Spectrahedral representations of the convex hull of $SO(n)$

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Orbitopes

\begin{align*}
\bullet (1, 2, 3) & \quad \bullet (1, 0) \\
\end{align*}

orbitope = \text{conv}_{g \in G} \{ g \cdot x_0 \}

- \text{G finite, orbitope polyhedral}
- \text{G infinite, orbitope usually non-polyhedral}
- \text{In many interesting examples, } G = SO(n) \text{ or } O(n)

Systematic study of algebraic and geometric properties
[Sanyal, Sottile, Sturmfels 2011]
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Orbitopes: examples

Polyhedral

- $\ell_1, \ell_\infty$ balls
- cut polytope, quadratic assignment polytope

Non-polyhedral

- sphere, nuclear norm, spectral norm balls
- non-negative polynomials
- moments of probability measures on spaces with a lot of symmetry
Spectrahedral representations

Convexity alone is not enough

\[ C := \text{conv}_{g \in G} \{ g \cdot x_0 \} \quad \text{convex but not useful description} \]

Would like a description that
- provides algorithms
- gives insight into geometry

Spectrahedral representation

\[ C = \{ x \in \mathbb{R}^n : A_0 + \sum_{i=1}^n A_i x_i \succeq 0 \} \]

Size of representation := size of symmetric matrices \( A_i \)
Main orbitope of interest

‘Tautological $SO(n)$ orbitope’

convex hull of $n \times n$ rotation matrices $= \text{conv}_{g \in SO(n)} \{ g \cdot \text{Identity} \}$

conv $SO(n)$

$\text{conv} \{ X : X^T X = I, \ \det(X) = 1 \}$

$SO(n)^\circ$

$\{ Y : \max_{X \in SO(n)} \langle Y, X \rangle \leq 1 \}$
Main questions

- Is the convex hull of $SO(n)$ a spectrahedron? [Sanyal et al. 2011]
- Is the polar of $SO(n)$ a spectrahedron?

Previous work:
- conv $SO(2)$ and $SO(2)^\circ$ are spectrahedra (circles)
- conv $SO(3)$ and $SO(3)^\circ$ are spectrahedra [Parrilo; Sanyal et al. 2011]
O(n) and SO(n) are different

O(n) has two connected components det(X) = 1 and det(X) = −1

Answers to main questions for O(n) case:

- Spectral norm ball = conv O(n) is a spectrahedron
- Nuclear norm ball = O(n)°, also a spectrahedron
  [Sanyal et al. 2011]

Distinguishing sign of determinant is more tricky!
Main result

Theorem

- \( SO(n)^{\circ} \) has a spectrahedral rep of size \( 2^{n-1} \)
- \( \text{conv} \ SO(n) \) has a spectrahedral rep of size \( 2^{n-1} + 2n \)

There are no smaller spectrahedral representations of these sets (apart for some easy simplifications for \( n = 2, 3 \)).

\[
\text{conv} \ SO(n) = \left\{ X : \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix} \preceq I, \sum_{ij} X_{ij} A_{ij}^{\text{odd}} \preceq (n-2)I \right\}
\]

\[
SO(n)^{\circ} = \left\{ Y : \sum_{ij} Y_{ij} A_{ij}^{\text{even}} \preceq I \right\}
\]
The matrices $A_{ij}$

$A^{\text{even}}_{ij}$ is $2^{n-1} \times 2^{n-1}$ symmetric matrix indexed by even subsets of \{1, 2, \ldots, n\}:

$$\left[A^{\text{even}}_{ij}\right]_{I,J} = \begin{cases} 
(-1)^{|I \cap \{i\}|} & \text{if } i = j \text{ and } I = J \\
(-1)^{|I \cap J \cap (i,j)|+1} & \text{if } i < j \text{ and } I \Delta J = \{i, j\} \\
(-1)^{|I \cap J \cap (j,i)_c|} & \text{if } j < i \text{ and } I \Delta J = \{i, j\} \\
0 & \text{otherwise} 
\end{cases}$$

where $(i, j) = \{k : i < k < j\}$. 

Quadratic parameterization of $SO(n)$

Two natural descriptions of $SO(n)$
- Defining polynomial equations
- Quadratic parameterization

Key fact
There is a (very special) subset $\text{Spin}(n) \subset \mathbb{R}^{2n-1}$ and a quadratic map $Q : \mathbb{R}^{2n-1} \to \mathbb{R}^{n \times n}$ such that
  - $SO(n) = Q(\text{Spin}(n))$
  - $\text{Spin}(n)$ is a subset of unit sphere in $\mathbb{R}^{2n-1}$

Example ($2 \times 2$ case)

$$Q(z_1, z_2) = \begin{bmatrix} z_1^2 - z_2^2 & -2z_1z_2 \\ 2z_1z_2 & z_1^2 - z_2^2 \end{bmatrix}$$
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$$Q(\cos(\theta/2), \sin(\theta/2)) = \begin{bmatrix} \cos^2(\theta/2) - \sin^2(\theta/2) & -2\sin(\theta/2)\cos(\theta/2) \\ 2\sin(\theta/2)\cos(\theta/2) & \cos^2(\theta/2) - \sin^2(\theta/2) \end{bmatrix}$$
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Proof: spectrahedral representation of $SO(2)^\circ$

**Aim:** describe

$$SO(2)^\circ = \{ Y : \max_{X \in SO(2)} \langle Y, X \rangle \leq 1 \}$$
Proof: spectrahedral representation of $SO(2)^\circ$

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**Since:** $SO(2) = Q(\text{unit circle})$

$$\max_{X \in SO(2)} \langle Y, X \rangle = \max_{z \in \text{unit circle}} \langle Y, Q(z) \rangle$$

$$= \max_{z \in \text{unit circle}} \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} Y_{11} + Y_{22} & Y_{21} - Y_{12} \\ Y_{21} - Y_{12} & -Y_{11} - Y_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$
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Spectrahedral representation of $SO(2)^\circ$

$$\max_{X \in SO(2)} \langle Y, X \rangle \leq 1 \text{ if and only if } \begin{bmatrix} Y_{11} + Y_{22} & Y_{21} - Y_{12} \\ Y_{21} - Y_{12} & -Y_{11} - Y_{22} \end{bmatrix} \preceq I$$
Proof sketch: spectrahedral representation of $SO(n)^\circ$

\[
\max_{X \in SO(n)} \langle Y, X \rangle = \max_{z \in \text{Spin}(n)} \langle Y, Q(z) \rangle \leq \max_{z \in \text{unit sphere}} \langle Y, Q(z) \rangle
\]

**Main technical contribution**: For any $n \times n$ matrix $Y$:

- all eigenvectors of quadratic form $z \mapsto \langle Y, Q(z) \rangle$ are in $\text{Spin}(n)$
- implies: largest eigenvector is in $\text{Spin}(n)$
- implies: equality instead of inequality!
Proof sketch: spectrahedral representation of $SO(n)^\circ$

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Spectrahedral representation of $SO(n)^\circ$

\[
\max_{X \in SO(n)} \langle Y, X \rangle \leq 1 \quad \text{if and only if} \quad \sum_{i,j} Y_{ij} A_{ij}^\text{even} \leq I
\]
Projected spectrahedral representation of $\text{conv } SO(n)$

**Polarity**

\[
\text{conv } SO(n) = (SO(n)^\circ)^\circ
\]

**Conic duality**

Easy *projected* spectrahedral representation of $\text{conv } SO(n)$ from spectrahedral representation of $SO(n)^\circ$.

This *does not* tell us whether $\text{conv } SO(n)$ is a spectrahedron.
Spectrahedral representation of \(\text{conv } SO(n)\)

Theorem (Miranda and Thompson, 1994)

\[
\text{conv } SO(n) = \left\{ X \in \mathbb{R}^{n \times n} : \sigma_1(X) \leq 1, \quad \sum_{i=1}^{n-1} \sigma_i(X) - \text{sgn det}(X)\sigma_n(X) \leq n - 2 \right\}.
\]

Spectrahedral representation:

\[
\text{conv } SO(n) = \left\{ X \in \mathbb{R}^{n \times n} : X \in \text{conv } O(n), \quad X \in (n - 2) SO^{-}(n) \right\}.
\]
Our spectrahedral representations are smallest possible

Diagonal elements of \( SO(n) \):

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

Diagonal slice of conv \( SO(n) \) = parity polytope

\[\begin{align*}
\text{parity polytope has } & 2^{n-1} + 2n \text{ facets (if } n \geq 4) \\
\implies \text{any spectrahedral rep of conv } SO(n) \text{ has size at least } & 2^{n-1} + 2n \text{ (if } n \geq 4)
\end{align*}\]

Diagonal slice of \( SO(n)^\circ \) = polar of parity polytope

\[\begin{align*}
\text{polar of parity polytope has } & 2^{n-1} \text{ facets} \\
\implies \text{any spectrahedral rep of } SO(n)^\circ \text{ has size at least } & 2^{n-1}
\end{align*}\]
Questions/Conclusions

Summary

- Gave explicit (minimum sized) spectrahedral representations for $SO(n)^\circ$ and conv $SO(n)$.

Questions

- Are there polynomial sized semidefinite (projected spectrahedral) representations of $SO(n)^\circ$ and conv $SO(n)$?
- When are a convex set and its polar both spectrahedra?
- When do orbitopes have (small) spectrahedral/semidefinite representations?