frac{{}^{\mathrm{O}}d}{dt}\left[(\cancelto{0}{{}^{\mathrm{A}}\dot{\boldsymbol{r}}_B} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B) + (\cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_M} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_M)\right] \\
&= \left[\cancelto{0}{{}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times (\cancelto{0}{{}^{\mathrm{A}}\dot{\boldsymbol{r}}_B} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B)\right] + \frac{{}^{\mathrm{O}}d}{dt}\left[({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_M\right] \\
&= {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B) + (\cancelto{0}{{}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}}} + \cancelto{0}{{}^{\mathrm{A}}\dot{\boldsymbol{\omega}}_{\mathrm{B}}} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_M + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times (\cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_M} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_M) \\
&= {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B) + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_M + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times \left[({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_M\right] \\
&= {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B) + \underline{({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_M} \\
&\quad + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{B}}\boldsymbol{r}_M) + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} \times ({}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_M) + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_M) + \underline{{}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} \times ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{B}}\boldsymbol{r}_M)}
\end{aligned}
$$

Using the vector identity $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} + \boldsymbol{B} \times (\boldsymbol{A} \times \boldsymbol{C})$, we can combine underlined terms:

$$
\begin{aligned}
{}^{\mathrm{O}}\boldsymbol{a}_M &= {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times \left[{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{A}}\boldsymbol{r}_B + {}^{\mathrm{B}}\boldsymbol{r}_M)\right] + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} \times ({}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_M) + 2\,{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_M) \\
&= -\omega_0^2(r + L\sin\theta)\hat{\mathbf{i}}_A - \dot{\theta}^2 L\hat{\mathbf{i}}_B - 2\omega_0\dot{\theta}L\cos\theta\hat{\mathbf{k}}_A \\
&= -\left[\omega_0^2(r + L\sin\theta) + \dot{\theta}^2 L\sin\theta\right]\hat{\mathbf{i}}_A + \dot{\theta}^2 L\cos\theta\,\hat{\mathbf{j}}_A - 2\omega_0\dot{\theta}L\cos\theta\hat{\mathbf{k}}_A
\end{aligned}
$$

The first and second terms represent the centripetal accelerations due to the rotations, while the third term is the Coriolis acceleration due to a rotation in a rotating frame. By symmetry, the left mass's acceleration should be the same, except the $\hat{\mathbf{i}}_A$ and $\hat{\mathbf{k}}_A$ components should be opposite in sign, thus:

$$
{}^{\mathrm{O}}\boldsymbol{a}_N = \left[\omega_0^2(r + L\sin\theta) + \dot{\theta}^2 L\sin\theta\right]\hat{\mathbf{i}}_A + \dot{\theta}^2 L\cos\theta\,\hat{\mathbf{j}}_A + 2\omega_0\dot{\theta}L\cos\theta\,\hat{\mathbf{k}}_A
$$

The only torques about O should be the torques exerted by each mass on the body and the torque $\boldsymbol{\tau}^O$ applied by the skater. These should all sum to zero as the angular velocity ω_0 of the skater is constant.

$$
\sum \boldsymbol{\tau} = \boldsymbol{\tau}^O + \sum \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\tau}^O + {}^{\mathrm{O}}\boldsymbol{r}_N \times (-m\,{}^{\mathrm{O}}\boldsymbol{a}_N) + {}^{\mathrm{O}}\boldsymbol{r}_M \times (-m\,{}^{\mathrm{O}}\boldsymbol{a}_M) = I\cancelto{0}{{}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}}} = \mathbf{0}
$$

Just like with the accelerations, the $\hat{\mathbf{i}}_A$ components of the lever arms should be negatives of each other: ${}^{\mathrm{O}}\boldsymbol{r}_M = (r + L\sin\theta)\hat{\mathbf{i}}_A - L\cos\theta\hat{\mathbf{j}}_A$ and ${}^{\mathrm{O}}\boldsymbol{r}_N = -(r + L\sin\theta)\hat{\mathbf{i}}_A - L\cos\theta\hat{\mathbf{j}}_A$. Because all the $\hat{\mathbf{i}}_A$ and $\hat{\mathbf{k}}_A$ components are negatives of each other, cross terms of the form $\hat{\mathbf{i}}_A \times \hat{\mathbf{j}}_A$, $\hat{\mathbf{j}}_A \times \hat{\mathbf{i}}_A$, and $\hat{\mathbf{j}}_A \times \hat{\mathbf{k}}_A$ should all cancel, leaving only the $\hat{\mathbf{i}}_A \times \hat{\mathbf{k}}_A$ terms, thus:

$$
\boldsymbol{\tau}^O = m({}^{\mathrm{O}}\boldsymbol{r}_N \times {}^{\mathrm{O}}\boldsymbol{a}_N + {}^{\mathrm{O}}\boldsymbol{r}_M \times {}^{\mathrm{O}}\boldsymbol{a}_M) = \boxed{4m\omega_0\dot{\theta}L\cos\theta(r + L\sin\theta)\hat{\mathbf{J}}}
$$

We have solved the problem directly by calculating accelerations, but perhaps an easier way to look at this problem is recognizing that the angular momentum is conserved.

Approach 2

Our second approach is to exploit the concept of angular momentum. Again, we are given that ω_0 and $\dot{\theta}$ are constant. Thus, the rotational form of Newton's second law becomes:

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}^O = \frac{{}^{\mathrm{O}}d}{dt}{}^{\mathrm{O}}\boldsymbol{H}^O + \cancelto{0}{{}^{\mathrm{O}}\boldsymbol{v}_O} \times {}^{\mathrm{O}}\boldsymbol{p}_M = \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{O}}\boldsymbol{h}_M^O + {}^{\mathrm{O}}\boldsymbol{h}_N^O)$$

Solving for the angular momentum of the right mass ${}^{\mathrm{O}}\boldsymbol{h}_M^O$ directly:

$$\begin{aligned}
{}^{\mathrm{O}}\boldsymbol{h}_M^O &= m\,{}^{\mathrm{O}}\boldsymbol{r}_M \times {}^{\mathrm{O}}\boldsymbol{v}_M \\
&= m\,{}^{\mathrm{O}}\boldsymbol{r}_M \times \frac{{}^{\mathrm{O}}d}{dt}(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{r}_A} + {}^{\mathrm{A}}\boldsymbol{r}_B + {}^{\mathrm{B}}\boldsymbol{r}_M) \\
&= m\,{}^{\mathrm{O}}\boldsymbol{r}_M \times (\cancelto{0}{{}^{\mathrm{A}}\dot{\boldsymbol{r}}_B} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + \cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_M} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_M) \\
&= m\,{}^{\mathrm{O}}\boldsymbol{r}_M \times \left[{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_M\right] \\
&= m\left[(r + L\sin\theta)\hat{\mathbf{i}}_A - L\cos\theta\hat{\mathbf{j}}_A\right] \times \left[-\omega_0(r + L\sin\theta)\hat{\mathbf{k}}_A + \dot{\theta}L(\cos\theta\hat{\mathbf{i}}_A + \sin\theta\hat{\mathbf{j}}_A)\right] \\
&= m\omega_0(r + L\sin\theta)^2\,\hat{\mathbf{j}}_A + m\omega_0 L\cos\theta(r + L\sin\theta)\,\hat{\mathbf{i}}_A + m\left[\dot{\theta}L^2\cos^2\theta + \dot{\theta}L\sin\theta(r + L\sin\theta)\right]\hat{\mathbf{k}}_A
\end{aligned}$$

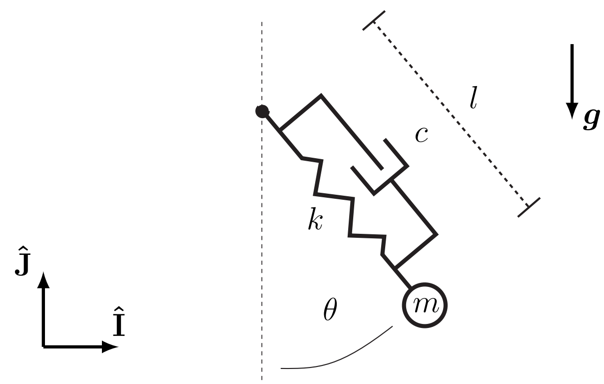
Again, due to symmetry, the left mass's angular momentum ${}^{O}\boldsymbol{h}_N^O$ should be the same as the right's except that the $\hat{\mathbf{i}}_A$ and $\hat{\mathbf{k}}_A$ components should be negative. Thus, when they add together, their $\hat{\mathbf{i}}_A$ and $\hat{\mathbf{k}}_A$ components must cancel:

$${}^{\mathrm{O}}\boldsymbol{H}^O = {}^{\mathrm{O}}\boldsymbol{h}_M^O + {}^{\mathrm{O}}\boldsymbol{h}_N^O = 2m\omega_0(r + L\sin\theta)^2\,\hat{\mathbf{j}}_A$$

Taking a time derivative yields the required torque:

$$\begin{aligned}
\boldsymbol{\tau}^O = \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{O}}\boldsymbol{h}_M^O + {}^{\mathrm{O}}\boldsymbol{h}_N^O) &= \frac{{}^{\mathrm{O}}d}{dt}(2m\omega_0(r + L\sin\theta)^2\,\hat{\mathbf{j}}_A) \\
&= \boxed{4m\omega_0\dot{\theta}L\cos\theta(r + L\sin\theta)\,\hat{\mathbf{J}}}
\end{aligned}$$

Problem 1



A small ball of mass m hangs from an elastic string which can be modeled as a spring with coefficient k in parallel with a dashpot with coefficient c. The spring has a natural length l. Consider the motion of this pendulum in a plane in the presence of gravity. Derive the equations of motion for the system. Are the equations of motion coupled in general? Are the equations of motion coupled for small angles?

Solution

Define inertial reference frame $\hat{\mathrm{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$ with origin at the pivot point. Define a rotating reference frame $\hat{\mathrm{A}} = (A \equiv O, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A \equiv \hat{\mathbf{K}})$ with origin also at the pivot point, with vector $\hat{\mathbf{i}}_A$ pointing toward the mass. Also, let r be the position of the mass in the $\hat{\mathbf{i}}_A$ direction, with l being the natural length of the spring in the same direction.

Since only one frame rotates with respect to the other, we can directly apply the formula for the acceleration of a particle in a single intermediate frame frame of reference, where p is the position of the mass:

$$^{\mathrm{O}}\boldsymbol{a}_m = {}^{\mathrm{O}}\ddot{\boldsymbol{r}}_A + {}^{\mathrm{A}}\ddot{\boldsymbol{r}}_m + 2\,{}^{\mathrm{O}}\boldsymbol{w}_A \times {}^{\mathrm{A}}\dot{\boldsymbol{r}}_m + {}^{\mathrm{O}}\dot{\boldsymbol{w}}_A \times {}^{\mathrm{A}}\boldsymbol{r}_m + {}^{\mathrm{O}}\boldsymbol{w}_A \times ({}^{\mathrm{O}}\boldsymbol{w}_A \times {}^{\mathrm{A}}\boldsymbol{r}_m)$$

Here, ${}^{\mathrm{O}}\ddot{\boldsymbol{r}}_A = \mathbf{0}$, ${}^{\mathrm{A}}\ddot{\boldsymbol{r}}_m = \ddot{r}\,\hat{\mathbf{i}}_A$, ${}^{\mathrm{O}}\boldsymbol{w}_A = \dot{\theta}\,\hat{\mathbf{K}}$, ${}^{\mathrm{A}}\dot{\boldsymbol{r}}_m = \dot{r}\,\hat{\mathbf{i}}_A$, and ${}^{\mathrm{A}}\boldsymbol{r}_m = r\,\hat{\mathbf{i}}_A$. This results in the polar coordinate equation for acceleration we've seen in class.

$$^{\mathrm{O}}\boldsymbol{a}_m = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{i}}_A + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{j}}_A$$

To find the equations of motion, first balance the forces in the radial direction $\hat{\mathbf{i}}_A$ taking into account the force from the spring, the dashpot, and gravity.

$$\begin{aligned} \sum \boldsymbol{F}_m \cdot \hat{\mathbf{i}}_A &= m\,{}^{\mathrm{O}}\boldsymbol{a}_m \cdot \hat{\mathbf{i}}_A \\ -k(r-l) - c\dot{r} + mg\cos\theta &= m(\ddot{r} - r\dot{\theta}^2) \end{aligned}$$

$$\boxed{0 = m(\ddot{r} - r\dot{\theta}^2) + c\dot{r} + kr - mg\cos\theta - kl}$$

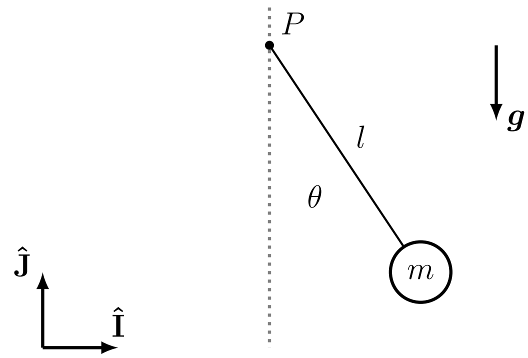
The second equation of motion comes from considering the tangential component $\hat{\mathbf{j}}_A$, where the only external force is due to gravity.

$$\begin{aligned} \sum \boldsymbol{F}_m \cdot \hat{\mathbf{j}}_A &= m\,{}^{\mathrm{O}}\boldsymbol{a}_m \cdot \hat{\mathbf{j}}_A \\ -mg\sin\theta &= mr\ddot{\theta} + 2m\dot{r}\dot{\theta} \end{aligned}$$

$$\boxed{0 = r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta}$$

Both equations of motion include both r and θ even after small angle approximations are used for the $\sin\theta$ and $\cos\theta$, implying that the two equations of motion are coupled.

Problem 2



Consider a pendulum consisting of a point mass m and rigid rod of length L with negligible mass moving in the presence of gravity. The pivot point P of the pendulum is not fixed but follows some non-zero trajectory ${}^{\mathrm{O}}\boldsymbol{r}_P(t)$. Derive the equations of motion for the system. Then derive the equations of motion for the specific pivot trajectory ${}^{\mathrm{O}}\boldsymbol{r}_P(t) = A\cos(\omega t)\hat{\mathbf{J}}$.

Solution

Define inertial reference frame $\hat{\mathrm{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$ with origin fixed at the initial location of the pivot. Define a rotating reference frame $\hat{\mathrm{P}} = (P, \hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P, \hat{\mathbf{k}}_P \equiv \hat{\mathbf{K}})$ with origin moving with the pivot, with vector $\hat{\mathbf{i}}_P = \sin\theta\hat{\mathbf{I}} - \cos\theta\hat{\mathbf{J}}$ pointing toward the mass. We write Newton's Second Law for the sum of the forces on mass m:

$$\sum \boldsymbol{F}_m = -mg\,\hat{\mathbf{J}} - F_T\,\hat{\mathbf{i}}_P = \frac{{}^{\mathrm{O}}d}{dt}{}^{\mathrm{O}}\boldsymbol{p}_m = m{}^{\mathrm{O}}\boldsymbol{a}_m$$

The sum of the forces on the mass should be gravity and a tension force from the rod. The tension force must lie in the direction of the rod as the sum of the torques on the massless rod about any point must be zero. From Kinematics:

$$\begin{aligned}
{}^{\mathrm{O}}\boldsymbol{a}_m = \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{r}_m\right)\right) &= \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{r}_P + {}^{\mathrm{P}}\boldsymbol{r}_m\right)\right) \\
&= \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\dot{\boldsymbol{r}}_P + \cancelto{0}{{}^{\mathrm{P}}\dot{\boldsymbol{r}}_m} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{P}}\boldsymbol{r}_m\right) \\
&= {}^{\mathrm{O}}\ddot{\boldsymbol{r}}_P + {}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}} \times {}^{\mathrm{P}}\boldsymbol{r}_m + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times (\cancelto{0}{{}^{\mathrm{P}}\dot{\boldsymbol{r}}_m} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{P}}\boldsymbol{r}_m) \\
&= {}^{\mathrm{O}}\ddot{\boldsymbol{r}}_P(t) + \ddot{\theta}L\,\hat{\mathbf{j}}_P - \dot{\theta}^2 L\,\hat{\mathbf{i}}_P
\end{aligned}$$

Taking the sum of the forces in the $\hat{\mathbf{j}}_P$ direction yields the general equation of motion.

$$\begin{aligned}
\hat{\mathbf{j}}_P \cdot [-mg\,\hat{\mathbf{J}} - F_T\,\hat{\mathbf{i}}_P = m({}^{\mathrm{O}}\ddot{\boldsymbol{r}}_P(t) + \ddot{\theta}L\,\hat{\mathbf{j}}_P - \dot{\theta}^2 L\,\hat{\mathbf{i}}_P)] \\
-g\sin\theta = {}^{\mathrm{O}}\ddot{\boldsymbol{r}}_P(t)\cdot\hat{\mathbf{j}}_P + \ddot{\theta}L
\end{aligned}$$

$$\boxed{\ddot{\theta} = -\frac{1}{L}\left({}^{\mathrm{O}}\ddot{\boldsymbol{r}}_P(t)\cdot\hat{\mathbf{j}}_P + g\sin\theta\right)}$$

For ${}^{\mathrm{O}}\boldsymbol{r}_P(t) = A\cos(\omega t)\hat{\mathbf{J}}$, we just plug in to our general equation.

$$\ddot{\theta} = -\frac{1}{L}\left((-A\omega^2\cos(\omega t)\hat{\mathbf{J}})\cdot\hat{\mathbf{j}}_P + g\sin\theta\right)$$

$$\boxed{\ddot{\theta} = \frac{\sin\theta}{L}\left(A\omega^2\cos(\omega t) - g\right)}$$