2.001 Fall 2011: Mechanics and Materials I — **Quiz #1 Review**
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2.001 Quiz #1 Review

1 Connections

Unknown reaction forces provided for different types of connections:

Table 1-1 Hibbeler (8th Edition)

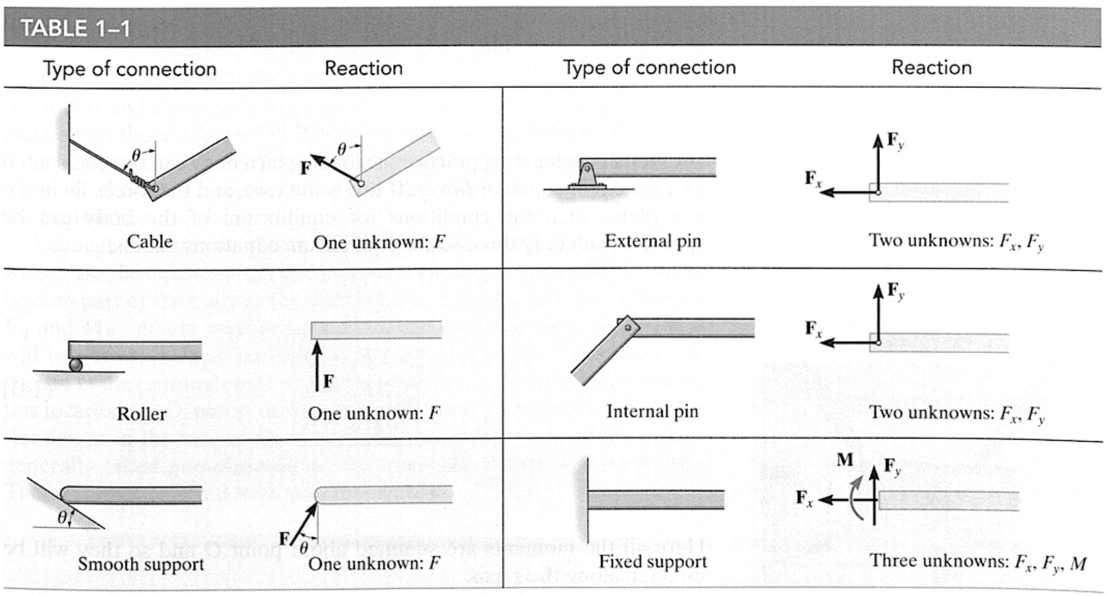
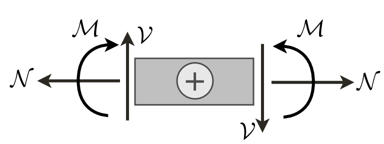


TABLE 1–1

Type of connection	Reaction	Type of connection	Reaction
Cable	One unknown: F	External pin	Two unknowns: F_x, F_y
Roller	One unknown: F	Internal pin	Two unknowns: F_x, F_y
Smooth support	One unknown: F	Fixed support	Three unknowns: F_x, F_y, M

2 Internal Forces & Sign Conventions



In order to analyze the internal forces in an object, we can imagine cutting it in two and analyzing what forces must exist in order to keep each piece in equilibrium. Just as with connections between objects, a certain number of internal forces are required to constrain the object to remain in equilibrium. In three dimensions, three reaction forces and three reaction moments are required to model the possible internal interactions. In two dimensions, two reaction forces (axial $\mathcal{N}$ and shear $\mathcal{V}$) and one reaction moment $\mathcal{M}$ are necessary. Our sign conventions for this course are shown above. The most important convention here is that positive internal axial force describes a beam in tension.

3 Equilibrium

Newton's second law yields multiple equations that describe equilibrium. For a body that can move in three dimensions, Newton's second law yields six equations (three for force balance and three for moment balance). For a body that can move only in two dimensions, Newton's second law yields three equations (two for force balance and one for moment balance).

$$\left[\sum \boldsymbol{F} = \boldsymbol{0} \quad , \quad \sum (\boldsymbol{M})_B = \boldsymbol{0} \right] \implies \left[\sum F_x = \sum F_y = \sum (M_z)_B = 0 \right]$$

Here, moments are taken about some arbitrary point B. Reactions between two objects in static equilibrium must be equal and opposite according to Newton's third law. Note that if only two forces and no moments act on an object, the forces must be equal in magnitude, opposite in direction, and collinear. We call such objects *two-force-members*.

4 Indeterminate Structures & Degrees of Freedom

Statically *determinate* structures are structures that are minimally supported. By that we mean that if you remove any piece or constraint from the structure, it can no longer support equilibrium and can collapse. We can solve for the internal and reaction forces of determinate structures in a straight-forward manner using equations of equilibrium.

Alternatively, structures that are over-supported and have redundant supports or constraints are called statically *indeterminate*. These systems share the property that there exists a constraint or element of the structure that can be removed, and the system will still be able to exist in equilibrium and not collapse. The system is over constrained, so equilibrium will not give enough equations to solve for the unknowns. *Constitutive* equations will provide the additional equations we require by relating deformations in the system through the structure's geometry. We relate the deformations of the system to the forces acting on the system through *constitutive relations*.

Degrees of freedom (DoF) are an independent set of movements a structure can make that describe all possible deformed configurations. If you fix all degrees of freedom, the system will not be able to move (it is a complete set), while if you fix any strict subset of them, the structure will be able to move (they are independent). The number of equilibrium equations you need in order to solve a statically indeterminate system is equal to the number of degrees of freedom for the structure.

5 Stress, Strain, & Constitutive Relations

Stress is a tensor that describes the state of loading in an object. The tensor properties of stress will be discussed later in the course. For now, we will deal primarily with average normal stress σ in an object produced by a force P distributed over an area A with units of [N/m^2]. More generally, a force is the integral of stress acting on an area.

$$\sigma = \frac{P}{A} \qquad\qquad P = \int\limits_A \sigma \, dA$$

The average strain ε in an object is the proportional elongation δ of the object with respect to the original length L of the object, with units [m/m] (unit-less). More generally, the elongation is the integral of strain over the length of the object.

$$\varepsilon = \frac{\delta}{L} \qquad\qquad \delta = \int\limits_L \varepsilon \, dL$$

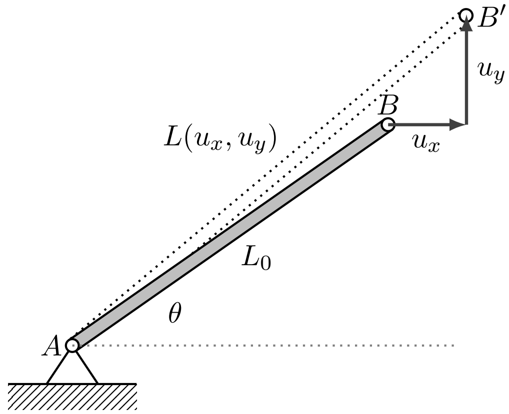
For materials deforming in the *elastic* regime, the stress σ and strain ε of a material can be modeled as being directly proportional to one another. We can relate these two quantities through the use of Hooke's Law:

$$\sigma = E\varepsilon$$

where E is a scalar property of the material that is deforming called the Young's modulus. In general, any of the quantities in the above equations (σ, ε, E, F, A, δ, L) could be functions of space or time, but typically one or more of them will be constant in some direction. For *axial loading*, strain ε is assumed to be only a function of a single axial direction (call it x), and we can write an object's elongation as an integral over the single variable x. If the beam is not a composite structure and the young's modulus is only a function x, we have a further simplification. If P, E, and A are all constant in x, then the integral reduces to the familiar elongation equation for a uniform two-force-member.

$$\overset{\varepsilon(x)}{\Longrightarrow} \quad \delta = \int_0^L \frac{P(x)}{\int\limits_{A(x)} E(x,y,z)dA} \, dx \quad \overset{E(x)}{\Longrightarrow} \quad \delta = \int_0^L \frac{P(x)}{E(x)A(x)} \, dx \quad \overset{P,E,A \text{ const in } x}{\Longrightarrow} \quad \delta = \frac{PL}{EA}$$

6 Compatibility



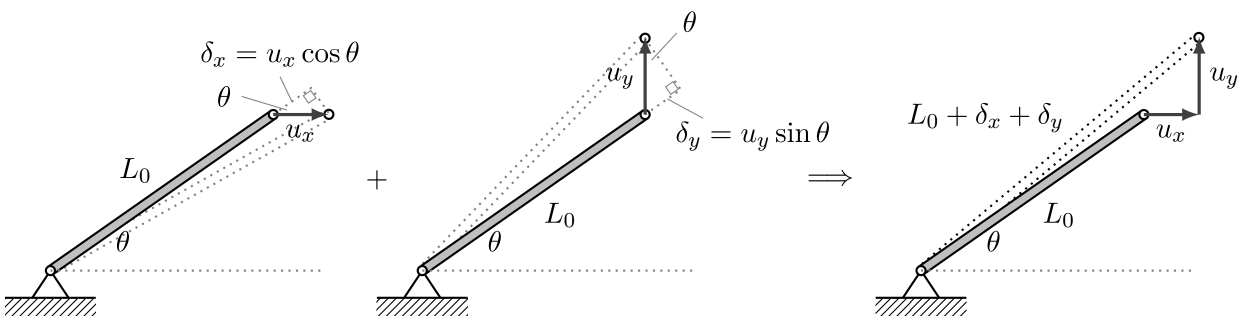
Compatibility refers to the geometric relationship between the deformations and displacements of beams and objects. This can get tricky when beams deflect at angles other than along their axis. For example, consider the bar AB of length L_0 above, with point A fixed and point B deflecting by (u_x, u_y). The exact stretched length L of the deformed member is given by geometry:

$$L(u_x, u_y) = \sqrt{L_0^2 + 2L_0(u_x \cos\theta + u_y \sin\theta) + u_x^2 + u_y^2}$$

This equation is messy to solve and superposition does not apply, so we typically assume small deflections so that a first order approximation of the elongations will suffice. The first order approximation of the Taylor series for the above function yields:

$$L(u_x, u_y) \approx L_0 + u_x \cos\theta + u_y \sin\theta$$

and we can see that to first order, we can treat the deflections in x and y as a linear system. In practice, we can use the result above to find elongations by drawing the beam displaced in each of its component directions independently, and then projecting the displaced point onto the line of the *original* beam by dropping a perpendicular. The distance from the original point to the intersection will represent the approximate elongation of the displaced point.



7 Displacement Method

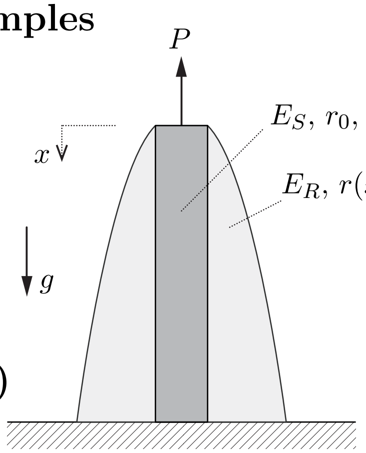
We can solve statically indeterminate structures defined by axial loading in two ways. So far, we have only discussed the *displacement method* where we displace the degrees of freedom of the system and analyze how the system reacts. This method is good for systems that are very indeterminate (systems that are highly over-constrained), but can be applied to any statically indeterminate system. The method has nine steps. First we analyze the system for its degrees of freedom (1), then we write down all the equations we need in order to solve the problem (2)-(4), and lastly, we go through the algebra to solve for our unknown variables (5)-(9).

(1) Identify degrees of freedom	# DoF = n	$u_x,\, u_y,\, \phi,\, \ldots \iff \xi_1, \xi_2, \ldots, \xi_n$
(2) Conjugate equilibrium equations	# Equations = n	$\sum \boldsymbol{F},\, \sum(\boldsymbol{M})_B$
for each DoF	# Unknowns = m	$\mathcal{N}_1,\, \mathcal{N}_2,\, \ldots,\, \mathcal{N}_m$
(3) Constitutive relations	# Equations = m	$\sigma = E\varepsilon,\, \delta = \int \varepsilon dL,\, \mathcal{N} = \int \sigma dA$
	# Unknowns = m	$\delta_1,\, \delta_2,\, \ldots,\, \delta_m$
(4) Compatibility	# Equations = m	$\delta = u_x \cos\theta + u_y \sin\theta$
	# Unknowns = n	$u_x,\, u_y,\, \phi,\, \ldots \iff \xi_1, \xi_2, \ldots, \xi_n$
(5) Backsubstitute	Combine $2m$ equations into n equations for DoF	
(6) Solve for DoF	$u_x,\, u_y,\, \phi,\, \ldots \iff \xi_1, \xi_2, \ldots, \xi_n$	
(7) Solve for beam elongations	$\delta_1,\, \delta_2,\, \ldots,\, \delta_m$	
(8) Solve for internal forces	$\mathcal{N}_1,\, \mathcal{N}_2,\, \ldots,\, \mathcal{N}_m$	
(9) Solve for reactions at supports	$\mathcal{R}_x^A,\, \mathcal{R}_y^B,\, \mathcal{M}_z^C,\, \ldots$	

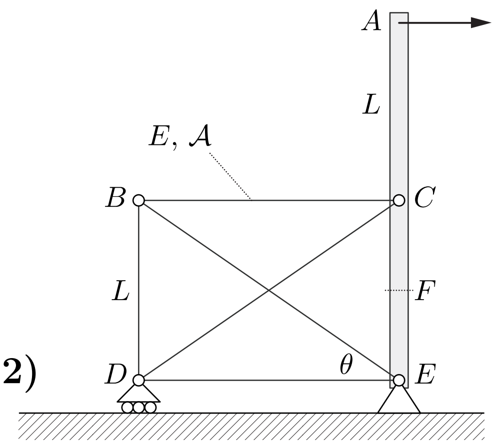
This table demonstrates that if our structure is statically determinate (minimally constrained), the number of degrees of freedom and the number of unknown internal forces in our equilibrium equations will be equal ($n = m$). However, if $m > n$, our system is statically indeterminate (over-constrained), while if $n > m$, our system is under-constrained and can move (i.e. 2.003).

8 Examples

(1)



(2)



Problem (1)

Consider a steel pillar with a rubber covering attached rigidly to the ground. The cylindrical steel pillar has constant radius r_0 and density ρ. The rubber covering has variable radius $r(x) = \sqrt{(x + r_0)r_0}$, with $x = 0$ at the top of the pillar and $x = H$ at the bottom of the pillar. Assume the weight of the rubber is negligible compared to the weight of the steel. Let steel and rubber have Young's moduli E_S and E_R respectively. Given that a force P is applied to the top of the pillar, determine the pillar's elongation from its *unstressed* length H. Note that gravity acts.

Solution: This composite structure is statically determinate, elongating under axial load. Force balance in the x direction of a chopped off section of the pillar of length x is given by:

$$\sum F_x = 0 = P - N - \rho g \pi r_0^2 x \qquad N(x) = P - \rho g \pi r_0^2 x$$

Taking our general equation relating uniaxial force to uniaxial elongation yields:

$$\delta = \int_0^H \frac{N(x)}{\int_A E\,dA}\,dx = \int_0^H \frac{N(x)}{E_S A_S + E_R A_R}\,dx = \int_0^H \frac{P - \rho g \pi r_0^2 x}{E_S \pi r_0^2 + E_R(\pi(\sqrt{(x + r_0)r_0})^2 - \pi r_0^2)}\,dx$$

$$= \int_0^H \frac{P - \rho g \pi r_0^2 x}{E_S \pi r_0^2 + E_R \pi r_0 x}\,dx = \int_0^H \frac{P}{E_S \pi r_0^2 + E_R \pi r_0 x}\,dx - \int_0^H \frac{\rho g r_0 x}{E_S r_0 + E_R x}\,dx$$

$$= \left[\frac{P}{E_R \pi r_0} \ln\left|E_S \pi r_0^2 + E_R \pi r_0 x\right| - \rho g r_0 \left(\frac{x}{E_R} - \frac{E_S r_0}{E_R^2} \ln\left|E_S r_0 + E_R x\right|\right)\right]_0^H$$

$$= \boxed{\left(\frac{P}{E_R \pi r_0} + \frac{\rho g r_0^2}{E_R}\frac{E_S}{E_R}\right) \ln\left|1 + \frac{E_R}{E_S}\frac{H}{r_0}\right| - \frac{\rho g r_0 H}{E_R}}$$

Problem (2)

Consider the truss above. Rod $\overline{AE}$ is rigid, while all other beams are deformable with Young's modulus E and cross-sectional area $\mathcal{A}$. A force P is applied to the right at point A. Determine the rotation of the rigid beam due to this loading. What are the internal forces acting at point F?

Solution: This structure is statically indeterminate as a deformable bar could be removed and the structure could remain in equilibrium. Let $\overline{BD}$ be bar 1, $\overline{BC}$ be bar 2, $\overline{DE}$ be bar 3, $\overline{BE}$ be bar 4, and $\overline{CD}$ be bar 5. Using the displacement method:

(1) DoF: $\{u_x^D, u_x^B, u_y^B, \phi^E\}$ (ϕ^E being the rotation of rigid bar about E)

(2) Equilibrium:

$$\sum F_x^D = 0 = N_3 + N_5 \cos\theta$$
$$\sum F_x^B = 0 = N_2 + N_4 \cos\theta$$
$$\sum F_y^B = 0 = -N_1 - N_4 \sin\theta$$
$$\sum (M_z)_E = 0 = (L)N_2 - (2L)P + (L\cos\theta)N_5$$

(3) C. R.:

$$\delta_1 = \frac{N_1 L}{E\mathcal{A}}, \quad \delta_2 = \frac{N_2 L}{E\mathcal{A}\tan\theta}, \quad \delta_3 = \frac{N_3 L}{E\mathcal{A}\tan\theta}, \quad \delta_4 = \frac{N_4 L}{E\mathcal{A}\sin\theta}, \quad \delta_5 = \frac{N_5 L}{E\mathcal{A}\sin\theta}$$

(4) Compatibility:

$$\delta_1 = u_y^B, \quad \delta_2 = \phi^E L - u_x^B, \quad \delta_3 = -u_x^D, \quad \delta_4 = u_y^B \sin\theta - u_x^B \cos\theta, \quad \delta_5 = -u_x^D \cos\theta + \phi^E L \cos\theta$$

We have 14 equations in 14 unknowns. Back-solving for ϕ^E yields:

$$\boxed{\phi^E = \frac{2P}{EA}\frac{(\cos^2\theta + 1)(\cos^3\theta + \sin^3\theta + 1)}{\cos^2\theta \sin\theta(2\cos^3\theta + \sin^3\theta + 2)}}$$

Internal forces at point F are given by equilibrium on section $\overline{ACF}$:

$$\sum F_x = 0 = P + \mathcal{V} - N_2 - N_5 \cos\theta$$
$$\sum F_y = 0 = -\mathcal{N} - N_5 \sin\theta$$
$$\sum (M_z)_C = 0 = (L/2)\mathcal{V} + \mathcal{M} - (L)P$$

Yields: $\boxed{\mathcal{N} = -N_5 \sin\theta}$ $\boxed{\mathcal{V} = N_2 + N_5\cos\theta - P}$ $\boxed{\mathcal{M} = (L/2)(3P - N_2 - N_5\cos\theta)}$

Where N_2 and N_5 are found from the above system.

Algebra is messy. Luckily, we can write this system of linear equations as a matrix and solve using MATLAB. Here, we have used the letters S and C to represent $\sin\theta$ and $\cos\theta$ respectively.

```
syms S C P L E A;
A = [
%N1,   N2,   N3, N4, N5,  d1,    d2,    d3,    d4,    d5, uDx, uBx, uBy,  phi
  0,    0,    1,  0,  C,   0,     0,     0,     0,     0,   0,   0,   0,    0;
  0,    1,    0,  C,  0,   0,     0,     0,     0,     0,   0,   0,   0,    0;
  1,    0,    0,  S,  0,   0,     0,     0,     0,     0,   0,   0,   0,    0;
  0,    1,    0,  0,  C,   0,     0,     0,     0,     0,   0,   0,   0,    0;
 -L,    0,    0,  0,  0, E*A,     0,     0,     0,     0,   0,   0,   0,    0;
  0, -L*C,    0,  0,  0,   0, E*A*S,     0,     0,     0,   0,   0,   0,    0;
  0,    0, -L*C,  0,  0,   0,     0, E*A*S,     0,     0,   0,   0,   0,    0;
  0,    0,    0, -L,  0,   0,     0,     0, E*A*S,     0,   0,   0,   0,    0;
  0,    0,    0,  0, -L,   0,     0,     0,     0, E*A*S,   0,   0,   0,    0;
  0,    0,    0,  0,  0,   1,     0,     0,     0,     0,   0,   0,  -1,    0;
  0,    0,    0,  0,  0,   0,     1,     0,     0,     0,   0,   1,   0,   -L;
  0,    0,    0,  0,  0,   0,     0,     1,     0,     0,   1,   0,   0,    0;
  0,    0,    0,  0,  0,   0,     0,     0,     1,     0,   0,   C,  -S,    0;
  0,    0,    0,  0,  0,   0,     0,     0,     0,     1,   C,   0,   0, -L*C];
b = [0,0,0,2*P,0,0,0,0,0,0,0,0,0,0]';
x = simplify(A\b);

% N1                          (2*S*conj(P)*(C^3 + 1))/(C*(2*C^3 + S^3 + 2))
% N2                                (2*conj(P)*(C^3 + 1))/(2*C^3 + S^3 + 2)
% N3                         -(2*conj(P)*(C^3 + S^3 + 1))/(2*C^3 + S^3 + 2)
% N4                           -(2*conj(P)*(C^3 + 1))/(C*(2*C^3 + S^3 + 2))
% N5                     (2*conj(P)*(C^3 + S^3 + 1))/(C*(2*C^3 + S^3 + 2))
% d1                   (2*L*S*conj(P)*(C^3 + 1))/(A*C*E*(2*C^3 + S^3 + 2))
% d2                   (2*C*L*conj(P)*(C^3 + 1))/(A*E*S*(2*C^3 + S^3 + 2))
% d3           -(2*C*L*conj(P)*(C^3 + S^3 + 1))/(A*E*S*(2*C^3 + S^3 + 2))
% d4                   -(2*L*conj(P)*(C^3 + 1))/(A*C*E*S*(2*C^3 + S^3 + 2))
% d5            (2*L*conj(P)*(C^3 + S^3 + 1))/(A*C*E*S*(2*C^3 + S^3 + 2))
% uDx           (2*C*L*conj(P)*(C^3 + S^3 + 1))/(A*E*S*(2*C^3 + S^3 + 2))
% uBx     (2*L*conj(P)*(C^3 + 1)*(S^3 + 1))/(A*C^2*E*S*(2*C^3 + S^3 + 2))
% uBy                  (2*L*S*conj(P)*(C^3 + 1))/(A*C*E*(2*C^3 + S^3 + 2))
% phi  (2*conj(P)*(C^3 + 1)*(C^3 + S^3 + 1))/(A*C^2*E*S*(2*C^3 + S^3 + 2))
```

2.001 Quiz #2 Review

1 3D Constitutive Relations and Thermal Expansion

Stress and strain are tensors. As such, they have multiple components that can be represented in a matrix. In this class, these matrices will always be symmetric.

$$\bar{\bar{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad \bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

The subscript notation for stress σ_{ij} refers to a stress in the j direction applied to a surface with surface normal in the i direction. We often use τ_{ij} instead of σ_{ij} for shear stresses. This is purely notational. Alternatively, each component of the strain tensor ε_{ij} refers to the derivative of the i component of displacement with respect to the j direction $\frac{du_i}{dj}$. The components of the strain tensor are related to the components of the stress tensor according to the following three dimensional constitutive equations:

$$\frac{du_i}{di} = \varepsilon_{ii} = \frac{1}{E}[\sigma_{ii} - \nu(\sigma_{jj} + \sigma_{kk})] + \alpha\Delta T$$

$$\gamma_{ij} = \frac{du_i}{dj} + \frac{du_j}{di} = 2\varepsilon_{ij} = \frac{\tau_{ij}}{G} \qquad G = \frac{E}{2(1+\nu)}$$

G is the shear modulus for the material and α is the material's coefficient of thermal expansion. Note that $\alpha\Delta T$ results directly in a change in strain and will not affect stress unless the strain of the object is constrained in some way.

2 Pressure Vessels

Pressure vessels are not a new topic, but a convenient application for equilibrium and the above three dimensional constitutive relations. For thin-walled vessels, we make the approximation that the wall thickness is very small compared to the radius of the vessel, i.e. $R/t \gg 1$.

This approximation justifies the assumption that the radial stress is negligible compared to the transverse stresses ($\sigma_{rr} \approx 0$). The hardest part about pressure vessels is drawing correct free body diagrams and approximating areas. Here are results for the state of stress in the cylindrical walls of a cylindrical thin-walled pressure vessel, both open and closed:

$$(\sigma_{rr}, \sigma_{xx}, \sigma_{\theta\theta})_{open} = (0,0,1)\,\frac{PR}{t} \qquad (\sigma_{rr}, \sigma_{xx}, \sigma_{\theta\theta})_{closed} = (0,1,2)\,\frac{PR}{2t}$$

Additionally, for a cylindrical pressure vessel, the axial strains are related to the macroscopic changes in length assuming uniform deformation:

$$\varepsilon_{xx} = \frac{\delta_L}{L} \qquad \varepsilon_{\theta\theta} = \frac{2\pi(R+\delta_R) - 2\pi R}{2\pi R} = \frac{\delta_R}{R} \qquad \varepsilon_{rr} = \frac{\delta_t}{t}$$

3 Torsion

Torsion is quite analogous to axial loading with torque T taking the place of axial forces $\mathcal{N}$, shear stress $\tau_{x\theta}$ taking the place of axial stress σ_{xx}, the shear modulus G taking the place of the Young's modulus E, shear strain $\varepsilon_{x\theta}$ taking the place of axial strain ε_{xx}, the polar moment of inertia $I_P$taking the place of an effective area, and deflection angle ϕ taking the place of axial displacement u_x.

Axial Load	Torsion
$\mathcal{N} = \int_A \sigma_{xx} dA$	$T = \int_A r\sigma_{x\theta} dA$
$\sigma_{xx} = E\varepsilon_{xx}$	$\tau_{x\theta} = 2G\varepsilon_{x\theta}$
$\varepsilon_{xx} = \frac{du_x}{dx}$	$\varepsilon_{x\theta} = \frac{r}{2}\frac{d\phi}{dx}$
$u_x = \int_L \frac{\mathcal{N}}{(EA)_{eff}} dx$	$\phi = \int_L \frac{T}{(GI_P)_{eff}} dx$
$(EA)_{eff} = \int_A E dA$	$(GI_P)_{eff} = \int_A Gr^2 dA$

Here are some derived values of polar moment of inertia I_P for different shaped cross-sections of constant shear modulus G.

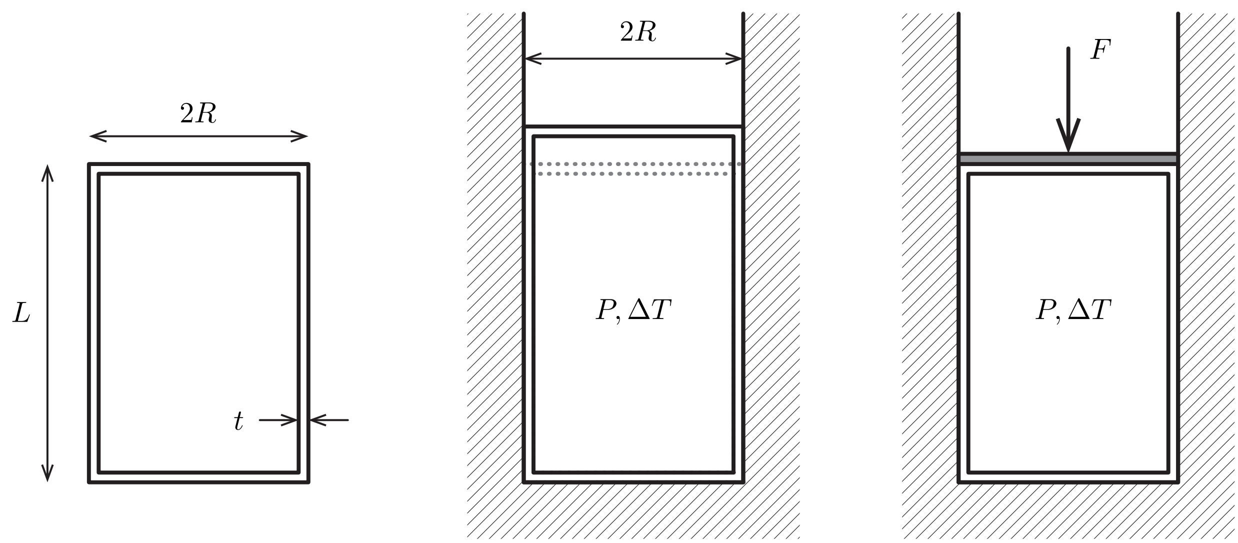
$$(I_P)_{circle} = \frac{\pi}{2}R^4 \qquad (I_P)_{annulus} = \frac{\pi}{2}(R^4 - r^4) \qquad (I_P)_{thin-shell} = 2\pi R^3 t$$

Note that just like for axial loading, we have a sign convention for talking about internal loads. For axial loading, positive internal axial force was normal to a cut surface, while positive beam deformation was elongation. For torsion, positive internal torque is given by the right hand rule for the direction of twist about the surface normal of a cut.

Exactly analogous to axial loading, torsional systems can be statically determinate or indeterminate. The displacement method works exactly the same in this context: identifying angular degrees of freedom, performing moment equilibrium, applying constitutive equations, the relating them through compatibility.

4 Examples

4.1 Pressure Vessel Example



A thin-walled cylindrical pressure vessel has material properties E, ν, and α with unpressurized length L, radius R, and thickness t at room temperature. It is placed unpressurized into a rigid pipe, also with radius R.

(a) The pressure vessel is pressurized to gauge pressure $P > 0$ and heated by an amount $\Delta T > 0$. Determine the state of stress in the cylindrical walls of the pressure vessel.

Solution: σ_{rr} is negligible because of our thin-walled assumption. σ_{xx} is the same as the axial stress in a closed cylinder (the vessel is not constrained in x). $\sigma_{\theta\theta}$ can be found directly from the constitutive relations noting that $\varepsilon_{\theta\theta} = 0$.

$$\varepsilon_{\theta\theta} = 0 = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{xx} + \sigma_{rr})\right] + \alpha\Delta T$$

$$\boxed{\sigma_{rr} = 0} \qquad \boxed{\sigma_{xx} = \frac{PR}{2t}} \qquad \boxed{\sigma_{\theta\theta} = \frac{\nu PR}{2t} - \alpha E\Delta T}$$

(b) A force F is then applied to the top of the pressure vessel to press the vessel down to its original length L. Assume the gauge pressure remains the same as in part (b). Determine the new state of stress in the cylindrical walls of the pressure vessel.

Solution: σ_{rr} is still negligible but both the circumferential and axial strains are now both constrained to zero strain. Thus:

$$\varepsilon_{xx} = 0 = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{\theta\theta} + \sigma_{rr})\right] + \alpha\Delta T \qquad \varepsilon_{\theta\theta} = 0 = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{xx} + \sigma_{rr})\right] + \alpha\Delta T$$

$$\boxed{\sigma_{rr} = 0} \qquad \boxed{\sigma_{xx} = \sigma_{\theta\theta} = -\frac{\alpha E\Delta T}{1-\nu}}$$

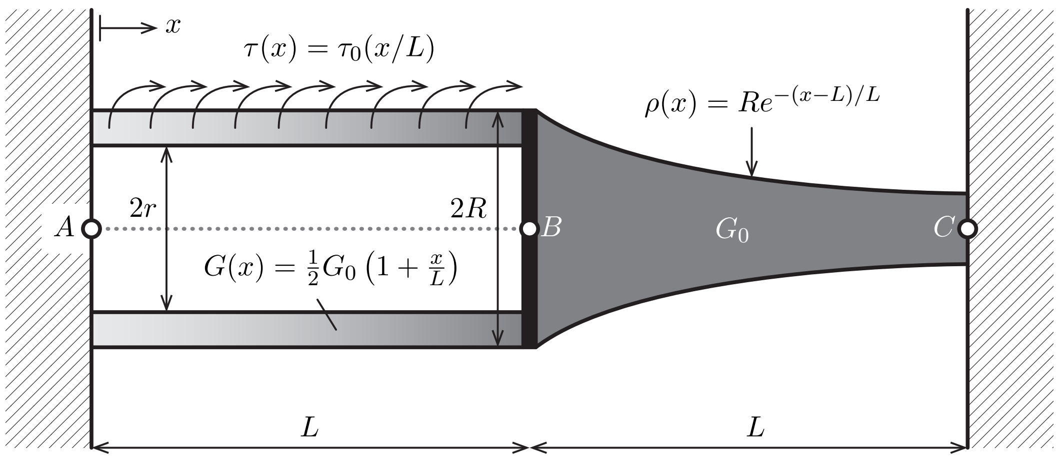
(c) How much force F had to be applied?

Solution: We can now solve for the force using equilibrium now that the internal axial stress is known. Taking equilibrium in the vertical direction for the top half of the pressure vessel yields:

$$\sum F_y = 0 = P(\pi R^2) - F - \sigma_{xx}(2\pi R t)$$

$$\boxed{F = P\pi R^2 + \frac{\alpha E\Delta T}{1-\nu}(2\pi R t)}$$

4.2 Torsion Example

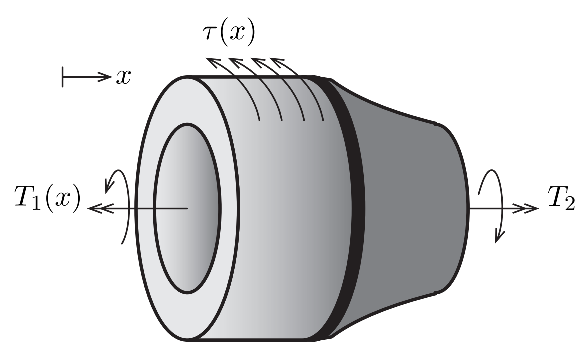


Two circularly symmetric beams, each of length L are joined by a thin rigid plate and rigidly attached on each end to stationary walls. Let the coordinate x measure the horizontal distance from the left wall. The left beam has a uniform cross sectional area in the shape of an annulus with outer radius R and inner radius r whose shear modulus varies along its length according to $G(x) = \frac{1}{2}G_0(1 + x/L)$. The right bar has uniform shear modulus G_0, and circular cross-section whose radius varies along its length according to $\rho(x) = Re^{-(x-L)/L}$. A distributed shear stress is applied to the left bar along its cylindrical outer surface with magnitude $\tau(x) = \tau_0 x/L$. Determine the reaction torque applied to the right wall.

Solution: This is a statically indeterminate system (it is supported by both walls) so we will not be able to solve for internal or reaction torques directly from equilibrium. We will need to write down constitutive relations and compatibility to solve the problem. Let's first apply equilibrium on a middle chunk of the beam by summing moments about the axial direction:

$$\sum (M_x)_A = 0 = T_2 - T_1(x) - \int_x^L r\left(\tau_0 \frac{x}{L}\right)(2\pi R)dx$$

$$\underline{T_1(x) = T_2 - \frac{\pi R^2}{L}\tau_0(L^2 - x^2)}$$



Here, $T_1(x)$ is the varying internal torque in the left beam while T_2 is the constant internal torque in the right beam. T_2 is also equal to the reaction torque the problem demands. This is one equation in two unknowns and the system is statically indeterminate. Constitutive relations relate the internal torques to the world of angles:

$$\Delta\phi_{AB} = \int_0^L \frac{T_1(x)}{(GI_P)^{(1)}_{eff}}dx \qquad (GI_P)^{(1)}_{eff} = \int_A G(x,r)r^2 dA = \left[\frac{1}{2}G_0\left(1+\frac{x}{L}\right)\right]\frac{\pi}{2}(R^4 - r^4)$$

$$\underline{\Delta\phi_{AB} = \frac{4}{G_0\pi(R^4 - r^4)}\int_0^L \frac{T_1(x)}{1+x/L}dx}$$

$$\Delta\phi_{BC} = \int_L^{2L} \frac{T_2}{(GI_P)^{(2)}_{eff}}dx \qquad (GI_P)^{(2)}_{eff} = \int_A G(x,r)r^2 dA = G_0\frac{\pi}{2}[\rho(x)]^4 = G_0\frac{\pi}{2}R^4 e^{-4(x-L)/L}$$

$$\Delta\phi_{BC} = \frac{2T_2}{G_0\pi R^4}\int_L^{2L} e^{4(x-L)/L}dx \qquad \underline{\Delta\phi_{BC} = \frac{T_2 L}{2G_0\pi R^4}\left(e^4 - 1\right)}$$

Lastly, compatibility requires that the angle of deflection from A to B plus the angle of deflection from B to C is zero (both walls are fixed):

$$\underline{\Delta\phi_{AB} + \Delta\phi_{BC} = 0}$$

We have four equations in four unknowns $(T_2, T_1(x), \Delta\phi_{AB}, \Delta\phi_{BC})$. Solving for T_2:

$$\frac{4}{G_0\pi(R^4 - r^4)}\int_0^L \frac{LT_2 - \pi R^2\tau_0(L^2 - x^2)}{L+x}dx + \frac{T_2 L}{2G_0\pi R^4}\left(e^4 - 1\right) = 0$$

$$T_2\frac{L}{G_0\pi}\left[\frac{4}{R^4 - r^4}\int_0^L \frac{1}{L+x}dx + \frac{1}{2R^4}\left(e^4 - 1\right)\right] + \frac{4R^2\tau_0}{G_0(R^4 - r^4)}\int_0^L -\frac{(L+x)(L-x)}{L+x}dx = 0$$

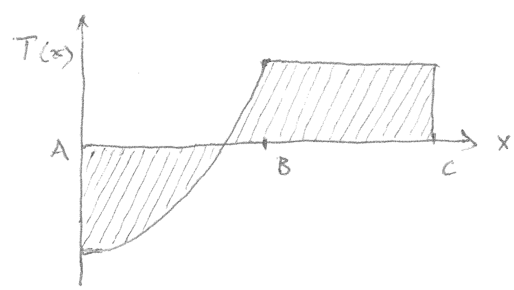
$$T_2\frac{L}{G_0\pi}\left\{8\left[\ln|L+x|\right]_0^L + \left(1 - r^4/R^4\right)\left(e^4 - 1\right)\right\} + \frac{8R^2\tau_0}{G_0}\left[-\frac{1}{2}(L-x)^2\right]_0^L = 0$$

$$\boxed{T_2 = \left[\frac{4\pi}{8\ln|2| + (1 - r^4/R^4)(e^4 - 1)}\right]R^2 L\tau_0}$$

Addendum: Sketch the internal torque $T(x)$ and the twist angle $\phi(x)$ as a functions of axial distance x.

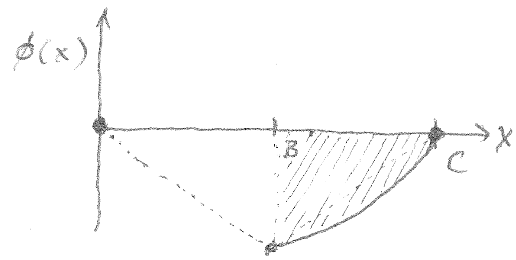
Solution: There are two ways of thinking about sketching these diagrams. The first is quantitative and involves solving all the equations for the internal torques and angles as functions of x. We will do that to get the exact values of each point on the curve. The second way is a good way to absorb the problem before you delve into the math by qualitatively constructing the shape of each curve just from logic, geometry, and intuition. Let's think about the sketches qualitatively first and see if our quantitative equations match up with our intuition.

Let's start with internal torque. The external distributed shear stress is applying a torque to our system in the global $-x$ direction. We would then expect the reaction torque at each wall to be in the opposite global direction (the x direction) to counteract this applied load. Since the normal to the cut face on the left points in the $-x$ direction, we expect the reaction torque at A to be negative, and because the normal to the cut face on the right points in the $+x$ direction, we expect the reaction torque at C to be positive. Additionally the internal torque from B to C should be constant (no additional load is applied), which the internal torque should increase from A to B. Also, because internal torque is an integral of applied shear and the applied shear is linear, we can expect the internal torque from A to B to be quadratic.

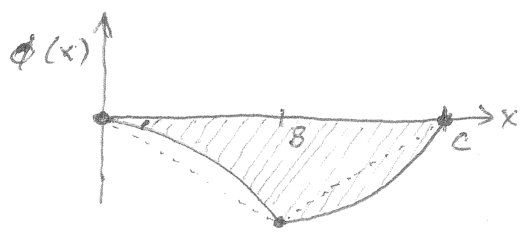


Now for the twist angle. Note that twist angle refers to a global twist of the beam, and we are taking positive ϕ to be about the $+x$ direction. Because each end is fixed, the twist angles at A and C are clearly zero. The external distributed shear stress is applying a torque to our system in the global $-x$ direction. It would make sense that the bar would then have a twist angle in the same direction with its maximum magnitude at B. Thus we have qualitative estimates for the twist angle at A, B, and C.

The actual shapes of the curves in sections AB and BC are a little more difficult to intuit, but we would certainly expect the curve from A to B to be monotonically decreasing and the curve from B to C to be monotonically increasing. For the right side of the bar, the smaller radius cross sections will be able to resist rotation less then the larger radius cross sections, thus we expect it to change its twist angle faster as the radius decreases.



On the left side, because both the shear modulus and the applied shear stress change with distance, it is not exactly intuitive as to which one wins out. It turns out that since the internal torque increases quadratically and the shear modulus increases linearly, the internal torque wins out and the beam will change its angle faster closer to B.



Now let's see if the equations agree with our qualitative arguments. Solving the above equations for the internal torque as a function of x yields:

$$T(x) = \begin{cases} T_2 - \frac{\pi R^2}{L}\tau_0(L^2 - x^2) & \text{for } 0 \leq x \leq L \\ T_2 & \text{for } L < x \leq 2L \end{cases}$$

This agrees with our estimate because $T_2 < \pi R^2 L \tau_0$, so $T(x)$ is negative at A. Solving the above equations for twist angle as a function of x yields:

$$\phi(x) = \begin{cases} \frac{4}{G_0\pi(R^4 - r^4)}\left[T_2 \ln|1 + x/L| - \frac{1}{2}\pi R^2 \tau_0 (L-x)^2\right] & \text{for } 0 \leq x \leq L \\ \phi(L) + \frac{T_2 L}{2G_0 \pi R^4}\left(e^{4(x-L)/L} - 1\right) & \text{for } L < x \leq 2L \end{cases}$$

This agrees with our estimate because the negative quadratic term $-x^2$ wins out over the logarithmic term $\ln|1 + x/L|$.

2.001 Final Review

1 Beam Bending

Beam bending is analogous to axial loading and torsion except that the loading is not axially symmetric. The first relationships we derived were derivative relationships between the loading per unit length q in the *downward* direction, the internal shear $\mathcal{V}$, and the internal moment $\mathcal{M}$:

$$-q(x) = \frac{d\mathcal{V}}{dx} \qquad \mathcal{V}(x) = \frac{d\mathcal{M}}{dx}$$

If we know the loading on the beam, we can directly draw shear and moment diagrams from either equilibrium or inspection, noting boundary conditions and discontinuities.

Next, we developed a link between the internal moment and the axial stresses in the material. We argued that the axial stress distribution would be linear and zero at the location of the neutral axis given no axial loading on the beam.

$$\mathcal{M}(x) = -\int_{A(x)} \sigma_{xx}(x, y, z) y dA$$

The 'y' here is the distance from the neutral axis $y = w - \overline{w}$ which is the location such that the first moment of the material stiffness is zero:

$$\int_{A(x)} E(x, y, z)(w - \overline{w}) dA = 0$$

We related the beam strain to a radius of curvature $\rho(x)$ which varies only in the axial direction, and then related the radius of curvature to the slope and displacement under small deformations of the beam.

$$\varepsilon_{xx}(x) = -y\frac{1}{\rho(x)} = -y\frac{d\theta}{dx} = -y\frac{d^2 v}{dx^2}$$

In combination with Hooke's law, we arrived at the following second order differential constitutive equation in terms of an effective bending stiffness for the cross section:

$$\frac{d^2 v}{dx^2} = \frac{\mathcal{M}(x)}{(EI)_{eff}(x)} \qquad (EI)_{eff} = \int_{A(x)} E(x, y, z) y^2 dA$$

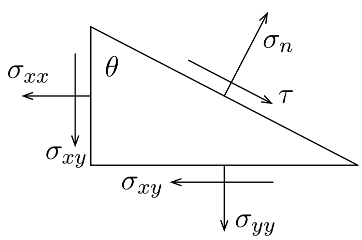
In order to solve this second order differential equation, two boundary conditions are needed.

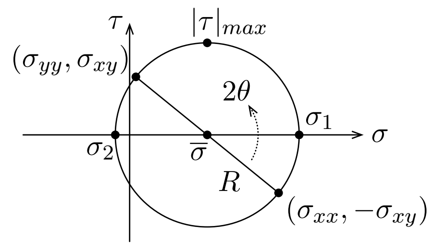
Lastly, just as the internal axial stress varies over the beam cross section, we also derived a relationship for the distribution of the transverse shear stress in a bending beam with uniform Young's modulus:

$$\sigma_{xy}(x,y) = \frac{V(x)Q(x,y)}{I(x)t(x,y)} \qquad \text{where} \qquad Q = \int_{A'(x,y)} y dA'$$

Here, A' is the subset of the cross sectional area further away from the neutral axis than y, and t is the width of the cross section at height y. Note that the transverse shear stress goes to zero at the top and bottom of a bending beam and is highest at the neutral axis.

2 Mohr's Circle





Mohr's circle provides a way to visualize the state of stress in a material. Because stress is a tensor quantity, it can manifest in different forms depending on the way we cut the material to analyze the internal stress. Our convention is to consider positive shear τ to run clockwise around a body. Given this convention, the equations describing stress transformations in two dimensions given an initial state of stress $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ and a rotation angle θ are:

$$\sigma_n(\theta) - \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos(2\theta) + \sigma_{xy}\sin(2\theta)$$
$$\tau(\theta) = \frac{\sigma_{xx}-\sigma_{yy}}{2}\sin(2\theta) - \sigma_{xy}\cos(2\theta)$$

Squaring both sides and adding them together yields the equation for Mohr's Circle:

$$(\sigma_n - \overline{\sigma})^2 + \tau^2 = R^2 \qquad \text{where} \quad \overline{\sigma} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \quad \text{and} \quad R^2 = \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2$$

Note that the principal stresses are always $\sigma_{1,2} = \overline{\sigma} \pm R$ and the magnitude of the maximum shear stress is always $|\tau|_{max} = R$.

3 General Review

3.1 Static Equilibrium

$$\sum \boldsymbol{F} = \boldsymbol{0} \quad , \quad \sum (\boldsymbol{M})_A = \boldsymbol{0}$$

3.2 3D Constitutive Relations and Thermal Expansion

$$\varepsilon_{ii} = \frac{du_i}{di} = \frac{1}{E}[\sigma_{ii} - \nu(\sigma_{jj} + \sigma_{kk})] + \alpha\Delta T$$

$$2\varepsilon_{ij} = \frac{du_i}{dj} + \frac{du_j}{di} = \gamma_{ij} = \frac{\sigma_{ij}}{G} \qquad G = \frac{E}{2(1+\nu)}$$

3.3 Pressure Vessels

State of stress for cylindrical pressure vessels given $R \gg t$:

$$(\sigma_{rr}, \sigma_{xx}, \sigma_{\theta\theta})_{open} = (0, 0, 1)\,\frac{PR}{t} \qquad (\sigma_{rr}, \sigma_{xx}, \sigma_{\theta\theta})_{closed} = (0, 1, 2)\,\frac{PR}{2t}$$

Strain deformation relations for cylindrical pressure vessels:

$$\varepsilon_{xx} = \frac{\delta_L}{L} \qquad \varepsilon_{\theta\theta} = \frac{2\pi(R + \delta_R) - 2\pi R}{2\pi R} = \frac{\delta_R}{R} \qquad \varepsilon_{rr} = \frac{\delta_t}{t}$$

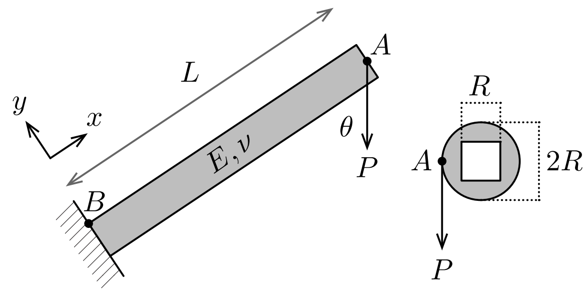
3.4 Beam Deformation and Combined Loadings

In this class, we have analyzed the deformation of beams under axial load, torsion, and bending. Our equations have only dealt with *small elastic deformations*. Because of this, if a beam undergoes multiple types of deformations, we can treat it as a linear superposition of the different modes of deformation. The analogous equations for axial loading, torsion, and bending are shown on the following page. The first four equations listed for each mode of deformation are the general equations derived from geometry, equilibrium, and Hooke's law. The second set of equations are useful derived equations from the first four. Note that we have assumed that $\sigma_{yy} = \sigma_{zz} = \Delta T = 0$ to simplify our constitutive equations.

Beam Deformation

Axial Load	Torsion	Bending
$\mathcal{N}(x) = \int_{A(x)} \sigma_{xx}(x,y,z)dA$	$\mathcal{T}(x) = \int_{A(x)} r\sigma_{x\theta}(x,r,\theta)dA$	$\mathcal{V}(x) = \frac{d\mathcal{M}}{dx} \quad \vert \quad \mathcal{M}(x) = -\int_{A(x)} \sigma_{xx}(x,y,z)ydA$
$\sigma_{xx}(x,y,z) = E(x,y,z)\varepsilon_{xx}(x)$	$\sigma_{x\theta}(x,r,\theta) = 2G(x,r,\theta)\varepsilon_{x\theta}(x,r)$	$\sigma_{xx}(x,y,z) = E(x,y,z)\varepsilon_{xx}(x)$
$\varepsilon_{xx} = \frac{du_x}{dx}$	$\varepsilon_{x\theta} = \frac{r}{2}\frac{d\phi}{dx}$	$\varepsilon_{xx}(x) = -y\frac{1}{\rho(x)} = -y\frac{d\theta}{dx} = -y\frac{d^2v}{dx^2}$
$u_x(x) = \int \frac{du_x}{dx}dx + C$	$\phi(x) = \int \frac{d\phi}{dx}dx + C$	$\theta(x) = \int \frac{1}{\rho(x)}dx + C_1 \quad \vert \quad v(x) = \int \theta(x)dx + C_2$
$(EA)_{eff} = \int_{A(x)} E(x,y,z)dA$	$(GI_P)_{eff} = \int_{A(x)} G(x,r,\theta)r^2dA$	$(EI)_{eff} = \int_{A(x)} E(x,y,z)y^2dA$
$A_{rect} = bh \qquad A_{circ} = \pi R^2$	$(I_P)_{rect} = \frac{bh}{12}(b^2+h^2) \quad (I_P)_{circ} = \frac{\pi}{2}R^4$	$I_{rect} = \frac{1}{12}bh^3 \qquad I_{circ} = \frac{\pi}{4}R^4$
$\frac{du_x}{dx} = \frac{\mathcal{N}(x)}{(EA)_{eff}(x)}$	$\frac{d\phi}{dx} = \frac{\mathcal{T}(x)}{(GI_P)_{eff}(x)}$	$\frac{d^2v}{dx^2} = \frac{\mathcal{M}(x)}{(EI)_{eff}(x)}$
$\sigma_{xx}(x,y,z) = \frac{\mathcal{N}(x)E(x,y,z)}{(EA)_{eff}(x)}$	———	$\sigma_{xx}(x,y,z) = -\frac{\mathcal{M}(x)E(x,y,z)y}{(EI)_{eff}(x)}$
———	$\sigma_{x\theta}(x,r,\theta) = \frac{\mathcal{T}(x)G(x,r,\theta)r}{(GI_P)_{eff}(x)}$	$\sigma_{xy}(x,y) = \frac{\mathcal{V}(x)}{I(x)t(x,y)}\int_{A'(x,y)} ydA' \quad (E = \text{const})$

4 Example: Beam Deformation

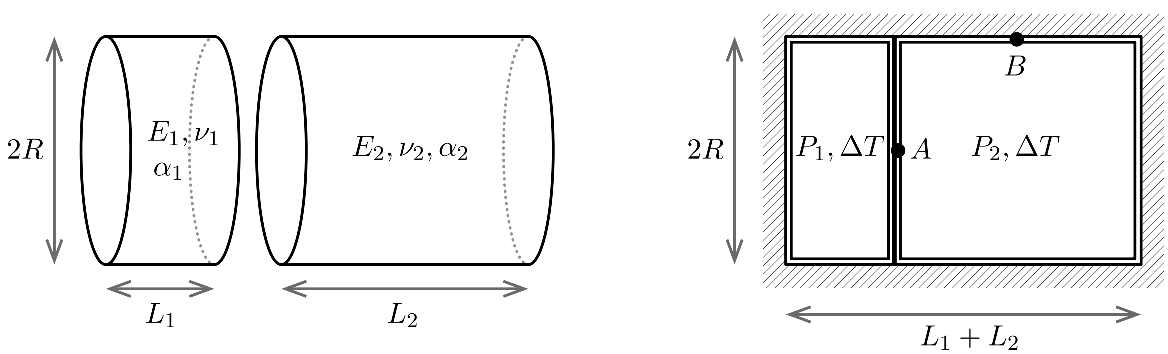


A uniform beam with Young's modulus E, poisson ratio ν, and geometry shown above is rigidly attached to a wall and deforms under an applied load P.

(a) Determine the displacement of point A assuming small deflections.

(b) Determine the state of stress at point B located at the top of the beam. Find the principal stresses $\sigma_{1,2}$ and maximum magnitude shear stress $|\tau|_{max}$.

5 Example: Statically Indeterminate Pressure Vessel



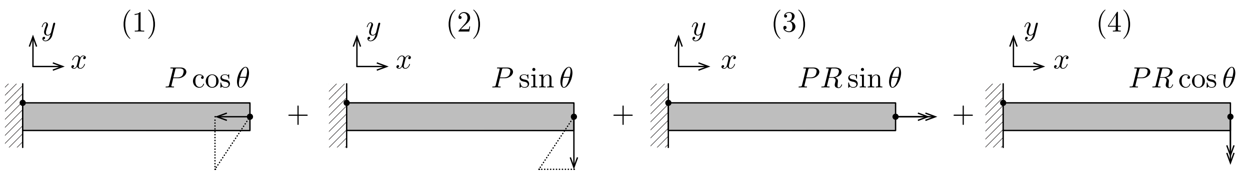
Two thin-walled cylindrical pressure vessels each with wall thickness t have unpressurized geometry shown on the left. They are enclosed in a rigid frictionless cylindrical chamber as shown on right, heated through a change in temperature ΔT, and pressurized to different pressures.

(a) Determine the displacement of point A assuming $\sigma_{rr} \approx 0$.

(b) Determine the state of stress at point B. Find the principal stresses $\sigma_{1,2}$ and maximum magnitude shear stress $|\tau|_{max}$.

Beam Deformation Example Solution:

For small deformations, this loading can be thought of as a superposition of multiple deformation modes. The load is axially compressing the bar and bending it down. Also, because point A is offset from the centerline, the load also twists the beam about its axis and bends the beam out of the page. These four loadings are shown below. Each deformation mode can contribute to the deflection of point A and the state of stress at point B. Let us analyze these for each mode.



(1) Axial loading with axial load $-P\cos\theta$ applied to the end in the x direction.

(2) Bending with point load $-P\sin\theta$ applied to the end in the y direction.

(3) Torsion with torque load $PR\sin\theta$ applied to the end in the x direction.

(4) Bending with concentrated moment $-PR\cos\theta$ applied to the end in the y direction.

(1) Axial Load

Contribution to deflection of A:

$$\mathcal{N}(x) = -P\cos\theta \qquad (EA)_{eff} = ER^2(\pi - 1) \qquad u_x^{A(1)}(L) = \frac{\mathcal{N}L}{(EA)_{eff}} = \boxed{-L\frac{P}{ER^2}\frac{\cos\theta}{\pi - 1}}$$

Contribution to stress at B:

$$\sigma_{xx}^{B(1)}(0,R) = \frac{\mathcal{N}E}{(EA)_{eff}} = \boxed{-\frac{P}{R^2}\frac{\cos\theta}{\pi - 1}}$$

(2) Vertical Bending

Contribution to deflection of A:

$$\mathcal{M}(x) = P\sin\theta(x-L) \qquad (EI)_{eff} = \frac{1}{12}ER^4(3\pi-1) \qquad \frac{d^2}{dx^2}v^{A(2)} = \frac{\mathcal{M}}{(EI)_{eff}} = -\frac{P(x-L)}{ER^4}\frac{12\sin\theta}{(3\pi - 1)}$$

$$\text{Given:} \quad v(0) = \theta(0) = 0 \qquad v^{A(2)}(x) = \frac{P(x^4 - 3Lx^2)}{ER^4}\frac{2\sin\theta}{(3\pi - 1)} \qquad u_y^{A(2)}(L) = \boxed{-L\frac{PL^2}{ER^4}\frac{4\sin\theta}{(3\pi - 1)}}$$

Contribution to stress at B:

$$\sigma_{xx}^{B(2)}(0,R) = -\frac{\mathcal{M}Ey}{(EI)_{eff}} = \boxed{\frac{PL}{R^3}\frac{12\sin\theta}{3\pi - 1}} \qquad \sigma_{xy}^{B(2)}(0,R) = \frac{\mathcal{V}(x)}{I(x)t(y)}\cancelto{0}{\int_{A'(x,y)} y\,dA'} = 0$$

(3) Torsion

Contribution to deflection of A:

$$\mathcal{T}(x) = PR\sin\theta \qquad (GI_P)_{eff} = \left[\frac{E}{2(1+\nu)}\right]\frac{1}{6}R^4(3\pi-1) \qquad \phi^{A(3)}(L) = \frac{\mathcal{T}L}{(GI_P)_{eff}} = \frac{PL}{ER^3}\frac{12\sin\theta(1+\nu)}{3\pi-1}$$

$$u_z^{A(3)} \approx 0 \qquad u_y^{A(3)} \approx -R\phi^{A(3)}(L) = \boxed{-L\frac{P}{ER^2}\frac{12\sin\theta(1+\nu)}{3\pi-1}}$$

Contribution to stress at B:

$$\sigma_{xz}^{B(3)}(0,R) = \frac{\mathcal{T}Gr}{(GI_P)_{eff}} = \boxed{\frac{P}{R^2}\frac{6\sin\theta}{3\pi-1}}$$

(4) Sideways Bending

Contribution to deflection of A:

$$\mathcal{M}(x) = PR\cos\theta \qquad (EI)_{eff} = \frac{1}{12}ER^4(3\pi-1) \qquad \frac{d^2}{dx^2}v^{A(4)} = \frac{\mathcal{M}}{(EI)_{eff}} = \frac{P}{ER^3}\frac{12\cos\theta}{(3\pi-1)}$$

$$\text{Given:} \quad v(0) = \theta(0) = 0 \qquad v^{A(4)}(x) = \frac{P}{ER^3}\frac{6\cos\theta}{(3\pi-1)}x^2 \qquad u_z^{A(4)}(L) = \boxed{L\frac{PL}{ER^3}\frac{6\cos\theta}{(3\pi-1)}}$$

Contribution to stress at B:

$$\sigma_{xx}^{B(2)}(0,0) = -\frac{\mathcal{M}E\cancelto{0}{y}}{(EI)_{eff}} = 0 \qquad \sigma_{xy}^{B(2)}(0,R) = \frac{\cancelto{0}{\mathcal{V}(x)}}{I(x)t(y)}\int_{A'(x,y)} y\,dA' = 0$$

Thus, deflection of point A due to loading is:

$$\boxed{\mathbf{u}^A = \frac{PL}{ER^2}\left[-\frac{\cos\theta}{\pi-1}, -\frac{4\sin\theta}{3\pi-1}\left(\frac{L^2}{R^2}+3(1+\nu)\right), \frac{L}{R}\frac{6\cos\theta}{3\pi-1}\right]}$$

And state of stress of material at point B is:

$$\boxed{\bar{\bar{\sigma}}_B = \frac{P}{R^2}\begin{bmatrix} \left(\frac{L}{R}\frac{12\sin\theta}{3\pi-1} - \frac{\cos\theta}{\pi-1}\right) & 0 & \frac{6\sin\theta}{3\pi-1} \\ 0 & 0 & 0 \\ \frac{6\sin\theta}{3\pi-1} & 0 & 0 \end{bmatrix}}$$

Thus, since $\overline{\sigma} = \frac{1}{2}\frac{P}{R^2}\left(\frac{L}{R}\frac{12\sin\theta}{3\pi-1} - \frac{\cos\theta}{\pi-1}\right)$ and $R = \frac{P}{R^2}\sqrt{\left(\frac{L}{R}\frac{12\sin\theta}{3\pi-1} - \frac{\cos\theta}{\pi-1}\right)^2 + \left(\frac{6\sin\theta}{3\pi-1}\right)^2}$,

$$\boxed{\sigma_{1,2} = \overline{\sigma} \pm R} \qquad \text{and} \qquad \boxed{|\tau|_{max} = R}$$

Statically Indeterminate Pressure Vessel Example Solution:

This problem is statically indeterminate, so we will have to solve a system of equations to find the displacement of point A. Note that there is only a single degree of freedom of the system: u_x^A, the x displacement of point A. First, let us perform equilibrium. Making a vertical cut on either side of point A, we can write down an equation of equilibrium relating σ_{xx}^1 to σ_{xx}^2:

$$\sum F_x = 0 = P_1(\pi R^2) - P_2)\pi R^2) + \sigma_{xx}^2(2\pi Rt) - \sigma_{xx}^1(2\pi Rt) \quad \Longrightarrow \quad \underline{\frac{R}{2t}(P_1 - P_2) = (\sigma_{xx}^1 - \sigma_{xx}^2)}$$

This is one equation in two unknowns, σ_{xx}^1 and σ_{xx}^2, thus the system is statically indeterminate. Constitutive relations relating the state of stress in the walls to the axial strain are:

$$\underline{\varepsilon_{xx}^1 = \frac{1}{E_1}\left[\sigma_{xx}^1 - \nu_1(\sigma_{\theta\theta}^1 + \cancelto{0}{\sigma_{rr}^1})\right] + \alpha_1 \Delta T}$$

$$\underline{\varepsilon_{xx}^2 = \frac{1}{E_2}\left[\sigma_{xx}^2 - \nu_2(\sigma_{\theta\theta}^2 + \cancelto{0}{\sigma_{rr}^2})\right] + \alpha_2 \Delta T}$$

This provides two equations but introduces an additional four new unknowns (ε_{xx}^1, ε_{xx}^2, $\sigma_{\theta\theta}^1$, $\sigma_{\theta\theta}^2$). However, we know that the circumferential strains ($\varepsilon_{\theta\theta} = \delta_R/R = 0$) of the vessels are constrained. Thus we have two more equations describing this constraint:

$$\underline{\varepsilon_{\theta\theta}^1 = 0 = \frac{1}{E_1}\left[\sigma_{\theta\theta}^1 - \nu_1(\sigma_{xx}^1 + \cancelto{0}{\sigma_{rr}^1})\right] + \alpha_1 \Delta T}$$

$$\underline{\varepsilon_{\theta\theta}^2 = 0 = \frac{1}{E_2}\left[\sigma_{\theta\theta}^2 - \nu_2(\sigma_{xx}^2 + \cancelto{0}{\sigma_{rr}^2})\right] + \alpha_2 \Delta T}$$

Lastly, we must relate the geometry of the deformations to the degrees of freedom of the system.

$$\underline{\delta_L^1 = L_1 \varepsilon_{xx}^1 = u_x^A} \qquad \underline{\delta_L^2 = L_2 \varepsilon_{xx}^2 = -u_x^A}$$

A symbolic algebraic solver can be used to solve this system of seven equations in seven unknowns yielding any of (σ_{xx}^1, σ_{xx}^2, $\sigma_{\theta\theta}^1$, $\sigma_{\theta\theta}^2$, ε_{xx}^1, ε_{xx}^2, u_x^A).

The state of stress at point B is given by $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = (\sigma_{xx}^2, \sigma_{\theta\theta}^2, 0)$. Thus, since $\overline{\sigma} = \frac{1}{2}(\sigma_{xx}^2 + \sigma_{\theta\theta}^2)$ and $R = \frac{1}{2}(\sigma_{xx}^2 - \sigma_{\theta\theta}^2)$,

$$\boxed{\sigma_{1,2} = \overline{\sigma} \pm R} \quad \text{and} \quad \boxed{|\tau|_{max} = R}$$