Exploiting the Structure of Two-Stage Robust Optimization Models with Exponential Scenarios

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This paper addresses a class of two-stage robust optimization models with an exponential number of scenarios given implicitly. We apply Dantzig-Wolfe decomposition to exploit the structure of these models and show that the original problem reduces to a single-stage robust problem. We propose a Benders algorithm for the reformulated single-stage problem. We also develop a heuristic algorithm that dualizes the linear programming relaxation of the inner maximization problem in the reformulated model and iteratively generates cuts to shape the convex hull of the uncertainty set. We combine this heuristic with the Benders algorithm to create a more effective hybrid Benders algorithm. Since the master problem and subproblem in the Benders algorithm are mixed integer programs, it is computationally demanding to solve them optimally at each iteration of the algorithm. Therefore, we develop novel stopping conditions for these mixed integer programs and provide the relevant convergence proofs. Extensive computational experiments on a nurse planning and a two-echelon supply chain problem are performed to evaluate the efficiency of the proposed algorithms.

Key words: Integer programming, Dantzig-Wolfe decomposition, two-stage robust optimization.

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1. Introduction

In the operations research literature there are many different methodologies to address uncertainty in optimization problems. Stochastic approaches are one of the main classes and are applicable if probability distributions of uncertain parameters are known. However, these approaches are usually criticized for requiring information on the probability distributions and also for computational complexities. Robust optimization, a more recent methodology, generally assumes that uncertain parameters belong to an uncertainty set, and aims to find a robust

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solution immunizing the decision maker against the worst-case scenario within this uncertainty set.

Robust optimization was initially proposed for single-stage optimization problems where the decision maker must choose a complete solution before the disclosure of information about the real values of uncertain parameters (Soyster 1973, Ben-Tal and Nemirovski 1999). Then it was extended to multi-stage problems where the values of uncertain parameters are revealed gradually in several stages (Ben-Tal et al. 2004, Delage and Iancu 2015). In multi-stage robust problems the decision maker does not choose a complete solution at the beginning, but instead makes partial decisions sequentially after observing the values of uncertain parameters over different stages.

In robust optimization problems choosing an appropriate uncertainty set is critical and can highly affect the robustness and the optimal objective value of the obtained solution. The decision maker should select a suitable uncertainty set to reasonably represent the randomness of the uncertain parameters while taking into account the computational issues arising in the solution algorithm. From the literature on robust optimization, the most prevalent uncertainty sets are box uncertainty sets (Soyster 1973), ellipsoidal uncertainty sets (Ben-Tal and Nemirovski 1999, El Ghaoui and Lebret 1997, El Ghaoui et al. 1998), polyhedral uncertainty sets and $\Gamma$-cardinality uncertainty sets (Bertsimas and Sim 2004). In box uncertainty sets, uncertain parameters are assumed to take their values from different intervals independently. Box uncertainty sets usually result in overly conservative solutions because all parameters are allowed to take their worst values simultaneously. Ellipsoidal uncertainty sets alleviate this issue by restricting the uncertain parameters to an ellipsoidal space and this prevents them from taking worst values at the same time. Polyhedral uncertainty sets confine the uncertain parameters to a polyhedral space and can be viewed as a special case of ellipsoidal uncertainty sets (Ben-Tal and Nemirovski 1999). In $\Gamma$-cardinality uncertainty sets, for each constraint the number of uncertain parameters deviating from their nominal values must be less than $\Gamma$.

In the literature, convex uncertainty sets are used to model robust problems. The main advantage of these uncertainty sets is that they can be simply formulated by continuous parameters and the problem remains tractable in many cases such as linear programs. However, it is sometimes unavoidable or desirable to use integer parameters to formulate the uncertainty set, which results in an exponential number of scenarios. Nguyen and Lo (2012) studied a single-stage robust portfolio problem where the weights of portfolios are fixed such that a generic objective function is optimized for the worst possible ranking of portfolios. Thus, in this application it is necessary to use integer parameters to formulate the ranking of portfolios. Feige et al. (2007)
and Gupta et al. (2014) also studied several classical covering problems where in their uncertainty sets, integer parameters were used to choose a set of active clients in a graph. Moreover, in some cases integer parameters are used to approximate non-convex uncertainty sets. For instance, Siddiq (2013) and Chan et al. (2017) studied a robust facility location problem and discussed how non-convex uncertainty sets can be approximated by discretization.

In this work we assume that the uncertainty appears on the right-hand side values and the corresponding technology matrix of recourse decision variables has a block-diagonal structure. The main contribution of our work is a novel reformulation exploiting the block-diagonal structure and three solution methods for a class of two-stage robust problems with an exponential number of scenarios given implicitly. This decomposition reduces the original two-stage problem to a single-stage problem. We then develop a Benders algorithm for the reformulated problem. We also develop a heuristic algorithm, and combine it with the Benders algorithm to create a more effective hybrid Benders algorithm. Since the master problem and subproblem in the Benders algorithm are mixed-integer programs, it is computationally expensive to solve them to optimality. Hence, we propose novel stopping conditions for these mixed integer programs and prove the convergence of the algorithm. We evaluate the computational performance of the proposed algorithms in a nurse planning and a two-echelon supply chain application.

We organize the remainder of this paper as follows. In Section 2, we provide a literature review on robust optimization with a focus of two-stage problems. In Section 3, we introduce the structure of the two-stage robust optimization problems studied in this paper and apply Dantzig-Wolfe decomposition to reformulate the original two-stage robust problem as a single-stage robust problem. In Section 4, we develop solution methods for the reformulated problem. In Section 5, we propose stopping conditions for the master problem and subproblem of the Benders algorithm. In Section 6, we show how to apply the proposed reformulation on a two-stage nurse planning problem and a two-echelon supply chain problem. We provide extensive computational results on these applications in Section 7. Finally we give concluding remarks and future research directions in Section 8. Omitted proofs are provided in the electronic supplement.

2. Literature review

In a single-stage robust optimization problem, constraints must be satisfied for all possible realizations of uncertain parameters. Therefore, by repeating constraints for different values of uncertain parameters, we can view a robust problem as a mathematical program with a large number of constraints. Depending on the structure of the uncertainty set, two techniques are usually applied to solve single-stage robust problems. The first approach is to iteratively generate violated constraints of the mathematical program explained above using a constraint
generation algorithm (Fischetti and Monaci 2012, Bertsimas et al. 2015). In the second approach, the problem is reformulated as its deterministic robust counterpart and then solved directly. Soyster (1973) presented such a deterministic counterpart model for robust linear problems with box uncertainty sets. Ben-Tal and Nemirovski (1999) proposed a second order cone program for uncertain linear programs with ellipsoidal uncertainty sets. They also showed that in the case of polyhedral uncertainty sets the robust counterpart model is a linear program. Bertsimas and Sim (2004) showed that robust linear programs with $\Gamma$-cardinality uncertainty sets can be reformulated as deterministic linear programs.

Multi-stage robust problems are more complicated than single-stage robust problems and are generally intractable (Ben-Tal et al. 2004). There are two common solution approaches for these problems. Both approaches transform the multi-stage problem to a single-stage problem and then apply the solution methods of the single-stage robust problem. In the first approach the recourse decisions are restricted to a function of uncertain parameters resulting in a single-stage robust problem. In this context, affine adaptability, also referred to as linear decision rules, assumes recourse decisions to be affine functions of uncertain parameters. This method is very popular and is applied in various areas such as supply chain management (Ben-Tal et al. 2005), inventory control (Ben-Tal et al. 2009), portfolio management (Fonseca and Rustem 2012), warehouse management (Ang et al. 2012), capacity management (Ouorou 2013) and network design (Poss and Raack 2013). Chen and Zhang (2009) introduced the extended affine adaptability by re-parameterizing the primitive parameters and then applying the affine adaptability. Bertsimas et al. (2011) proposed a more accurate approximation of recourse decisions using polynomial adaptability. A drawback of the functional adaptability is its inability to handle problems with integer recourse decisions. Another approach is finite adaptability in which the uncertainty set is split into a number of smaller subsets, each with its own set of recourse decisions. The number of these subsets can be either fixed a priori or decided by the optimization model (Vayanos et al. 2011, Bertsimas and Caramanis 2010, Hanasusanto et al. 2015, Postek and Den Hertog 2014, Bertsimas and Dunning 2016). An important advantage of the finite adaptability is that, in contrast to the functional adaptability approach, it easily handles problems with integer recourse variables.

There are many papers in the literature that have proposed Benders algorithms to solve two-stage robust optimization problems (Zheng et al. 2012, Bertsimas et al. 2013, Remli and Rekik 2013, Zhang et al. 2015). In these papers, assuming that the problem is set as a min-max-min problem, the authors have dualized the inner minimization to reformulate the problem to a min-max problem with bilinear terms in the objective function. Then, they have applied a
Benders algorithm to solve the first-stage problem together with cuts generated from an outer approximation algorithm which solves the maximization problem.

Column-and-constraint generation is another exact algorithm to solve a two-stage robust optimization problem (Zeng and Zhao 2013, Zhao and Zeng 2012b, Danandeh et al. 2014, Ding et al. 2016, Wang et al. 2014, Lee et al. 2014, 2015, Li et al. 2015, 2017, Chen et al. 2016, Wang et al. 2016, Neyshabouri and Berg 2017). The underlying idea of this approach is to make copies of recourse decision variables and also second-stage constraints for each possible realization of uncertain parameters which results in a large-scale mixed-integer programming model. As it is impossible to solve this model directly, a column-and-constraint generation algorithm is essential to generate critical uncertain scenarios and their corresponding recourse decision variables and second-stage constraints. The tricky part of this approach is the reformulation of the max-min subproblem to a max problem using Karush-Kuhn-Tucker (KKT) conditions. The reformulated subproblem includes bilinear terms in constraints which are linearized later by introducing a set of binary variables and adding big-M constraints.

The reformulation approach proposed in this paper is different from the ones used in the aforementioned Benders and column-and-constraint generation algorithms as it does not result in any bilinear term in our models. Therefore, our solution methodology does not require an outer approximation algorithm (Bertsimas et al. 2013) or any linearization by introducing extra binary variables and big-M constraints (Zeng and Zhao 2013). Moreover, our modeling approach is capable of handing second-stage integer variables, while these algorithms work only on problems with continuous recourse variables. To handle second-stage integer variables, Zhao and Zeng (2012a) extended the original column-and-constraint generation algorithm to a tri-level algorithm. However, the extended algorithm only works on special problems satisfying three restrictive assumptions: 1) including at least one continuous recourse variable, 2) holding the extended relative complete recourse property for recourse problem when the second-stage integer variables are ignored, and 3) satisfying the quasiconvex property for the inner max-min problem. The first and third conditions are not satisfied in applications studied in this paper.

The reformulation approach that we propose is inspired from the one proposed in Siddiq (2013) that presented a reformulation for a specific facility location problem. The advantage of our reformulation is that it is more general and applicable to any two-stage robust problem with block-diagonal structure in the technology matrix of recourse decision variables. Here, we emphasize that the block-diagonal structure of uncertainty sets addressed in Ben-Tal et al. (2006) and Ben-Tal and Nemirovski (2002) are different from the block-diagonal structure in the technology matrix of recourse decision variables considered in this work. Moreover, the reformulation proposed in this work is completely different from the reformulation proposed
by Zhang (2017) that provides augmented-Lagrangian lower and upper bound for a two-stage robust optimization problem with objective uncertainty.

3. Model and reformulation

We study a class of two-stage robust optimization problems with the following structure:

\[
(P1) \quad \min_{x \in \mathcal{X}} \left( c_1^T x + \max_{u \in \mathcal{U}} \left( \min_{y \in \mathcal{Y}(x,u)} c_2^T y \right) \right).
\]

In the above formulation, \(x\) and \(y\) are the vector of decision variables in the first and the second stage, respectively. \(u\) is the vector of uncertain parameters that are restricted to the uncertainty set \(\mathcal{U}\). \(c_1\) and \(c_2\) are given cost vectors. In the second stage problem, we have \(\mathcal{Y}(x,u) = \{y \in \tilde{\mathcal{Y}} | Cy \leq b - Ax - Bu\}\) where \(A, B\) and \(C\) are known matrices with appropriate dimensions and \(b\) denotes the known vector of right-hand side values. \(x \in \mathcal{X}\) and \(y \in \tilde{\mathcal{Y}}\) represent the integrality and bound constraints that we may have for variables in the first and second stages. Objective (1) minimizes the sum of the first- and second-stage costs. In this model, \(x \in \mathcal{X}\) must be selected such that for all realizations \(u \in \mathcal{U}\) there is at least one \(y \in \mathcal{Y}(x,u)\). We assume that \(\mathcal{U}\) is a finite uncertainty set. This assumption is necessary for the convergence proofs of the proposed Benders algorithms discussed in Section 5. The first- and second-stage variables can be continuous, integer or mixed, but without loss of generality all uncertain parameters are supposed to be integer. In fact, our method generalizes to finite set of fractional parameters (as shown in EC.14) and for the sake of clarity, we present the overall approaches over integer parameters. We also assume that \(C\) is a block-diagonal matrix. The main focus of this research is to exploit this block-diagonal structure and develop algorithms to solve the reformulated problem efficiently.

In the reminder of this section, we propose a reformulation of model (P1) and use it to develop solution methods in Section 4. For this we need the following additional notation, used throughout the rest of the paper.

- \(\mathcal{K}\) : The index set of blocks in matrix \(C\).
- \(C_k\) : The \(k\)-th block in matrix \(C\).
- \(\text{Row}_k\) : The number of rows in block \(C_k\).
- \(\text{Col}_k\) : The number of columns in block \(C_k\).
- \(y_k\) : The subset of variables \(y\) involved in block \(C_k\).
- \(\mathcal{Y}_k\) : The set of integrality and bound constraints corresponding to variables \(y_k\).
- \(c_{2k}\) : The subset of \(c_2\) corresponding to variables \(y_k\).
- \(b_k\) : The right-hand side values in front of block \(C_k\).
$A_k$: The rows in matrix $A$ in front of block $C_k$.

$B_k$: The rows in matrix $B$ in front of block $C_k$.

With respect to the block-diagonal structure of matrix $C$ we can rewrite constraint $Cy \leq b - Ax - Bu$ included within the second-stage feasible space $\mathcal{Y}(x, u)$ as follows.

$$C_k y_k \leq b_k - A_k x - B_k u \quad k \in \mathcal{K}$$  \hfill (2)

Furthermore, we define some notation related to $B_k u$ as:

$S'_k$: The set of all realizations for $B_k u$, i.e., $S'_k = \{ v \in \mathbb{R}^{\text{Row}_k} | v = B_k u, u \in \mathcal{U} \}$.

$S_k$: The index set of $S'_k$, i.e., $S_k = \{ 1, 2, ..., |S'_k| \}$.

$e_{ks}$: The $s$-th member of $S'_k$ (defined for $s \in S_k$).

$w_{ks}$: A binary variable that takes 1 if $B_k u$ is equal to $e_{ks}$, 0 otherwise.

Using all the above notation, we reformulate model (P1) as:

$$\begin{align*}
(P2) \quad \min_{x \in \mathcal{X}} & \left( c_1^T x + \max_{(u, w) \in (\mathcal{U}, \mathcal{W})} \left( \min_{y \in \mathcal{Y}(x, w)} \sum_{k \in \mathcal{K}} c_2^T k y_k \right) \right) \\
(\mathcal{U}, \mathcal{W}) = & \left\{ (u, w) | u \in \mathcal{U}, \right. \\
& B_k u = \sum_{s \in S_k} e_{ks} w_{ks} \quad k \in \mathcal{K}, \\
& \sum_{s \in S_k} w_{ks} = 1 \quad k \in \mathcal{K}, \\
& w_{ks} \in \{0, 1\} \quad k \in \mathcal{K}, s \in S_k \left\} \right. \\
(\mathcal{Y}(x, w) = & \left\{ y | y_k \in \mathcal{Y}_k \quad k \in \mathcal{K}, \\
& C_k y_k \leq b_k - A_k x - \sum_{s \in S_k} e_{ks} w_{ks} \quad k \in \mathcal{K} \right\} \right. \\
\end{align*}$$

In the following we introduce a new model that is equivalent to model (P2) as we will show later in Theorem 1 and Corollary 1. To introduce model (P3), for each $k \in \mathcal{K}$ we make $|S_k|$ copies of variables $y_k \in \mathbb{R}^{\text{Col}_k}$ and define variables $y'_{ks} \in \mathbb{R}^{\text{Col}_k} (s \in S_k).$ Model (P3) is given by (10)-(12).

$$\begin{align*}
(P3) \quad \min_{x \in \mathcal{X}} & \left( c_1^T x + \max_{(u, w) \in (\mathcal{U}, \mathcal{W})} \left( \min_{y' \in \mathcal{Y}'(x)} \sum_{k \in \mathcal{K}} \sum_{s \in S_k} c_2' y'_{ks} w_{ks} \right) \right) \\
\mathcal{Y}'(x) = & \left\{ y' | y'_{ks} \in \mathcal{Y}_k \quad k \in \mathcal{K}, s \in S_k, \\
& C_k y'_{ks} \leq b_k - A_k x - e_{ks} \quad k \in \mathcal{K}, s \in S_k \right\} \right. \\
\end{align*}$$
The structure of model (P3) is such that, if $w_{ks}$ takes 1, $y'_{ks}$ is equal to the optimal solution of $y_k$ in model (P2). Indeed by introducing constraint (12) we have made $|S_k|$ copies of constraint (9) to compute the values of $y'_{ks}$ independently. Moreover, to ensure that the optimal objective values of models (P2) and (P3) are the same, $c^T_2 y'_{ks}$ in (10) is multiplied by $w_{ks}$.

**Theorem 1.** Suppose that model (P2) is feasible. Then,

(a) $\hat{x}$ is a first-stage feasible solution of model (P2) if and only if it is a first-stage feasible solution of model (P3),

(b) the objective values of models (P2) and (P3) for the first-stage solution $\hat{x}$ are the same if $\max_{u,w}$ and $\min_{y'}$ are solved optimally, and

(c) for this first-stage solution, the optimal values of variables $y'_{ks}$ in model (P3) represent the second-stage optimal policies in model (P2).

The following corollary states the relations between models (P2) and (P3).

**Corollary 1.** Models (P2) and (P3) are equivalent, that is, either

- both models are unbounded, or
- both models are infeasible, or
- both models are feasible and bounded with the same optimal objective value and the same optimal solution for the first-stage variables. In this case the optimal solution of variables $y'_{ks}$ in model (P3) represents the optimal policies for variables $y_k$ in model (P2).

**Proof.** Theorem 1 directly results in cases 1 and 3. To prove case 2, we note that with respect to Theorem 1 for any feasible solution in model (P2) there is an equivalent feasible solution in model (P3). Therefore, model (P3) is infeasible if and only if model (P2) is infeasible. \hfill \Box

The next theorem shows that model (P3) can be reduced to a single-stage problem.

**Theorem 2.** In model (P3) the objective function $\max_{u,w} (\min_{y'} (\cdot))$ can be replaced by $\min_{y'} (\max_{u,w} (\cdot))$.

Therefore, we can rewrite model (P3) as:

$$
(P4) \min_{(x,y') \in (\mathcal{X},\mathcal{Y}')} \left( c^T_1 x + \max_{(u,w) \in (\U,\W)} \left( \sum_{k \in \mathcal{K}} \sum_{s \in S_k} c^T_{2k} y'_{ks} w_{ks} \right) \right)
$$

In fact, the reformulation presented in this section shows that we can transform the two-stage robust problem (P1) to a single-stage robust problem. In the above model, we have $(\mathcal{X}, \mathcal{Y}') = \{(x,y') | x \in \mathcal{X}, y' \in \mathcal{Y}'(x)\}$. We use the latter model to present our solution methods in Section 4.
4. Solution methods

In this section we propose three solution methods for model (P4). We present a Benders algorithm that iterates between a master problem and a subproblem to tighten the optimality gap. We also propose a heuristic that dualizes the linear programming relaxation of the inner max problem in model (P4). Then it iteratively generates cuts to shape the convex hull of the uncertainty set. We also present a hybrid Benders algorithm that applies the heuristic within the framework of the Benders algorithm.

4.1. Benders algorithm

In our Benders algorithm, valid lower and upper bounds are obtained by solving the master problem and the subproblem, respectively. The algorithm iterates between these problems until the bounds converge. In the following we present the master problem and subproblem. Then we explain the framework of the Benders algorithm.

Suppose that \( m \) scenarios \((\hat{\mathcal{u}}^j, \hat{\mathcal{w}}^j) \in (\mathcal{U}, \mathcal{W}), j = 1, 2, ..., m \) are already generated by solving the subproblem. We define the master problem of the Benders algorithm as:

\[
\begin{align*}
\text{(MP)} & \quad \min_{(x, y') \in (X, Y'), \theta} \theta \\
& \quad \theta \geq c_1^T x + \sum_{k \in \mathcal{K}} \sum_{s \in \mathcal{S}_k} c_2^{ks} y'_{ks} \hat{\mathcal{w}}^j_{ks} \quad j = 1, 2, ..., m 
\end{align*}
\]

Theorem 3. The optimal objective value of model (MP) is a valid lower bound for model (P4).

For a feasible solution \((\hat{x}', \hat{y}') \in (\mathcal{X}', \mathcal{Y}')\) a valid upper bound is obtained by solving the inner max problem in model (P4). We refer to the following problem as the subproblem of the Benders algorithm.

\[
\begin{align*}
\text{(SP)} & \quad \max_{(u, w) \in (\mathcal{U}, \mathcal{W})} \left( c_1^T \hat{x} + \sum_{k \in \mathcal{K}} \sum_{s \in \mathcal{S}_k} c_2^{ks} \hat{y}'_{ks} w_{ks} \right) 
\end{align*}
\]

Algorithm 1 provides the pseudo code of the Benders algorithm. In this algorithm, \( UB \) and \( LB \) respectively denote the best upper and lower bounds found during the algorithm. In Line 3, we obtain an initial solution \((\hat{x}', \hat{y}')\) by a heuristic algorithm that is explained at the end of Section 4.2. In Line 6, the algorithm sets \( UB \) equal to the optimal objective value of the subproblem if it is less than the current \( UB \). The stopping conditions of the Benders algorithm are then checked in Line 9 where \( \delta_{\text{acc}}^{\text{Benders}} \) and \( \text{AlgTimeLimit} \) respectively denote the maximum acceptable optimality gap and the available computational time.
Algorithm 1. Benders algorithm

1: Input parameters: $\delta_{\text{Benders}}^{\text{acc}}$ and AlgTimeLimit.
2: Set $UB=\infty$, $LB=-\infty$, and $m = 0$.
3: Find an initial solution $(\hat{x}, \hat{y}')$ by a heuristic.

4: repeat
5: Modify the objective function of subproblem (SP) using $(\hat{x}, \hat{y}')$.
6: Solve the subproblem and update $UB$ if it is necessary.
7: Add a new optimality cut (15) to the master problem and set $m = m + 1$.
8: Solve the master problem and update $LB$.
9: until $(100(UB - LB)/LB \leq \delta_{\text{Benders}}^{\text{acc}}$ or time limit AlgTimeLimit is reached)

4.2. Heuristic algorithm

In this section we present a heuristic algorithm. This algorithm dualizes the linear programming relaxation of the inner max problem in model (P4) to transform the min-max problem to a single minimization problem. Let us assume that constraints forming the convex hull of the inner max problem in model (P4) are as:

$$Du + Ew \leq b_2$$

In constraint (17), $u$ and $w$ are the vectors of uncertainty variables, $D$ and $E$ are technology matrices with appropriate dimensions and $b_2$ is the known vector of right-hand side values. $D$, $E$ and $b_2$ are independent from the values of $(\hat{x}, \hat{y}')$ that are fixed in the outer min problem in model (P4). This is because the solution space of uncertainty variables does not depend on the variables in the outer min problem. If we have constraints (17) we can replace constraints $(u, w) \in (U, W)$ with them. In this case, the inner max problem is a linear programming model for fixed values of $(\hat{x}, \hat{y}')$ in the outer min problem. Therefore by dualizing the inner max problem we obtain the following model.

$$(\text{D-P4}) \quad \min_{\hat{x}, \hat{y}', \gamma} (c_1^T x + b_2^T \gamma)$$

$$(x, y') \in (X, Y')$$

$$E_{ks}^T \gamma \geq c_{ks}^T y'$$ \quad $k \in K, s \in S_k$

$$D^T \gamma = 0$$

$$\gamma \geq 0$$

In model (D-P4), $\gamma$ is the vector of dual variables for constraint (17) and $E_{ks}$ is the column in $E$ that includes coefficients of variable $w_{ks}$. We can observe that the min-max problem in
model (P4) reduces to a single min problem and can be solved directly as a mixed-integer programming model. In the literature, the above dualization technique is prevalent to simplify single-stage robust problems where the inner max problem is a linear programming model. However, in our model, the inner max problem is a mixed integer program and constraints (17) forming the convex hull of the uncertainty set are unknown. In the following we present a heuristic algorithm that relaxes the integrality constraints of the variables in the inner max problem of model (P4). Then by iteratively generating cuts, it attempts to shape the solution space of the relaxed inner max problem into its convex hull before the relaxation. To present this heuristic we first need to define models (P5) and (P6).

\[
\text{(P5)} \quad \min_{(x,y') \in (X,Y')} \left( c_1^T x + \max_{(u,w) \in (U,W)'} \left( \sum_{k \in K} \sum_{s \in S_k} c_{2k}^T y'_{ks} w_{ks} \right) \right)
\]

\[
(U,W)' = \left\{ (u, w) \in (U, W) \mid Fu + Gw \leq b_3 \right\} \tag{25}
\]

In model (P5), constraint (25) is the set of valid cuts that the heuristic algorithm generates iteratively. This constraint set is empty at the beginning of the algorithm. In this constraint, \(F\) and \(G\) are technology matrices with appropriate dimensions and \(b_3\) the known vector of right-hand side values. We obtain the following model (P6) by relaxing the integrality constraints of variables \(u\) and \(w\) in model (P5) and then dualizing the inner max problem.

\[
\text{(P6)} \quad \min_{x,y',\pi,\lambda,\alpha,\beta} \left( c_1^T x + b_3^T \pi + b_4^T \lambda + \sum_{k \in K} \alpha_k \right)
\]

\[
(x,y') \in (X,Y') \tag{27}
\]

\[
B_k^T \beta_k + H^T \lambda + F^T \pi = 0 \tag{28}
\]

\[
\alpha_k + e_{k,s}^T \beta_k + G_{ks}^T \pi \geq c_{2k}^T y'_{ks} \geq k \in K, s \in S_k \tag{29}
\]

In model (P6), \(\beta_k\) and \(\pi\) are vectors of dual variables for constraints (5) and (25) respectively, \(\alpha_k\) is the dual variable of constraint (6) defined for each \(k \in K\) and \(G_{ks}\) is the column in \(G\) that includes coefficients of variable \(w_{ks}\). To write the dual of the inner max problem in model (P5) we have supposed that linear constraints hidden in uncertainty set \(U\) in constraint (4) are represented by \(Hu \leq b_4\). In model (P6), \(\lambda\) denotes the vector of dual variables for \(Hu \leq b_4\).

Algorithm 2 provides the pseudo code of our heuristic algorithm. In Line 2, we suppose that no instance of constraint (25) is available at the beginning of the algorithm and \(F\), \(G\) and \(b_3\) are empty. “Ite” is the iteration counter of the loop starting in Line 3. In Line 5, we solve model (P6) to obtain a feasible solution for \((\hat{x}, \hat{y}')\). For a fixed solution \((\hat{x}, \hat{y}')\) in model (P5),
the inner max problem is an integer program and we denote it by InnerMax(\(\hat{x}, \hat{y}'\)). In Line 7, we call Procedure 1. In each iteration of this procedure, we solve the linear programming relaxation of InnerMax(\(\hat{x}, \hat{y}'\)) and obtain a new fractional scenario (\(\hat{u}, \hat{w}\)). Then this procedure generates a number of valid cuts to remove this fractional scenario. This procedure continues until it cannot detect any other violated cut or time limit \(\text{AlgTimeLimit}\) is reached. We use an integer programming solver to perform Procedure 1 and let it generate valid cuts as explained above. In calling the integer programming solver, we limit the maximum number of nodes to be explored in the branch and bound tree to one. In Line 8 in Algorithm 2, we extract the cuts generated by the integer programming solver and update \(F, G, b_3\) in models (P5) and (P6). In Lines 9, the algorithm checks stopping criteria. One of these stopping criteria checks if the percentage of the objective value improvement obtained in the current iteration is less than or equal to parameter \(\delta_{\text{acc}}^H\).

**Algorithm 2. Heuristic algorithm**

1: Input parameters: \(\text{LocalTimeLimit}, \text{AlgTimeLimit}\) and \(\delta_{\text{acc}}^H\).
2: Set \(\text{Ite} = 0\) and empty \(F, G, b_3\) in models (P5) and (P6).
3: repeat
4: \(\text{Ite}++\).
5: Solve model (P6) with time limit \(\text{LocalTimeLimit}\) to obtain a feasible solution (\(\hat{x}, \hat{y}'\)).
6: Set \(\text{Obj}_{\text{Ite}}\) equal to the objective value of model (P6).
7: Apply Procedure 1 to generate several cuts (25).
8: Extract the generated cuts and update \(F, G, b_3\) in InnerMax(\(\hat{x}, \hat{y}'\)).
9: until (No cut is generated in Line 7 in this iteration or time limit \(\text{AlgTimeLimit}\) is reached or \((100(\text{Obj}_{\text{Ite}} - \text{Obj}_{\text{Ite}-1})/\text{Obj}_{\text{Ite}-1} \leq \delta_{\text{acc}}^H))\)

**Procedure 1. Cut generation for the proposed heuristic algorithm**

1: repeat
2: Solve the linear programming relaxation of InnerMax(\(\hat{x}, \hat{y}'\)) to obtain (\(\hat{u}, \hat{w}\)).
3: Detect some valid cuts to remove the fractional solution (\(\hat{u}, \hat{w}\)).
4: Update \(F, G, b_3\) in InnerMax(\(\hat{x}, \hat{y}'\)).
5: until (No valid cut is generated or time limit \(\text{AlgTimeLimit}\) is reached)

The proposed heuristic algorithm does not necessarily find the optimal solution. Appendix EC.4 presents an example to demonstrate that the heuristic algorithm does not guarantee
optimality. In the Benders algorithm presented by Algorithm 1, in Line 3 we solve model (P6) with empty $F$, $G$ and $b_3$ to find an initial solution $(\hat{x}, \hat{y}')$.

4.3. Hybrid Benders algorithm

In this section, we combine the Benders and heuristic algorithms to create a more efficient algorithm. In this hybrid algorithm, the Benders algorithm guarantees the convergence of the algorithm. The proposed heuristic algorithm improves the overall efficiency by generating valid cuts for the inner max problem and also by finding better solutions $(\hat{x}, \hat{y}')$ by solving model (P6).

Algorithm 3. Hybrid Bender algorithm

1: Input parameters: $WarmupTimeLimit$, $AlgTimeLimit$, $LocalHeuristicTimeLimit$, $EvaTimeLimit$, $\delta_{acc}^H$ and $\delta_{acc}^{Benders}$.
2: Set $UB = \infty$, $LB = -\infty$, $m = 0$ and empty $F$, $G$ and $b_3$ in models (P5) and (P6).
3: Find an initial solution $(\hat{x}, \hat{y}')$ by a heuristic.
4: repeat
5: Modify the objective function of subproblem (SP) using solution $(\hat{x}, \hat{y}')$.
6: Solve the subproblem and update $UB$ if it is necessary.
7: Add a new optimality cut to the master problem and set $m = m + 1$.
8: Solve the master problem, update $LB$ and save $(\hat{x}, \hat{y}')$ in the solution pool.
9: if ($WarmupTimeLimit$ is reached) then
10: Set $Ite = 0$.
11: repeat
12: $Ite + +$.
13: Apply Procedure 1 using solution $(\hat{x}, \hat{y}')$ to generate several cuts (25).
14: Extract the generated cuts and update $F$, $G$ and $b_3$ in models (P5) and (P6).
15: Solve model (P6) to obtain a solution $(\hat{x}, \hat{y}')$ and save it in the solution pool.
16: Set $Obj_{Ite}$ to the objective value of model (P6).
17: until (No cut is generated in Line 14 or time limit $LocalHeuristicTimeLimit$ is reached or $100(Obj_{Ite} - Obj_{Ite-1})/Obj_{Ite-1} \leq \delta_{acc}^H$)
18: Choose the best solution $(\hat{x}, \hat{y}')$ from the solution pool.
19: end if
20: until $(100(UB - LB)/LB \leq \delta_{acc}^{Benders}$ or $AlgTimeLimit$ is reached)

Algorithm 3 provides the pseudo code of the hybrid Benders algorithm. In Line 3 of this algorithm, we obtain an initial solution $(\hat{x}, \hat{y}')$ by solving model (P6) while $F$, $G$ and $b_3$ are
ignored. Lines 4 to 8 together with Line 20 are the same as the main loop of the Benders algorithm given in Algorithm 1. The algorithm finds a scenario by solving the subproblem in Line 6. We then add a new optimality cut to the master problem and solve it to find a new solution \((\hat{x}, \hat{y}')\). Then if time limit \(WarmupTimeLimit\) is already reached, the algorithm enters an inner loop starting in Line 11. This loop is taken from the heuristic algorithm and improves the current solution \((\hat{x}, \hat{y}')\) by iteratively generating cuts (25) in Line 13 and then solving model (P6) in Line 15. Then we check the stopping criteria of the heuristic algorithm in Line 17. To check if time limit \(LocalHeuristicTimeLimit\) is reached, the algorithm tracks the time from the start of the inner loop in Line 11.

After leaving the inner loop in Line 17 and before starting a new iteration of the algorithm, we have to decide on the new solution \((\hat{x}, \hat{y}')\) to modify the objective function of the subproblem in Line 5. Therefore, during the algorithm we save all generated solutions \((\hat{x}, \hat{y}')\) in a solution pool. Then in Line 18 among all solutions in the pool, we choose the one with the lowest worst objective value against all generated scenarios as the current solution \((\hat{x}, \hat{y}')\). In Line 9 of Algorithm 3, we have a time limit \(WarmupTimeLimit\) to prevent from entering the inner loop in Line 11 before this time limit. This is because, in small instances, the Benders algorithm converges very fast without any need of the heuristic algorithm.

There are generally two advantages for combining the Benders and heuristic algorithms. First, by generating cuts (25) in the heuristic algorithm and including them in the subproblem, we hope that the algorithm can solve the subproblem faster in next iterations. Also it is possible to improve the best solution \((\hat{x}, \hat{y}')\) by solving model (P6) in Line 17 in Algorithm 3.

5. Stopping conditions
The main shortcoming of the Benders and hybrid Benders algorithms is that their master problem and subproblem are mixed integer programs (MIPs) and therefore it would be very time consuming to optimally solve them in all iterations of the algorithms. In the following we present novel stopping conditions for these MIPs. Before explaining these conditions we define “\(\varepsilon\)-dominant incumbents” for the master problem and subproblem as follows: For a constant \(\varepsilon > 0\), an \(\varepsilon\)-dominant incumbent of the master problem is a feasible solution in the master problem with an objective value that is less than the lower bound of the recent subproblem by a margin of \(\varepsilon\). Similarly, an \(\varepsilon\)-dominant incumbent of the subproblem is a feasible solution in the subproblem with an objective value that is at least \(\varepsilon\) higher than the upper bound of the recent master problem. The stopping conditions are presented as follows.

**Stopping condition for the master problem (subproblem):** The mixed integer program terminates when the optimal solution is found or at least \(Time_{MP}\) seconds.
Table 1. A numerical example to explain the stopping conditions of MIPs in the Benders algorithm.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Subproblem</th>
<th>Master problem</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Order LB  UB</td>
<td>Order LB UB</td>
</tr>
<tr>
<td></td>
<td>1 120 470</td>
<td>2 45 110</td>
</tr>
<tr>
<td>2</td>
<td>3 340 650</td>
<td>4 30 300</td>
</tr>
<tr>
<td>3</td>
<td>5 320 360</td>
<td>6 155 165</td>
</tr>
<tr>
<td>4</td>
<td>7 170 185</td>
<td>8 157 160</td>
</tr>
<tr>
<td>5</td>
<td>9 160 160</td>
<td>10 160 160</td>
</tr>
</tbody>
</table>

seconds) is passed from the moment that the first $\varepsilon$-dominant incumbent of the master problem (subproblem) is found.

In Table 1, we present a numerical example with $\varepsilon = 5$ to explain this stopping condition for both the master problem and the subproblem. In this table, the results of the master problem and of the subproblem are presented in separate columns. We report the lower and upper bounds of the related mixed integer programs. Since these MIPs are not optimally solved there are gaps between the lower and upper bounds. Columns “Order” also give the order in which these MIPs are solved. In this example, in iterations 1 to 4, the upper bound of the master problem is at least $\varepsilon = 5$ units less than the lower bound of the previous subproblem. Moreover, in iterations 2 to 4, the lower bound of the subproblem is at least $\varepsilon = 5$ units higher than the upper bound of the master problem in the previous iteration. In Iteration 5, when solving the subproblem, we observe that the lower bound does not increase to $\varepsilon = 5$ units higher than the upper bound of the master problem in Iteration 4. Therefore, the stopping condition is not met and the subproblem has to be solved optimally. Similarly, in the same iteration, when we solve the master problem, the upper bound does not decrease to $\varepsilon = 5$ units less than the lower bound of the subproblem in that iteration. Thus, the stopping condition is not satisfied and the master problem has to be solved to optimality.

The upper bound in the master problem and the lower bound in the subproblem correspond to feasible solutions of MIPs. Therefore, the stopping condition for the subproblem means that the subproblem terminates before reaching the optimality if we find a critical uncertain scenario. We refer to a scenario as a critical one if by adding its corresponding cut (15) to the master problem, the objective value of solution $(\hat{x}, \hat{y}')$ found in the previous iteration, increases by at least $\varepsilon$ units.

Lower bounds in the master problem and the upper bounds in the subproblem are valid lower and upper bounds of the original robust problem, respectively. Therefore, we can impose the following constraints based on the best obtained lower and upper bounds as the Benders algorithm proceeds.

$$\theta \geq LB$$ (30)
Constraint (31) is valid because the optimal objective value of the subproblem is an upper bound on the optimal objective value of the original robust problem. As it will be discussed later, constraints (30)-(31) are vital for proving the convergence of the Benders algorithm with stopping conditions for the master problem and subproblem.

When we apply the stopping conditions, most of the time the subproblem is not solved optimally. Therefore, the best upper bound obtained by Algorithms 1 and 3 is poor if the algorithm times out. In this case, we call Procedure 2 at the end of Algorithms 1 and 3 to improve the quality of the best upper bound. This procedure sorts all solutions \((\hat{x}, \hat{y}')\) found by the master problem based on their upper bounds. The upper bound of each solution \((\hat{x}, \hat{y}')\) is the upper bound of its corresponding subproblem obtained in Line 6 of Algorithms 1 and 3. Procedure 2 evaluates these solutions separately by solving the subproblem without any stopping condition. When solving a subproblem if we obtain a feasible solution with an objective value higher than the best upper bound \(UB\), the subproblem terminates and Procedure 2 evaluates the next solution \((\hat{x}, \hat{y}')\) in the sorted list. This is because in this case, another solution with a better upper bound is already known. We consider a time limit \(EvaTimeLimit\) for this procedure.

**Procedure 2.** Evaluation of the generated solutions \((\hat{x}, \hat{y}')\)

1: Input parameters: \(EvaTimeLimit\) and \(\delta_{\text{acc}}^{\text{Benders}}\).

2: if \((100(UB - LB)/LB > \delta_{\text{acc}}^{\text{Benders}})\) then

3: Sort solutions \((\hat{x}, \hat{y}')\) in the solution pool.

4: for \(i=1\) to \(NumberSolutions\) do

5: Solve the subproblem for \(i\)-th solution \((\hat{x}, \hat{y}')\) and update \(UB\) if necessary.

6: if \((100(UB - LB)/LB \leq \delta_{\text{acc}}^{\text{Benders}}\) or \(EvaTimeLimit\) is reached) then

7: break;

8: end if

9: end for

10: end if

In the following we discuss the convergence of the Benders algorithm with and without stopping conditions for the subproblem and master problem. We use the following notation to present the next lemmas and theorems.

- \(W\): The set of vectors \(w\) for which there is \(u \in \mathcal{U}\) such that \((u, w) \in (\mathcal{U}, W)\).
- \(n\): The number of scenarios in \((\mathcal{U}, W)\).
$n'$: The number of unique vectors $w$ that the algorithm visits in the subproblem before it converges.

$n''$: The number of times that the algorithm visits an already encountered vector $w$ before it converges.

$\varepsilon$: A positive constant used in stopping conditions of the master problem and subproblem.

$Opt$: The optimal objective value of the original robust problem.

$O_{SP}^{i}$: The optimal objective value of the subproblem in iteration $i$.

$U_{i}^{MP}$: The upper bound of the master problem in iteration $i$.

$f(j)$: The iteration in which for the $j$-th times the algorithm generates a scenario with a new vector $w$ in the subproblem.

$g(i)$: The iteration in which for the $i$-th times the algorithm re-visits any of the generated vectors $w$ in the subproblem.

$I_{i}$: An indicator that is equal to 1 if in iteration $i$ the algorithm generates a scenario with a repeated vector $w$, 0 otherwise.

**Observation 1.** The Benders algorithm without stopping conditions for the master problem and subproblem converges in at most $|W| + 1 \leq n + 1$ iterations.

In the following, we present Lemmas 1-4 where Lemma 1 is a basis in the proofs of other lemmas and Lemmas 2-4 are used in the proof of Theorem 4.

**Lemma 1.** In the Benders algorithm with stopping conditions for the master problem and subproblem, if the algorithm finds a scenario with a repeated vector $w$ in the subproblem of iteration $i$, then it is the optimal solution of the subproblem and the optimal objective value of the subproblem is equal to the upper bound of the recent master problem in iteration $i-1$, i.e. $U_{i-1}^{MP} = O_{i}^{SP}$.

**Lemma 2.** In the Benders algorithm with the stopping conditions for the master problem and subproblem, if the algorithm finds a scenario with a repeated vector $w$ in the subproblem of iteration $i$ and $O_{i}^{SP} - Opt > \varepsilon$ holds, then in at most $k = \lceil (O_{i}^{SP} - Opt) / \varepsilon \rceil$ iterations either the algorithm finds a scenario with a new vector $w$ or $O_{i+k}^{SP} - Opt \leq \varepsilon$ holds.

**Lemma 3.** In the Benders algorithm with the stopping conditions for the master problem and subproblem, if the algorithm finds a scenario with a repeated vector $w$ in the subproblem of iteration $i$ and $O_{i}^{SP} - Opt \leq \varepsilon$ holds, then in the next iteration either the Benders algorithm converges or a scenario with a new vector $w$ is found.

**Lemma 4.** In the Benders algorithm with the stopping conditions, relation $O_{g(i_1)}^{SP} \geq O_{g(i_2)}^{SP}$ holds for any integer numbers $i_1$ and $i_2$ satisfying $1 \leq i_1 < i_2 \leq n''$. 
Theorem 4. The Benders algorithm with the stopping conditions converges in at most  
\[ \sum_{j=1}^n (1 + \left\lfloor (O_{SP}^{j+1} - \text{Opt})/\varepsilon \right\rfloor + 1)I_{f(j+1)} \]  
iterations that is bounded above by  
\[ |W| \left\lfloor (O_{g(1)} - \text{Opt})/\varepsilon \right\rfloor + 2 \]  
iterations.

\( O_{g(1)} \) is bounded as a result of the boundedness of the feasible area in subproblem (SP). Therefore, Theorem 4 proves the convergence of the Benders algorithm in a finite number of iterations.

6. Applications

In this section, we demonstrate how to apply the proposed reformulation on a nurse planning and a two-echelon supply chain problem.

6.1. Two-stage nurse planning problem

In a two-stage nurse planning problem, we plan wards’ nurses of a hospital for a medium term. The daily workloads of nurses depend on the number of patients brought from operating rooms to wards. Patients are already scheduled in operating rooms over the planning horizon. Before transferring patients from operating rooms to wards they may stay in ICUs for several days. The lengths of stays in ICUs and wards are uncertain and discrete. For each patient a number of local scenarios about the lengths of stays in ICUs and wards are available.

In the first stage of this problem, we assign a number of nurses to wards over the planning horizon. In the second stage if the nurses’ workload on a day is more than the service capacity of nurses assigned to that day, some extra nurses are hired. Nurses hired in the second-stage are paid more than those hired in the first-stage. Nurse staffing based on the workloads of patients transferred from operating rooms to wards is studied in the literature (Beliën and Demeulemeester 2008). The problem is formulated as follows:

Parameters:

- \( c_1 \) : The daily cost of a nurse hired in the first stage.
- \( c_2 \) : The daily cost of a nurse hired in the second stage.
- \( M_d \) : The maximum number of nurses available for hiring on day \( d \) in the second-stage.
- \( \delta \) : The amount of service time provided by a first- or second-stage nurses per day (in hours).
- \( \rho \) : The average of required service time for each patient per day (in hours).
- \( l_{tp}^{ICU} \) : The length of stay in ICUs for patient \( t \) in local scenario \( p \in P_t \).
- \( l_{tp}^{Ward} \) : The length of stay in wards for patient \( t \) in local scenario \( p \in P_t \).
- \( d_t' \) : The surgery day for patient \( t \).

Set:

- \( \mathcal{D} \) : The set of days in the planning horizon.
\( \mathcal{T} \): The set of patients already scheduled in operating rooms over the planning horizon.

\( \mathcal{T}_d \): The set of patients scheduled on day \( d \).

\( \mathcal{P}_t \): The set of local scenarios for patient \( t \). Each local scenario gives information on the lengths of stays in ICUs and wards.

\( \mathcal{P}_{td} \): The subset of local scenarios in \( \mathcal{P}_t \) where patient \( t \) is in wards on day \( d \), i.e.,

\[ \mathcal{P}_{td} = \{ p \in \mathcal{P}_t : d^\prime_t + l^\text{ICU}_{tp} \leq d, d^\prime_t + l^\text{ICU}_{tp} + l^\text{Ward}_{tp} > d \} \]

Variables:

\( x_d \): The number of nurses assigned to day \( d \) in the first stage.

\( u_{tp} \): 1 if patient \( t \) follows local scenario \( p \) after its surgery, 0 otherwise. (uncertainty variable)

\( y_d \): The number of nurses hired on day \( d \) in the second stage.

\[
\min_{x \in \mathcal{X}} \left( \sum_{d \in \mathcal{D}} c_1 x_d + \max_{u \in \mathcal{U}} \left( \min_{y \in \mathcal{Y}(x,u)} \sum_{d \in \mathcal{D}} (c_2 y_d) \right) \right) \\
\mathcal{X} = \{ x \mid x_d \geq 0, \text{integer} \} \\
\mathcal{U} = \{ u \mid \sum_{p \in \mathcal{P}_t} u_{tp} = 1 \text{ for } t \in \mathcal{T}, \ u_{tp} \in \{0,1\} \text{ for } t \in \mathcal{T}, p \in \mathcal{P}_t \} \\
\mathcal{Y}(x,u) = \{ y \mid \delta y_d \geq \rho \sum_{t \in \mathcal{T}} \sum_{p \in \mathcal{P}_{td}} u_{tp} - \delta x_d \in \mathcal{D}, \ 0 \leq y_d \leq M_d, \text{integer} \text{ for } d \in \mathcal{D} \}
\]

Constraints (33) and (37) represent the bounds and integrality constraints for first- and second-stage variables, respectively. (34) and (35) define the discrete uncertainty set. Constraint (36) is the daily demand constraints over the planning horizon.

In the following, we give the corresponding nurse planning problem reformulation in the form of model (P4). The definitions of variables \( x_d \) and \( u_{tp} \) from the nurse planning problem remain unchanged.

New sets:

\( \mathcal{S}_d \): The set of all possible realizations for the number of patients in wards on day \( d \).

New variables:

\( w_{ds} \): 1 if exactly \( s \) patients are in wards on day \( d \), 0 otherwise.

\[
\min_{(x,y) \in (\mathcal{X},\mathcal{Y})} \left( \sum_{d \in \mathcal{D}} c_1 x_d + \max_{(u,w) \in (\mathcal{U},\mathcal{W})} \left( \sum_{d \in \mathcal{D}} \sum_{s \in \mathcal{S}_d} c_2 w_{ds} y_{ds} \right) \right) \\
(\mathcal{U},\mathcal{W}) = \{ (u,w) \mid \sum_{s \in \mathcal{S}_d} w_{ds} = 1 \text{ for } d \in \mathcal{D}, \ w_{ds} \in \{0,1\} \text{ for } d \in \mathcal{D} \}
\]
\[ \sum_{t \in T} \sum_{p \in P_t} u_{tp} = \sum_{s \in S_d} s w_{ds} \quad d \in D, \quad (40) \]
\[ \sum_{p \in P_t} u_{tp} = 1 \quad t \in T, \quad (41) \]
\[ w_{ds} \in \{0, 1\} \quad d \in D \quad s \in S_d, \quad (42) \]
\[ u_{tp} \in \{0, 1\} \quad t \in T \quad p \in P_t \} \quad (43) \]

\[ (X, Y') = \left\{ (x, y') \mid \delta x_d + \delta y'_{ds} \geq \rho \times s \quad d \in D \quad s \in S_d \right\} \quad (44) \]
\[ x_d \geq 0, \text{integer} \quad d \in D, \quad (45) \]
\[ 0 \leq y'_{ds} \leq M_d, \text{integer} \quad d \in D \quad s \in S_d \right\} \quad (46) \]

6.2. Two-echelon supply chain problem

We consider a two-echelon supply chain problem where each customer’s order requires different numbers of various products. The second-layer facilities make the products and send them to the first-layer facilities that consolidate the products corresponding to each customer before shipping. There are several uncertain local scenarios for the demand of each customer. Similar two-echelon supply chain problems are studied in the literature (Amiri 2006, Gendron and Semet 2009, Sadjady and Davoudpour 2012, Pan and Nagi 2013).

In a two-stage robust optimization setting, the decision maker chooses which facilities to open in both layers in the first stage. Then the worst-case scenario for customers’ demands realizes. In the second-stage, the decision maker decides about the transportation of products from the second-layer facilities to first-layer facilities and from them to the customers. We formulate the problem as follows:

Set:
\[ \mathcal{F}_1 : \text{The set of first-layer facilities.} \]
\[ \mathcal{F}_2 : \text{The set of second-layer facilities.} \]
\[ \mathcal{I} : \text{The set of customers.} \]
\[ \mathcal{K} : \text{The set of products.} \]
\[ \mathcal{P}_i : \text{The set of local scenarios for customer } i. \] Each local scenario gives information on the demands of the customer for various products.

Parameters:
\[ c_f : \text{The opening cost of facility } f. \]
\[ d_{ikp} : \text{The demand of customer } i \text{ for product } k \text{ in the local scenario } p \in \mathcal{P}_i. \]
\[ t_{kif} : \text{The per unit transportation cost of product } k \text{ from first-layer facility } f \text{ to customer } i. \]
$t'_{kff'}$: The per unit transportation cost of product $k$ from first-layer facility $f$ to second-layer facility $f'$.

c$_{ifp}$: The transportation cost of products from first-layer facility $f$ to customer $i$ if local scenario $p$ happens for the customer. We have $c_{ifp} = \sum_{k \in K} d_{ikp} t_{kif}$.

c$_{ikff'}p$: The transportation cost of all products $k$ demanded by customer $i$, from first-layer facility $f$ to second-layer facility $f'$ if local scenario $p$ happens for the customer. We have $c_{ikff'}p = d_{ikp} t_{kff'}$.

$b_k$: The maximum number of product $k$ that can be demanded by all customers.

Variables:

$x_f$: 1 if facility $f$ is opened; 0 otherwise.

$u_{ip}$: 1 if local scenario $p$ realizes for customer $i$, 0 otherwise. (uncertainty variable)

$y_{1if}^1$: 1 if first-layer facility $f$ supplies the demand of customer $i$, 0 otherwise.

$y_{2ikff'}^1$: 1 if customer $i$’s demand for product $k$ is transported from second-layer facility $f'$ to first-layer facility $f$, 0 otherwise.

$$\min_{x \in X} \left( \sum_{f \in F_1 \cup F_2} c_f x_f + \max_{u \in U} \left( \min_{y \in Y(x,u)} \sum_{i \in I} \sum_{f \in F_1} \sum_{p \in P_i} (c_{ifp} u_{ip} y_{1if}^1) + \right. \right. $$

$$\left. \left. \sum_{i \in I} \sum_{p \in P_i} \sum_{k \in K} \sum_{p \in P_i} \left( c_{ikfp} u_{ip} y_{2ikff'}^2 \right) \right) \right)$$

$$X = \{ x | x_f \in \{0,1\} \quad f \in F_1 \cup F_2 \}$$

$$U = \{ u | \sum_{p \in P_i} u_{ip} = 1 \quad i \in I, \}$$

$$\sum_{i \in I} \sum_{p \in P_i} d_{ikp} u_{ip} \leq b_k$$

$$u_{ip} \in \{0,1\}$$

$$Y(x,u) = \{ y | \sum_{f \in F_1} y_{1if}^1 = 1 \quad i \in I \}$$

$$y_{1if}^1 \leq x_f$$

$$\sum_{f' \in F_2} y_{2ikff'}^2 = y_{1if}^1$$

$$y_{2ikff'}^2 \leq x_f$$

$$y_{1if}^1 \in \{0,1\}$$

$$y_{2ikff'}^2 \in \{0,1\}$$

Constraint (49) implies that exactly one of the local scenarios realizes for each customer. Constraint (50) is a budget constraint that makes the uncertainty set more general. Constraint
(52) states that each customer receives his/her order exactly from one of the first-layer facilities. First- and second-stage variables are linked by constraints (53) and (55) that let $y_{1f}$ and $y_{ikff'}$ take 1 only if $x_f = 1$ and $x_{ff'} = 1$ hold, respectively. Constraint (54) links the second-stage variables $y_{1f}$ and $y_{ikff'}$ to each other.

Model (47)-(57) is not in the format of model (P1) because, in objective function (47), the second-stage variables are multiplied by the uncertainty variables $u_{ip}$. After performing the reformulation explained in Appendix EC.10, we obtain the following model that is in the format of model (P4).

New variables:

$y_{1ifp}$: 1 if first-layer facility $f$ supplies the demand of customer $i$ assuming that local scenario $p$ has happened for the customer, 0 otherwise.

$y_{ikff'p}$: 1 if customer $i$’s demand for product $k$ is transported from second-layer facility $f'$ to first-layer facility $f$ assuming that local scenario $p$ has happened for the customer, 0 otherwise.

\[
\begin{align*}
\min_{(x,y) \in (\mathcal{X}, \mathcal{Y})} & \left( \sum_{f \in \mathcal{F}_1 \cup \mathcal{F}_2} c_f x_f + \max_{u \in \mathcal{U}} \left( \sum_{i \in \mathcal{I}} \sum_{f \in \mathcal{F}_1} \sum_{p \in \mathcal{P}_i} (c_{ifp} u_{ip} y_{1ifp}^1) + 
\sum_{f \in \mathcal{F}_1} \sum_{f' \in \mathcal{F}_2} \sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}_i} (c_{ikff'p} u_{ip} y_{ikff'p}^2) \right) \right) \\
& \text{subject to:} \quad (49) - (51) \\
& (\mathcal{X}, \mathcal{Y}) = \left\{ (x, y) \mid \sum_{f \in \mathcal{F}_1} y_{1ifp}^1 = 1, \quad i \in \mathcal{I}, \quad p \in \mathcal{P} \right\} \\
& \sum_{f \in \mathcal{F}_2} y_{1ifp}^1 \leq x_f, \quad i \in \mathcal{I}, \quad f \in \mathcal{F}_1, \quad p \in \mathcal{P} \\
& \sum_{f' \in \mathcal{F}_2} y_{ikff'p}^2 = y_{ikff'p}^1, \quad i \in \mathcal{I}, \quad k \in \mathcal{K}, \quad f \in \mathcal{F}_1, \quad p \in \mathcal{P} \\
& y_{ikff'p}^2 \leq x_{ff'}, \quad i \in \mathcal{I}, \quad k \in \mathcal{K}, \quad f \in \mathcal{F}_1, \quad f' \in \mathcal{F}_2, \quad p \in \mathcal{P} \\
& y_{1ifp}^1 \in \{0, 1\}, \quad i \in \mathcal{I}, \quad f \in \mathcal{F}_1, \quad p \in \mathcal{P} \\
& y_{ikff'p}^2 \in \{0, 1\}, \quad i \in \mathcal{I}, \quad k \in \mathcal{K}, \quad f \in \mathcal{F}_1, \quad f' \in \mathcal{F}_2, \quad p \in \mathcal{P} \right\}
\end{align*}
\]

7. Computational results

In this section, we present extensive computational results for the nurse planning and two-echelon supply chain problem introduced in Section 2. We implemented all algorithms in C++ and used IBM ILOG CPLEX 12.6 to solve the mixed integer programs. We ran experiments on a computer with two Intel Xeon X5675 processors, 3.07 Ghz, and a total of 12 cores. We ran each instance on a single core.
For all computational experiments, we set AlgTimeLimit to 2 hours in Algorithms 1 to 3. In the hybrid Benders algorithm, we fix the convergence limits $\delta_{acc}^{Benders}$ and $\delta_{acc}^H$ at 0.1%. We use the same values $\delta_{acc}^{Benders}$ and $\delta_{acc}^H$ for the Benders and heuristic algorithms, respectively. We also consider 5 seconds for TimeLB and TimeUB in the stopping conditions of the master problem and the subproblem in Algorithms 1 and 3. Furthermore, to run Procedure 2 for Algorithms 1 and 3 we set EvaTimeLimit to 2 hours. Therefore, considering parameters AlgTimeLimit and EvaTimeLimit, we run a problem instance for at most 4 hours by Algorithms 1 and 3 and 2 hours by Algorithm 2. In Algorithms 2 and 3, we fix LocalHeuristicTimeLimit at 30 seconds. In the hybrid Benders algorithm, we consider 20 minutes for WarmupTimeLimit.

The generated data sets for both applications are available as an online supplement.

7.1. Nurse planning instances

We generated 750 instances with different parameter settings. The parameters considered in the generation of the instances include the length of the planning horizon ($L$), the incentive factor ($IF$) and the number of operating rooms over the planning horizon ($OR$). We set the number of weeks in the planning horizon to $\{2, 3, 4\}$. We also assume that surgeries are scheduled only on workdays. We define the incentive factor ($IF$) as the ratio of $c_2/c_1$ where $c_1$ and $c_2$ are the daily cost of first-stage and second-stage nurses in objective function (7). A higher value of the incentive factor shows that the hospital pays more to second-stage nurses than first-stage ones. We set the incentive factor to $\{1.1, 1.3, 1.5, 1.7, 1.9\}$. We suppose that first-stage nurses are paid 1 unit cost per hour which for 8 work hours results in $c_1 = 8$. Furthermore, we also fix the number of operating rooms over the planning horizon at $\{1, 2, 3, 4, 5\}$. For each operating room we generate 3, 4, or 5 surgeries randomly with a uniform distribution. Considering a full factorial experiment, 75 combinations of $L$, $IF$ and $OR$ are possible and we generate 10 instances for each problem setting for a total of 750 instances. For each patient, we generate two scenarios for the lengths of stays in ICU and wards. In each scenario, both lengths of stays are uniformly generated from interval $[1 \text{ day}, 10 \text{ days}]$. The total number of global scenarios which include information for all patients can be computed by $2^{|T|}$ where $|T|$ is the number of patients in the planning horizon. It is worth noting that in our small-sized instances with 39 surgeries we have $2^{39} \approx 5.4 \times 10^{11}$ scenarios. We also assume that each nurse works for 8 hours a day ($\delta = 8$) and the average daily service time for each patient is 2 hours ($\rho = 2$).

7.2. Results of nurse planning instances

In this Section, we present computational results for three sets of experiments performed on nurse planning instances. In the first set of experiments, we aim at evaluating the computational performance of our proposed heuristic, Benders, and hybrid Benders algorithms. In our
computational experiments, we also consider a tri-level Benders algorithm that is inspired from Chen (2013). We give the details about the recent algorithm in Appendix EC.11. In Appendix EC.12, we have provided some computational experiments to tune $\varepsilon$ for the stopping conditions of the Benders and hybrid Benders algorithms that resulted in $\varepsilon = 5$.

In Table 2, we report the results for our heuristic, Benders, and hybrid Benders algorithms and the tri-level algorithm inspired from Chen (2013). In this table, each row gives the average results for 50 instances with different values of the incentive factor. For the heuristic algorithm, we do not report “LB” as this algorithm does not provide any lower bound. To compute the optimality gap values for the heuristic algorithm, we use the lower bound values of the Benders algorithm. Moreover, under Column “Hybrid Benders algorithm”, we have reported $\Delta(UB)(\%)$ that presents the gap between the upper bounds of the Benders and the hybrid Benders algorithms. We compute it by $\Delta(UB)(\%) = 100(UB_B - UB_{HB})/UB_B$ where $UB_B$ and $UB_{HB}$ denotes the upper bound values of the Benders and hybrid Benders algorithms.

In Table 2, we observe that for most instances the heuristic algorithm converges quickly after only a few iterations and the average optimality gaps are worse than those of the Benders and hybrid Benders algorithms. This is because the heuristic algorithm is a heuristic, while the two other algorithms are exact algorithm and converge to the optimal solution. We also observe that the averages of optimality gaps for the hybrid Benders algorithm are 0.61%, 3.29%, and 5.46%. These averages are higher than the averages of optimality gaps for the Benders algorithm. However, the averages of “$\Delta(UB)$” are -0.46%, 0.09%, and 0.94%, respectively. These values show that the hybrid Benders algorithm finds better upper bounds than the Benders algorithm in instances with planning horizons of 3 and 4 weeks. Moreover, the average optimality gaps for the tri-level Benders algorithm, proposed in the literature, are 2.74%, 25.45%, and 37.53% that are significantly higher than those of our Benders and hybrid Benders algorithms. It is also noteworthy that the tri-level Benders algorithm does not find feasible solutions for $L = 3, OR = 5$, and $L = 4, OR = 4, 5$.

In the second set of experiments, we evaluate the computational efficiency of the Benders algorithm for different levels of block-diagonal decomposition in the nurse planning problem. In Table 3, $R$ stands for the percent of the smallest-size blocks (day blocks) that are merged with other blocks to form larger ones. $R = 0$ represents an extreme case where the block-diagonal structure of the nurse planning problem is decomposed as much as possible and each block is corresponding to a single day. Higher values of $R$ means that the Benders algorithm benefits less from the block-diagonal structure of the problem. This table shows that, for the largest set of instances with $L = 4$, the average optimality gap increases from 5.11% to 12.13% as $R$ increases. Figure 1 depicts the bound values and the number of variables in the subproblem.
Table 2. Computational results for the proposed heuristic, Benders, and hybrid Benders algorithms and the tri-level Benders algorithm from the literature

<table>
<thead>
<tr>
<th>Data Info.</th>
<th>Heuristic algorithm</th>
<th>Benders algorithm</th>
<th>Hybrid Benders algorithm</th>
<th>Tri-Level Benders algorithm from the literature</th>
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versus $R$. This figure shows that both $LB$ and $UB$ values deteriorates as $R$ increases. This is because higher values of $R$ lead to more complicated subproblems (more variables) due to less benefiting from the block-diagonal structure. Figure 2 shows the improvement of the lower bound during the run time for different value of $R$. We can see that, during the run time, the lower bound is stronger for cases with smaller $R$.

![Figure 1](image1.png)

**Figure 1** Lower bound, upper bound, and the number of variables in the subproblem versus $R$.

![Figure 2](image2.png)

**Figure 2** Lower bound trends for different values of $R$ during the run time.

In the third set of experiments, we intend to evaluate the performance of our stopping conditions proposed in Section 5. In Table 4, we report the results for three implementations of our Benders algorithm. The first implementation is the one with our proposed stopping condition for $\varepsilon = 5$. In the second implementation, we deactivated our proposed stopping conditions and added the conventional stopping condition. This stopping condition terminates the subproblem as soon as it finds a solution that defines an inequality cutting off the master problem’s solution. We can view this conventional stopping condition as a special case of our proposed
stopping condition with $\varepsilon = 0$ that works only for the subproblem. The third algorithm reported in Table 4 is the branch-and-cut implementation of our Benders algorithm. In this algorithm, the master problem is solved only once and whenever a new incumbent solution is found within the branch-and-bound tree, the algorithm solves the subproblem and add the optimality cuts to the tree.

In Table 4, each row gives the average results for 50 instances with different values of the incentive factor. The results of "Time (sec)", "Ite.", $LB$, $UB$, and "Gap(\%)" for the first implementation are the same as those presented for the Benders algorithm in Table 2. In fact, in Table 4, we report the same results for the first implementation for ease of comparison with the two other algorithms. Moreover, in this table, for the Benders algorithm with our proposed stopping conditions, we have presented additional results. "$LB_1$" and "$UB_1$" give the lower and upper bounds in the first iteration of the Benders algorithm and $Gap_1$ computes the gap between these bounds. Moreover, "Imp", gives the percentage of the upper bound improvement obtained during the Benders algorithm. We compute it by $100(UB_1 - UB)/UB_1$.

We observe that the average optimality gaps for the Benders algorithm when we apply the proposed stopping conditions are 0.05%, 2.57%, and 5.11% for instances with $L = 2$, $L = 3$, and $L = 4$. However, the average optimality gaps for the Benders algorithm with the conventional stopping condition and the branch-and-cut algorithm are 0.29%, 5.59%, 9.33% and 2.32%, 27.06%, 40.28%, respectively. This observation indicates that the proposed stopping conditions are essential for the efficiency of the proposed Benders algorithm. For small instances such as those with $L = 2$, $OR = 1, 2, 3$, the algorithm with the conventional stopping condition repeats more iterations than the algorithm with the proposed stopping conditions. However, for larger instances such as those with $L = 4$, $OR = 2, 3, 4, 5$, the former algorithm repeats significantly fewer iterations than the latter algorithm does. There are two reasons for this behaviour: 1) the algorithm with the conventional stopping condition stops the subproblem as soon as it finds a second-stage solution that cuts off the current first-stage solution. As a result, the generated cuts are generally less effective than the cuts that the algorithm with the proposed stopping conditions generates and therefore the first algorithm requires more iterations for convergence. 2) For small instances the master problem is simpler and the algorithm with the conventional stopping condition can optimally solve it fairly quickly. However, for larger instances proving the optimality of the master problem becomes the bottleneck of the algorithm, while the algorithm with the proposed stopping conditions avoids this issue by terminating the master problem when the $\varepsilon$-stopping condition is satisfied.

Furthermore, large optimality gaps of the branch-and-cut algorithm are due to the fact that, whenever the algorithm finds a first-stage feasible solution, it solves a subproblem that is a
Table 3. Computational results of the Benders algorithm for different level of decomposing the block-diagonal structure.

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Table 4. Computational results to evaluate the performance of the proposed stopping conditions.

<table>
<thead>
<tr>
<th>Data Info.</th>
<th>Benders algorithm with the proposed stopping conditions</th>
<th>Benders algorithm with the conventional stopping condition</th>
<th>Branch-and-cut algorithm</th>
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<tr>
<td></td>
<td>LB$_1$ UA$_1$ Gap$_1$ (%) Imp$_1$ (%) Time (sec) Iter. LB UB Gap (%) Time (sec) Iter. LB UB Gap (%) Time (sec) LB UB Gap (%)</td>
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<tr>
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<td>Average 2268 3097 47.48 3.30 11697 363 2835 3019 5.11 11708 67 2757 3087 9.33 13350 2615 3828 40.28</td>
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</tbody>
</table>
mixed-integer programming model. This requires solving a larger number of mixed-integer programs that is computationally expensive. We also observe that the averages of initial gaps in the first iteration of the Benders algorithm with the proposed stopping conditions (\(\text{Gap}_1\)) are 66.29\%, 55.10\%, and 47.56\% that are considerably higher than the final optimality gaps. This demonstrates that the Benders algorithm significantly improves the optimality gap. Moreover, the averages of “Imp” are 5.14\%, 4.80\%, and 3.31\%. These averages show that the Benders algorithm improves the upper bound during the algorithm and the improvement of optimality gap is not only because of improving the lower bound. We also observe that the upper bound improvement decreases as the length of the planning horizon increases. This observation confirms that instances with longer planning horizons are more difficult and the Benders algorithm becomes less effective in solving them. Similarly instances with more operating rooms are more difficult and the Benders algorithm performs more iterations before stopping for such instances.

7.3. Supply chain instances

We generated 600 instances for the supply chain problem. The parameters that we considered to generate the instances include the number of customers (\(|\mathcal{I}|\)), the number of first- and second-layer facilities (\(|\mathcal{F}_1|\) and \(|\mathcal{F}_2|\)), customers’ demands (\(d_{ikp}\)), and transportation costs (\(t_{kif}\) and \(t_{kff'}\)). For each instance, we uniformly generate the coordinates of customers and facilities in a square with a side length of 100 kilometers. Then, we set \(t_{kif} = \alpha_k [(x_i - x_f)^2 + (y_i - y_f)^2]^{0.5}\) and \(t_{kff'} = \alpha_k [(x_f - x_{f'})^2 + (y_f - y_{f'})^2]^{0.5}\) where \(\alpha_k\) is the per kilometer transportation cost for product \(k \in \mathcal{K} = \{1, 2, 3\}\) and is uniformly chosen from \([0.7, 1.3]\). Also, we uniformly generate the demands \(d_{ikp}\) from \([100, 200]\). Moreover, we assume \(c_f = \lambda \beta_f\) where \(\beta_f\) is uniformly generated from \([100, 200]\) and \(\lambda\) is a parameter to tune the relative magnitude of facilities fixed costs compared to the transportation costs. We set the number of customers (\(|\mathcal{I}|\)) to \(\{50, 60, 70\}\). For the number of facilities, we consider five cases of \((|\mathcal{F}_1|, |\mathcal{F}_2|) \in \{(5, 5), (5, 10), (10, 10), (10, 20), (20, 20)\}\). Finally, we set the relative cost parameter \(\lambda\) to \(\{1, 10, 100, 1000\}\). We generated 600 instances by considering 10 instances for each combination of \(|\mathcal{I}|, (|\mathcal{F}_1|, |\mathcal{F}_2|),\) and \(\lambda\). For each test instance, we set \(b_k\) in constraint (50) equal to \(1.5 \sum_{i \in \mathcal{I}} \bar{d}_{ik}/|\mathcal{I}|\) where \(\bar{d}_{ik}\) represents the average demand of product \(k\) by customer \(i\).

7.4. Results of supply chain instances

As demonstrated in Table 2, the Benders algorithm outperforms the heuristic and hybrid Benders algorithms in terms of the optimality gap. Therefore, in Table 5, we have provided the computational results to compare the proposed Benders algorithm with the tri-level Benders algorithm proposed by Chen (2013). We have explained the different components of the tri-level algorithm for the supply chain problem in Appendix EC.13. In Table 5, Columns \(LB_1, UB_1,\)
Gap_1(\%), and Imp(\%), Time (sec), LB, UB, and Gap(\%) are the same as those in Tables 2 and 4. Furthermore, under Column “Tri-Level Benders algorithm”, we have reported \( \Delta(UB)(\%\) that gives the gap between the upper bounds of the Benders and tri-level algorithms. We compute it by \( \Delta(UB)(\% = 100(UB_T - UB_B)/UB_T) \) where \( UB_B \) and \( UB_T \) respectively denote the upper bound values of the Benders and tri-level Benders algorithms. Table 5 shows that the average optimality gap of the proposed Benders algorithm for instances with \( L =, 2, 3, and 4 \) is 0.78\%, 1.25\%, and 1.23\%, respectively. However, the optimality gaps for the tri-level Benders algorithm are very poor that is mainly due to very weak lower bounds. As explained at the end of Appendix EC.13, this is because the structure of the supply chain problem is such that optimality cuts for the outer master problem cannot be enhanced. Also, the average values of \( \Delta(UB)(\%\) are 30.47\%, 28.57\%, and 28.90\% implying that the solutions found by the proposed Benders algorithm are significantly superior than those of the tri-level Benders algorithm.

There are also two noteworthy points about Gap_1(\%) and Imp(\%). Comparison of Gap_1(\%) and Gap(\%) for the Benders algorithm shows that the algorithm significantly improves the optimality gap from the first iteration to the last one. Also, the values of Imp(\%) shows that the final upper bound values are around 60\% stronger than the initial upper bound values. This demonstrates that the improvement of the optimality gap from the first iteration to the last iteration of the Benders algorithm, is not just because of improving the lower bound.

8. Conclusion

We have considered a class of two-stage robust optimization models with an exponential number of scenarios. We exploited the structure of the problem using Dantzig-Wolfe decomposition and reduced the original two-stage robust problem to a single-stage robust problem. We then proposed a Benders and a heuristic algorithm for the reformulated problem and combined them to create a more effective hybrid algorithm capable of finding solutions with better objective values. Since the master problem and subproblem of the Benders algorithm are mixed integer programs, it is computationally demanding to optimally solve them in each iteration of the algorithm. Therefore, we presented novel stopping conditions for them and provided the relevant convergence proofs. We performed extensive computational experiments to evaluate the performance the proposed algorithms in a nurse planning and a supply chain problem. For the nurse planning problem, the computational results demonstrated that the Benders and hybrid Benders algorithms find solutions with an average optimality gap of less than 3\% over all instances with planning horizons up to four weeks. Moreover, our experiments showed that the proposed Benders algorithm is capable of finding quality solutions with an average optimality gap of less than 1.25\% for the supply chain instances with up to 70 customers and 40 facilities. A possible future research direction would be to explore the extension of the proposed...
algorithms to multi-stage robust problems with exponential scenarios. Moreover, applying the proposed algorithms to other applications should be of interest.
Table 5. Computational results to the proposed Benders algorithm and the tri-level Benders algorithm from the literature for the supply chain problem.

<table>
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<tr>
<th>Data Info.</th>
<th>Proposed Benders algorithm</th>
<th>Tri-Level Benders algorithm from the literature</th>
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Acknowledgments

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References


Electronic Companion
“Exploiting the Structure of Two-Stage Robust Optimization Models with Exponential Scenarios”

Contents

EC.1 Proof of Theorem 1 ................................................. ec2
EC.2 Proof of Theorem 2 ................................................. ec3
EC.3 Proof of Theorem 3 ................................................. ec4
EC.4 An example to show the local optimality of the heuristic algorithm .......... ec5
EC.5 Proof of Lemma 1 .................................................. ec7
EC.6 Proof of Lemma 2 .................................................. ec7
EC.7 Proof of Lemma 3 .................................................. ec9
EC.8 Proof of Lemma 4 .................................................. ec10
EC.9 Proof of Theorem 4 ................................................ ec10
EC.10 Details of the reformulation for the supply chain problem ......................... ec11
EC.11 Tri-level Benders algorithm for the nurse scheduling problem ............... ec14
EC.12 Experiments on the tuning of $\varepsilon$ for nurse planning instances .......... ec18
EC.13 Tri-level algorithm for the supply chain problem ............................... ec20
EC.14 Generalization to finite sets of fractional parameters .......................... ec22
EC.1. Proof of Theorem 1

We use the following notation.

\((\mathcal{U}, \mathcal{W})\) : The uncertainty set that is defined by (4)-(7).

\(\mathcal{J}\) : The index set of \((\mathcal{U}, \mathcal{W})\) that is defined as 
\[ \mathcal{J} = \{1, 2, \ldots, |(\mathcal{U}, \mathcal{W})|\} \] where \(|(\mathcal{U}, \mathcal{W})|\) represents the cardinality of \((\mathcal{U}, \mathcal{W})\).

\((\omega^j, \omega^j)\) : The \(j\)-th member of \((\mathcal{U}, \mathcal{W})\).

We also define \(f_k(x, w)\) and \(g_k(x, e_{ks})\) as follows.

\(f_k(x, w) = \min_{y_k} c_{2k}^t y_k \) \tag{EC.1.1}

\[ C_k y_k \leq b_k - A_k x - \sum_{s \in S_k} e_{ks} w_{ks} \quad k \in \mathcal{K} \] \tag{EC.1.2}

\[ y_k \in \mathcal{Y}_k \quad k \in \mathcal{K} \] \tag{EC.1.3}

\[ g_k(x, e_{ks}) = \min_{y'_{ks}} c_{2k}^t y'_{ks} \] \tag{EC.1.4}

\[ C_k y'_{ks} \leq b_k - A_k x - e_{ks} \quad k \in \mathcal{K} \] \tag{EC.1.5}

\[ y'_{ks} \in \mathcal{Y}_k \quad k \in \mathcal{K} \] \tag{EC.1.6}

For each scenario \((y, w) \in (\mathcal{U}, \mathcal{W})\) with index \(j \in \mathcal{J}\), with respect to constraint (6)-(7), exactly one of the variables \(w^j_{ks}, s \in S_k\) is equal to 1 for each \(k \in \mathcal{K}\). Let \(s_j\) denote the index in \(S_k\) for which \(w^j_{ks_j}\) is equal to 1. Therefore we have the following relations.

\[ w^j_{ks_j} = 1 \quad j \in \mathcal{J} \] \tag{EC.1.7}

\[ w^j_{ks} = 0 \quad j \in \mathcal{J}, s \neq s_j \] \tag{EC.1.8}

In the following we prove the if-statement of Theorem 1. The only if-statement of this theorem can be proven in a reverse direction. Assume that \(\hat{x}\) is a first-stage feasible solution of model (P2). In the following we separately prove that

- \(\hat{x}\) is also a first-stage feasible solution of model (P3).
- The objective values of (P2) and (P3) for this fixed first-stage solution are the same if \(\max\) and \(\min\) are solved optimally.

**Proof of Part 1:** Since model (P2) is feasible, there is at least a feasible second-stage policy \(\{\alpha_{kj}\}_{k \in \mathcal{K}}\) for each \(j \in \mathcal{J}\) such that

\[ C_k \alpha_{kj} \leq b_k - A_k \hat{x} - \sum_{s \in S_k} e_{ks} w^j_{ks} \quad k \in \mathcal{K}, j \in \mathcal{J} \] \tag{EC.1.9}

\[ \alpha_{kj} \in \mathcal{Y}_k \quad k \in \mathcal{K}, j \in \mathcal{J} \] \tag{EC.1.10}
Using (EC.1.7) and (EC.1.8), we can rewrite relations (EC.1.9)-(EC.1.10) as follows.

\[ C_k \alpha_{kj} \leq b_k - A_k \hat{x} - e_{ks} \quad k \in K, j \in J \quad (EC.1.11) \]

\[ \alpha_{kj} \in Y_k \quad k \in K, j \in J \quad (EC.1.12) \]

Relations (EC.1.11)-(EC.1.12) demonstrate that for each \( k \in K \) and \( s \in S_k \) there is at least one \( j \in J \) such that for \( y'_{ks} = \alpha_{kj} \) constraints \( C_k y'_{ks} \leq b_k - A_k x - e_{ks} \) and \( y'_{ks} \in Y_k \) are satisfied. Therefore, \( \hat{x} \) is also a first-stage feasible solution of model (P3).

**Proof of Part 2:** To prove that the objective values of (P2) and (P3) for the fixed first-stage solution \( \hat{x} \) are the same, it is enough to prove that relation (EC.1.13) or its equivalent, relation (EC.1.14), holds.

\[
c_1^T \hat{x} + \max_{(u, w) \in (U, W)} \left( \sum_{k \in K} f_k(\hat{x}, w) \right) = c_1^T \hat{x} + \max_{(u, w) \in (U, W)} \left( \sum_{k \in K} \sum_{s \in S_k} g_k(\hat{x}, e_{ks}) w_{ks} \right) \quad (EC.1.13)
\]

\[
\max_{(u, w) \in (U, W)} \left( \sum_{k \in K} f_k(\hat{x}, w) \right) = \max_{(u, w) \in (U, W)} \left( \sum_{k \in K} \sum_{s \in S_k} g_k(\hat{x}, e_{ks}) w_{ks} \right) \quad (EC.1.14)
\]

Moreover, regarding (EC.1.7) and (EC.1.8), in constraint (EC.1.2) of \( f_k(\hat{x}, w^j) \) we can substitute \( \sum_{s \in S_k} e_{ks} w_{ks}^j \) by \( e_{ks} \). It is then clear that mathematical programs corresponding to \( g_k(\hat{x}, e_{ks}) \) and \( f_k(\hat{x}, w^j) \) have the same structure and following relations hold.

\[
g_k(\hat{x}, e_{ks}) = f_k(\hat{x}, w^j) \quad k \in K, j \in J \quad (EC.1.15)
\]

\[
\arg \min_{y'_{ks}} \left( g_k(\hat{x}, e_{ks}) \right) = \arg \min_{y_k} \left( f_k(\hat{x}, w^j) \right) \quad k \in K, j \in J \quad (EC.1.16)
\]

The following stream of equalities proves the validity of (EC.1.14). In the following relations the second equality is obtained using (EC.1.15). The third equality is valid because of (EC.1.7)-(EC.1.8).

\[
\max_{(u, w) \in (U, W)} \left( \sum_{k \in K} f_k(\hat{x}, w) \right) = \max_{j \in J} \left( \sum_{k \in K} f_k(\hat{x}, w^j) \right) = \max_{j \in J} \left( \sum_{k \in K} g_k(\hat{x}, e_{ks}) \right) =
\]

\[
\max_{j \in J} \left( \sum_{k \in K} \sum_{s \in S_k} g_k(\hat{x}, e_{ks}) w_{ks}^j \right) = \max_{(u, w) \in (U, W)} \left( \sum_{k \in K} \sum_{s \in S_k} g_k(\hat{x}, e_{ks}) w_{ks} \right)
\]

In addition, (EC.1.16) shows that we can obtain the second-stage optimal policies for variables \( y_k \) in model (P2) from the optimal values of variables \( y'_{ks} \).

**EC.2. Proof of Theorem 2**

As discussed in Appendix EC.1, we can present the inner max problem in model (P3) by

\[
\max_{(u, w) \in (U, W)} \left( \sum_{k \in K} \sum_{s \in S_k} g_k(\hat{x}, e_{ks}) w_{ks} \right) \quad (EC.2.1)
\]
where $g_k(\hat{x}, e_{ks})$ is defined as:

$$g_k(x, e_{ks}) = \min_{y'_{ks}} c^T_{2k} y'_{ks}$$  \hspace{1cm} (EC.2.2)

$$C_k y'_{ks} \leq b_k - A_k \hat{x} - e_{ks} \quad k \in K$$  \hspace{1cm} (EC.2.3)

$$y'_{ks} \in Y_k \quad k \in K$$  \hspace{1cm} (EC.2.4)

It is clear that the optimal values of vectors $y'_{ks}$ for $k \in K, s \in S_k$ are independent of $(u, w) \in (U, W)$ and are defined by

$$y'_{ks} = \arg \min_{y'_{ks} \in G_{ks}} (c^T_{2k} y'_{ks})$$  \hspace{1cm} (EC.2.5)

where $G_{ks} = \{y'_{ks} \in Y_k | C_k y'_{ks} \leq b_k - A_k \hat{x} - e_{ks}\}$. Therefore, because of the independence of $y'_{ks}, k \in K, s \in S_k$ from $(u, w) \in (U, W)$, we can swap max and min in model (P3) and Theorem 2 is proven.

**EC.3. Proof of Theorem 3**

Consider the following problem.

$$(MP') \min_{(x,y') \in (X,Y')}(c^T_1 x + \max_{(y,w) \in (U,W')} \left( \sum_{k \in K} \sum_{s \in S_k} c^T_{2k} y'_{ks} w_{ks} \right))$$  \hspace{1cm} (EC.3.1)

where $(U, W') = \{(w^j, w^j), j = 1, 2, ..., m\}$. Since $(U, W') \subseteq (U, W)$ the optimal objective value of model $(MP')$ is a valid lower bound for the optimal objective value of the original robust problem $(P4)$. In the following we demonstrate that $(MP')$ is equivalent to $(MP)$. By writing the convex combination of $m$ scenarios $(w^j, w^j)$, model $(MP')$ can be rewritten as:

$$(MP'') \min_{(x,y') \in (X,Y')} \left( c^T_1 x + \max_{\lambda} \left( \sum_{j=1}^{m} \lambda_j \left( \sum_{k \in K} \sum_{s \in S_k} c^T_{2k} y'_{ks} \hat{w}_{ks}^j \right) \right) \right)$$  \hspace{1cm} (EC.3.2)

$$\sum_{j=1}^{m} \lambda_j = 1$$  \hspace{1cm} (EC.3.3)

$$\lambda_j \geq 0 \quad j = 1, 2, ..., m.$$  \hspace{1cm} (EC.3.4)

In model $(MP'')$, for a fixed value of $(x, y')$, the inner max problem is a linear programming model and one of its extreme points will be the optimal solution. Each extreme point of this model corresponds to one of the scenarios $(w^j, w^j)$. Therefore, model $(MP'')$ is equivalent to model $(MP')$. By dualizing the inner max problem in model $(MP'')$ and assuming $\theta$ as the dual variables of constraint (EC.3.3) we obtain model $(MP)$ and Theorem 3 is proven.
EC.4. An example to show the local optimality of the heuristic algorithm

Consider the problem \( \min_{(x_1, x_2) \in \mathcal{X}} (2x_1 + 1.5x_2 + \max_{(u_1, u_2) \in \mathcal{U}} (x_1u_1 + x_2u_2)) \) where

\[
\mathcal{U} = \{ (u_1, u_2) \in \mathbb{N}^2 \mid u_1 \leq 2, u_2 \geq 1, 0.99u_1 + 2u_2 \leq 5.98, 1.99u_1 + u_2 \geq 2.99 \}
\]

and

\[
\mathcal{X} = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 1, (x_1, x_2) \in \{0, 1\}^2 \}.
\]

The solution space of the variables \((u_1, u_2)\) are four points \(A, B, C, \) and \(D\) in Figure EC.1. The optimal solution of this problem is \((x_1, x_2) = (0, 1)\). For this solution the objective line in \(\max_{(u_1, u_2) \in \mathcal{U}} \) is Line \(L_1\). This objective line shows that scenarios \(A\) and \(B\) in the problem are optimal with a total objective value of 3.5. If we relax the integrality constraints on variables \(u_1\) and \(u_2\) the solution space in the problem extends to polytope \(E-B-C-D\). In this case, for solution \((x_1, x_2) = (0, 1)\) the optimal scenario is Point \(E\) with an objective value of 4.49. However, for solution \((x_1, x_2) = (1, 0)\) the objective line \(L_2\) represents the objective function of the inner max problem. This objective line finds points \(B\) and \(C\) as the optimal scenarios with an objective value of 4. In this example, if we apply the heuristic algorithm to solve this problem the algorithm converges in the first iteration by finding the non-optimal solution \((x_1, x_2) = (1, 0)\).

![Figure EC.1](image-url)  
**Figure EC.1**  
The solution space of the uncertainty variables in the example presented to show the non-optimality of the heuristic algorithm.
Notation used in EC.5 to E.9

We use the following notation in the proofs of Appendices EC.5 to EC.9.

**W**: The set of vectors $w$ for which there is $u \in U$ such that $(u, w) \in (U, W)$.

$n$: The number of scenarios in $(U, W)$.

$n'$: The number of unique vectors $w$ that the algorithm visits in the subproblem before it converges.

$n''$: The number of times that the algorithm visits an already encountered vector $w$ in the subproblem before it converges.

$\varepsilon$: A positive constant used in stopping conditions of the master problem and subproblem.

$MP(i)$: The master problem in iteration $i$.

$SP(i)$: The subproblem in iteration $i$.

$Opt$: The optimal objective value of the original robust problem.

$U^i_{MP}$: The upper bound of the master problem in iteration $i$.

$O^i_{MP}$: The optimal objective value of the master problem in iteration $i$.

$L^i_{MP}$: The lower bound of the master problem in iteration $i$.

$U^i_{SP}$: The upper bound of the subproblem in iteration $i$.

$O^i_{SP}$: The optimal objective value of the subproblem in iteration $i$.

$L^i_{SP}$: The lower bound of the subproblem in iteration $i$.

$f(j)$: The iteration in which for the $j$-th times the algorithm generates a scenario with a new vector $w$ in the subproblem.

$g(i)$: The iteration in which for the $i$-th times the algorithm re-visits any of the generated vectors $w$ in the subproblem.

$I_i$: An indicator that is equal to 1 if in iteration $i$ the algorithm generates a scenario with a repeated vector $w$, 0 otherwise.
EC.5. Proof of Lemma 1

Let $(\hat{x}, \hat{y}')$ and $\hat{\theta}$ respectively denote the solution and the objective value of the master problem in iteration $i - 1$. Furthermore, let $(\hat{u}, \hat{\bar{w}})$ denote the scenario with the repeated vector $w = \hat{\bar{w}}$ found in the subproblem in iteration $i$. Since vector $w = \hat{\bar{w}}$ is repeated, we have already included an instance of constraint (15) corresponding to this vector in the master problem in iteration $i - 1$ and the following relation holds.

$$\hat{\theta} \geq c_1^T \hat{x} + \sum_{k \in K} \sum_{s \in S_k} c_{2k}^T \hat{y}'_{ks} \hat{\bar{w}}_{ks} \quad (EC.5.1)$$

The Benders algorithm applies solution $(\hat{x}, \hat{y}')$ to modify the objective function of the subproblem in iteration $i$. If $(\hat{u}, \hat{w})$ is not the optimal solution of subproblem then it means that in the subproblem the following stopping condition is satisfied.

$$c_1^T \hat{x} + \sum_{k \in K} \sum_{s \in S_k} c_{2k}^T \hat{y}'_{ks} \hat{w}_{ks} \geq \hat{\theta} + \varepsilon \quad (EC.5.2)$$

Obviously relation (EC.5.2) is in contrast with (EC.5.1) and we conclude that if the algorithm visits a scenario with a repeated vector $\hat{\bar{w}}$ in the subproblem, this scenario is the optimal solution of the subproblem. To prove that the optimal objective value of the subproblem is equal to the upper bound of the master problem in iteration $i - 1$, we have to show that in the master problem, an instance of constraint (15) corresponding to the repeated vector $\hat{\bar{w}}$ is binding. If for another scenario with a different repeated vector $w = w'$, constraint (15) is binding, then we must have $c_1^T \hat{x} + \sum_{k \in K} \sum_{s \in S_k} c_{2k}^T \hat{y}'_{ks} \hat{w}_{ks} < c_1^T \hat{x} + \sum_{k \in K} \sum_{s \in S_k} c_{2k}^T \hat{y}'_{ks} \hat{w}'_{ks}$, which is a contradiction regarding the optimality of $(\hat{u}, \hat{\bar{w}})$ in the subproblem in iteration $i$. Therefore, if the algorithm finds a scenario with a repeated vector $\hat{\bar{w}}$ in the subproblem, the optimal objective value of the subproblem is equal to the upper bound of the recent master problem.

EC.6. Proof of Lemma 2

Equivalently this lemma states that if in $k = \lfloor (O_{SP}^i - Opt)/\varepsilon \rfloor$ iterations after iteration $i$ the algorithm does not find any scenario with a repeated vector $w$ then $O_{MP}^i - O_{SP}^i \leq \varepsilon$ holds. In iteration $i$, since the algorithm found a scenario with a repeated vector $w$ in subproblem $SP(i)$, regarding Lemma 1 this scenario is the optimal solution of the subproblem and $L_{SP}^i = O_{SP}^i$ holds. Furthermore, in the master problem $MP(i)$ that is solved after subproblem $SP(i)$, two cases are possible.

Case 1) $O_{MP}^i > O_{SP}^i - \varepsilon$ holds. First note that $O_{MP}^i < Opt$ is a valid regarding Theorem 3. $O_{MP}^i > O_{SP}^i - \varepsilon$ together with $O_{MP}^i < Opt$ results in $Opt > O_{SP}^i - \varepsilon$. The latter relation contradicts with the initial assumption $O_{SP}^i - Opt > \varepsilon$. Therefore, this case does not happen.
Case 2) $O^{MP}_i \leq O^{SP}_i - \varepsilon$ holds. This relation is equivalent to $O^{MP}_i \leq L^{SP}_i - \varepsilon$ with respect to relation $L^{SP}_i = O^{SP}_i$. Regarding $O^{MP}_i \leq L^{SP}_i - \varepsilon$, the stopping condition in master problem $MP(i)$ is satisfied and the master problem stops when it finds a feasible solution with an upper bound $U^{MP}_i$ satisfying the following relation.

$$U^{MP}_i \leq L^{SP}_i - \varepsilon = O^{SP}_i - \varepsilon \quad (EC.6.1)$$

We have assumed that no scenario with a new vector $w$ is generated in $k = \lfloor (O^{SP}_i - Opt)/\varepsilon \rfloor$ iterations after iteration $i$. Therefore, in iteration $i+1$ a scenario with a repeated vector $w$ is generated and regarding Lemma 1 we have $O^{SP}_{i+1} = U^{MP}_i$. The recent relation together with (EC.6.1) results in the following relation.

$$O^{SP}_{i+1} \leq O^{SP}_i - \varepsilon \quad (EC.6.2)$$

Similarly we can show that for $k = \lfloor (O^{SP}_i - Opt)/\varepsilon \rfloor$ relation (EC.6.3) holds. This is because it is supposed form iteration $i$ to iteration $i + \lfloor (O^{SP}_i - Opt)/\varepsilon \rfloor$ all visited scenarios have repeated vectors $w$.

$$O^{SP}_{i+h} \leq O^{SP}_{i+h-1} - \varepsilon \quad h \in \{1, 2, ..., k\} \quad (EC.6.3)$$

Relation (EC.6.3) is equivalent to (EC.6.4).

$$\frac{O^{SP}_{i+h} - Opt}{\varepsilon} \leq \frac{O^{SP}_{i+h-1} - Opt}{\varepsilon} - 1 \quad h \in \{1, 2, ..., k\} \quad (EC.6.4)$$

From (EC.6.4) we can simply obtain

$$\frac{O^{SP}_{i+k} - Opt}{\varepsilon} \leq \frac{O^{SP}_{i+h-1} - Opt}{\varepsilon} - k \quad (EC.6.5)$$

For $k = \lfloor (O^{SP}_i - Opt)/\varepsilon \rfloor$ we will have:

$$\frac{O^{SP}_{i+k} - Opt}{\varepsilon} \leq \frac{O^{SP}_{i+h-1} - Opt}{\varepsilon} - \frac{O^{SP}_i - Opt}{\varepsilon} \quad (EC.6.6)$$

which is equivalent to

$$O^{SP}_{i+k} - Opt \leq \varepsilon \quad (EC.6.7)$$

Therefore, we proved that if in $k = \lfloor (O^{SP}_i - Opt)/\varepsilon \rfloor$ iterations after iteration $i$ the algorithm does not find any scenario with a repeated vector $w$ then $O^{SP}_{i+k} - Opt \leq \varepsilon$ holds.
EC.7. Proof of Lemma 3

Three cases are possible.

Case 1) \( O_{i}^{SP} - \text{Opt} \leq \epsilon \) and \( O_{i}^{SP} - O_{i}^{MP} \geq \epsilon \) hold. We show that in this case in the next iteration the algorithm generates a scenario with a new vector \( w \). Because of \( O_{i}^{SP} - O_{i}^{MP} \geq \epsilon \), the stopping condition in the master problem in iteration \( i \) is satisfied and the following relation holds.

\[
U_{i}^{MP} \leq O_{i}^{SP} - \epsilon \leq \text{Opt} \quad (\text{EC.7.1})
\]

If the algorithm visits a scenario with a repeated vector \( w \) in the subproblem in iteration \( i + 1 \), we must have \( U_{i}^{MP} = O_{i+1}^{SP} \) regarding Lemma 1. Then with respect to (EC.7.1), \( O_{i+1}^{SP} < \text{Opt} \) holds which is a contradiction because the optimal objective value of the subproblem is an upper bound of the optimal objective of the robust problem. Therefore, in this case in the next iteration a scenario with a new vector \( w \) will be generated.

Case 2) \( O_{i}^{SP} - \text{Opt} \leq \epsilon \), \( O_{i}^{SP} - O_{i}^{MP} \leq \epsilon \) and \( O_{i}^{MP} < \text{Opt} \) hold. We show in the next iteration the algorithm generates a scenario with a new vector \( w \). Because of \( O_{i}^{SP} - O_{i}^{MP} \leq \epsilon \), in the master problem in iteration \( i \) there is not any scenario satisfying the stopping condition. Thus, the master problem is solved optimally and we will have the following relation.

\[
O_{i}^{MP} = U_{i}^{MP} \quad (\text{EC.7.2})
\]

In the subproblem of next iteration, if the algorithm visits a scenario with a repeated vector \( w \), then regarding Lemma 1 we must have relation (EC.7.3).

\[
U_{i}^{MP} = O_{i+1}^{SP} \quad (\text{EC.7.3})
\]

Considering the primary assumption \( O_{i}^{MP} < \text{Opt} \) and relations (EC.7.2)- (EC.7.3) we must have \( O_{i+1}^{SP} < \text{Opt} \) which is a contradiction because the optimal objective value of the subproblem is an upper bound of the optimal objective value of the robust problem. Therefore, in this case in iteration \( i + 1 \) the algorithm generates a scenario with a new vector \( w \).

Case 3) \( O_{i}^{SP} - \text{Opt} \leq \epsilon \), \( O_{i}^{SP} - O_{i}^{MP} \leq \epsilon \) and \( O_{i}^{MP} = \text{Opt} \) hold. We show that in this case in the next iteration either the Benders algorithm converges or it generates a scenario with a new vector \( w \). Because of \( O_{i}^{SP} - O_{i}^{MP} \leq \epsilon \), in the master problem in iteration \( i \) there is not any scenario satisfying the stopping condition. Therefore, the master problem is solved optimally and relation (EC.7.4) holds.

\[
L_{i}^{MP} = O_{i}^{MP} = U_{i}^{MP} \quad (\text{EC.7.4})
\]
In the subproblem of iteration $i + 1$, the algorithm generates a scenario with either a new vector $w$ or a repeated vector $w$. In the later case regarding Lemma 1 we must have relation (EC.7.3). Considering the primary assumption $O_{i}^{MP} = Opt$ and relations (EC.7.3)-(EC.7.4) we have $L_{i}^{MP} = Opt = O_{i+1}^{SP}$. This relation demonstrates that the optimal solution of the robust problem is obtained and the Benders algorithm is converged. Therefore, in this case, in the next iteration either the Benders algorithm converges or it generates a scenario with a repeated vector $w$.

**EC.8. Proof of Lemma 4**

Regarding constraint (31) since the algorithm visits a scenario with a repeated vector $w$ in the subproblem of iteration $g(i_{1})$, in any iteration $j \geq g(i_{1})$, the inequality $U_{j}^{MP} \leq O_{g(i_{1})}^{SP}$ holds and by setting $j = g(i_{2}) - 1 \geq g(i_{1})$ we obtain the following relation.

$$U_{g(i_{2})-1}^{MP} \leq O_{g(i_{1})}^{SP}$$

(EC.8.1)

Note that $g(i_{2}) - 1 \geq g(i_{1})$ holds because $i_{1} < i_{2}$. Also regarding Lemma 1, in the subproblem of iteration $g(i_{2})$ that the algorithm has visited a scenario with a repeated vector $w$, we have $U_{g(i_{2})-1}^{MP} = O_{g(i_{2})}^{SP}$. This relation together with (EC.8.1) demonstrates the validity of $O_{g(i_{2})}^{SP} \leq O_{g(i_{1})}^{SP}$.

**EC.9. Proof of Theorem 4**

To prove that the Benders algorithm converges in at most $\sum_{j=1}^{n'} (1 + \left\lfloor (O_{f(j+1)}^{SP} - Opt)/\varepsilon \right\rfloor + 1)I_{f(j+1)}$ iterations it is enough to show it takes at most $1 + \left\lfloor (O_{f(j+1)}^{SP} - Opt)/\varepsilon \right\rfloor + 1)I_{f(j+1)}$ iterations between visiting $j$-th and $(j + 1)$-th new vector $w$ in the subproblem. Let us assume $j < n'$. Two cases are possible.

**Case 1** we have $I_{f(j+1)} = 0$ that means in the iteration $f(j) + 1$ the algorithm finds a scenario with a new vector $w$. In this case the number of between visiting $j$-th and $(j + 1)$-th new scenarios is 1.

**Case 2** we have $I_{f(j+1)} = 1$ that means in iteration $f(j) + 1$ the algorithm visits a scenario with a repeated vector $w$. In this case, after visiting the a scenario with a repeated vector $w$ in iteration $f(j) + 1$, with respect to Lemma 2 it takes at most $k = \left\lfloor (O_{f(j+1)}^{SP} - Opt)/\varepsilon \right\rfloor$ to find a scenario with a new vector $w$ or to have $O_{f(j)+1+k}^{SP} - Opt \leq \varepsilon$. In the later case, regarding Lemma 3, we know that in the next iteration $f(j) + k + 2$ either the Benders algorithm converges or a scenario with a new vector $w$ is found. Since it is assumed that $j < n'$, the Benders algorithm does not converge before finding the $(j + 1)$-th scenario with a new vector $w$. Thus, we expect that the algorithm generates $(j + 1)$-th new vector $w$ by iteration $f(j) + k + 2$. In other words, the number of iterations between visiting $j$-th and $(j + 1)$-th new vector $w$ is at
most \( [(O_{f(j)+1}^{SP} - Opt)/\varepsilon] + 2 \). Therefore, for \( j < n' \) the number of iterations between visiting \( j \)-th and \( (j+1) \)-th new vector \( w \) is computed by relation (EC.9.1).

\[
(1 - I_{f(j)+1}) + \left( \left\lfloor \frac{O_{f(j)+1}^{SP} - Opt}{\varepsilon} \right\rfloor + 2 \right) I_{f(j)+1} \tag{EC.9.1}
\]

For \( j = n' \) we can use a similar reasoning as presented above for \( j < n' \). The difference is that only Case 2 is applicable because regarding the definition of \( n' \) no new vector \( w \) is visited after visiting the \( n' \)-th new vector \( w \). Moreover, when we use Lemmas 2 and 3 in Case 2, the generation of a scenario with a new vector \( w \) is not an option and we are sure that after finding the \( n' \)-th new vector \( w \), the Benders algorithm converges in at most \( \left( \left\lfloor (O_{f(j)+1}^{SP} - Opt)/\varepsilon \right\rfloor + 2 \right) \) iterations that is the same as (EC.9.1) with respect to \( I_{f(j)+1} = 1 \) for \( j = n' \). Therefore, by summing the number of iterations computed by (EC.9.1) from \( j = 1 \) to \( j = n' \) we obtain the following maximum number of iterations.

\[
\sum_{j=1}^{n'} \left( 1 + \left\lfloor (O_{f(j)+1}^{SP} - Opt)/\varepsilon \right\rfloor + 1 \right) I_{f(j)+1} = n' + \sum_{j=1}^{n'} \left( \left\lfloor (O_{g(j)}^{SP} - Opt)/\varepsilon \right\rfloor + 1 \right) I_{f(j)+1} \\
\leq n' + \sum_{j=1}^{n'} \left( \left\lfloor (O_{g(j)}^{SP} - Opt)/\varepsilon \right\rfloor + 1 \right) = n' \left\lfloor (O_{g(1)}^{SP} - Opt)/\varepsilon \right\rfloor + 2 \]

**Proof of the first inequality:** We know that in \( n' \) iterations the algorithm visits at least one scenario with a repeated vector \( w \). \( g(1) \) denotes the iteration in which a repeated vector \( w \) is visited for the first time. To prove the first inequality it is enough to show the validity of the following relation (EC.9.2).

\[
\left\lfloor \frac{O_{g(1)}^{SP} - Opt}{\varepsilon} \right\rfloor + 1 \geq \left( \left\lfloor \frac{O_{f(j)+1}^{SP} - Opt}{\varepsilon} \right\rfloor + 1 \right) I_{f(j)+1} \quad j \in \{1, 2, ..., n'\} \tag{EC.9.2}
\]

As \( O_{g(1)}^{SP} \geq Opt \) is a valid relation, (EC.9.2) holds when \( I_{f(j)+1} \) equal 0. In the case that \( I_{f(j)+1} \) is equal to 1, regarding the definition of \( g(1) \) and \( I_{f(j)+1} \) we know that \( g(1) \leq f(j) + 1 \). Thus, with respect to Lemma 4, we have \( O_{g(1)}^{SP} \geq O_{f(j)+1}^{SP} \) that results in \( \left\lfloor (O_{g(1)}^{SP} - Opt)/\varepsilon \right\rfloor + 1 \geq \left\lfloor (O_{f(j)+1}^{SP} - Opt)/\varepsilon \right\rfloor + 1 \). Therefore, relation (EC.9.2) is valid.

**EC.10. Details of the reformulation for the supply chain problem**

First, we have to make model (47)-(57) consistent with model (P1), and then we can apply the proposed reformulation. To this end, we replace \( u_{ip'y_{ij}} \) and \( u_{ip'y_{kff'}} \) by new second-stage variables \( v_{ijp}^1 \) and \( v_{kff'}^2 \) in the objective function as in (EC.10.1) and add the new constraints (EC.10.2)-(EC.10.3) to \( \mathcal{Y}(x, y) \) defined by (52)-(57).

\[
\min_{x \in \mathcal{X}} \left( \sum_{f \in F_1 \cup F_2} c_f x_f + \max_{y \in \mathcal{Y}(x, y)} \sum_{i \in I} \sum_{j \in J} \sum_{p \in P_i} (c_{ip} v_{ijp}^1) + \right.
\]
\[
\begin{align*}
&+ \sum_{f \in F_1} \sum_{f' \in F_2} \sum_{i \in I} \sum_{k \in K_i} \sum_{p \in P_i} (c_{ikffp} u_{ikfpp}^2) \bigg) \\
v_{ip}^1 \geq u_{ip}^1 \quad i \in I \quad f \in F_1 \quad p \in P_i \\
v_{ikfpp}^2 \geq u_{ip}^2 \quad i \in I \quad k \in K \quad f \in F_1 \quad f' \in F_2 \quad p \in P_i
\end{align*}
\] (EC.10.1)

(1) Then, we linearize (EC.10.2)-(EC.10.3) as (EC.10.4)-(EC.10.5) to make the structure of \( \mathcal{Y}(x,y) \) consistent with that of \( \mathcal{Y}(x,y) \) in model (P1).

\[
\begin{align*}
v_{ip}^1 \geq u_{ip}^1 + y_{ip}^1 - 1 & \quad i \in I \quad f \in F_1 \quad p \in P_i \\
v_{ikfpp}^2 \geq u_{ip}^2 + y_{ikfpp}^2 - 1 & \quad i \in I \quad k \in K \quad f \in F_1 \quad f' \in F_2 \quad p \in P_i
\end{align*}
\] (EC.10.4) (EC.10.5)

After the above modifications, set \( \mathcal{Y}(x,y) \) will be as follows:

\[
\begin{align*}
\mathcal{Y}(x,y) &= \left\{ y \mid (52) - (57), (EC.10.4) - (EC.10.5) \right\}
\end{align*}
\] (EC.10.6)

The identifier \( i \in I \) is common in all relations defining \( \mathcal{Y}(x,y) \) in (EC.10.6). Therefore, we can define the block-diagonal structure for \( i \in I \). In this case, the new binary variable \( w_{ip} \) for the uncertainty set is defined and linked to \( u_{ip} \) as (EC.10.7)-(EC.10.8).

\[
\begin{align*}
[u_{ip}]_{p \in P_i} &= \sum_{p \in P_i} e_i w_{ip} & i \in I \\
\sum_{p \in P_i} w_{ip} &= 1 & i \in I
\end{align*}
\] (EC.10.7) (EC.10.8)

In (EC.10.7), \( e_i \) is \( i \)-th unit vector with 1 as the \( i \)-th entry and 0 as other entries. Constraint (EC.10.7) shows that \( u_{ip} = w_{ip} \) holds in this problem. Comparison of (EC.10.7) with (5) shows that \( B_k u \) in (5) is equivalent to \( [u_{ip}]_{p \in P_i} \) in (EC.10.7). Therefore, to follow the reformulations presented by models (P3) and (P4), we makes copies of the second-stage variables by adding index \( p' \in P_i \) to them and also multiply the second-stage variables in the objective function by their corresponding \( w_{ip'} \) (or equivalently \( u_{ip'} \)). In this case, the revised model will be as:

\[
\begin{align*}
\min_{x \in \mathcal{X}} & \left( \sum_{f \in F_1 \cup F_2} c_{fxf} + \max_{u \in \mathcal{U}} \left( \min_{y \in \mathcal{Y}(x,y)} \sum_{i \in I} \sum_{f \in F_1} \sum_{p \in P_i} \sum_{p' \in P_i} (c_{ipu_{ip}w_{ip}u_{ip'}v_{ip}^1 + \}
\right. \right.
\left. + \sum_{f \in F_1} \sum_{f' \in F_2} \sum_{i \in I} \sum_{k \in K_i} \sum_{p \in P_i} \sum_{p' \in P_i} (c_{ikfpp} u_{ip}^2 v_{ikfpp}^2) \bigg) \right) \\
\mathcal{X} &= \left\{ x \mid x_f \in \{0,1\} \quad f \in F_1 \cup F_2 \right\} \\
\mathcal{U} &= \left\{ u \mid \sum_{p \in P_i} u_{ip} = 1 \quad i \in I \right\}
\end{align*}
\] (EC.10.9) (EC.10.10) (EC.10.11)

\[
\sum_{i \in I} \sum_{p \in P_i} d_{ikp} u_{ip} \leq b_k & \quad k \in K
\] (EC.10.12)
\[
\begin{align*}
\mathcal{Y}(x, y) &= \left\{ y \mid \sum_{f \in \mathcal{F}_1} y_{1f'p'}^1 = 1 \right\} \\
&\quad \quad \quad i \in \mathcal{I}, p \in \mathcal{P}_i \right\} \quad \text{(EC.10.13)} \\
y_{1f'p'}^1 &\leq x_f \quad i \in \mathcal{I}, f \in \mathcal{F}_1, p' \in \mathcal{P} \quad \text{(EC.10.14)} \\
\sum_{f' \in \mathcal{F}_2} y_{ikf'p'}^2 &= y_{1f'p'}^1 \quad i \in \mathcal{I}, k \in \mathcal{K}, f \in \mathcal{F}_1, p' \in \mathcal{P} \quad \text{(EC.10.15)} \\
y_{ikf'p'}^2 &\leq x_{f'} \quad i \in \mathcal{I}, k \in \mathcal{K}, f \in \mathcal{F}_1, f' \in \mathcal{F}_2, p' \in \mathcal{P} \quad \text{(EC.10.16)} \\
y_{1f'p'}^2 &\in \{0, 1\} \quad i \in \mathcal{I}, f \in \mathcal{F}_1, p' \in \mathcal{P} \quad \text{(EC.10.17)} \\
y_{ikf'p'}^2 &\in \{0, 1\} \quad i \in \mathcal{I}, k \in \mathcal{K}, f \in \mathcal{F}_1, f' \in \mathcal{F}_2, p' \in \mathcal{P} \quad \text{(EC.10.18)} \\
v_{1f'p'}^{ip} &\geq e_{ipp'} + y_{1f'p'}^1 - 1 \quad i \in \mathcal{I}, f \in \mathcal{F}_1, p \in \mathcal{P}_i, p' \in \mathcal{P} \quad \text{(EC.10.19)} \\
v_{ikf'p'}^{ip} &\geq e_{ipp'} + y_{ikf'p'}^2 - 1 \quad i \in \mathcal{I}, k \in \mathcal{K}, f \in \mathcal{F}_1, f' \in \mathcal{F}_2, p \in \mathcal{P}_i, p' \in \mathcal{P} \quad \text{(EC.10.20)}
\end{align*}
\]

Comparison of constraints (EC.10.21)-(EC.10.22) with constraints (EC.10.4)-(EC.10.5) implies that we have replaced \( u_{ip} \) with \( e_{ipp'} \), that is a parameter and is equal to 1 for \( p = p' \) and 0 otherwise. The reason for the replacement of \( u_{ip} \) is that when we make copies of the second-stage constraints, we replace the uncertainty variables with their realizations (e.g. compare (9) with (12)). In constraints (EC.10.21)-(EC.10.22), we have \( v_{1f'p'}^{ip} = y_{1f'p'}^1 \) and \( v_{1f'p'}^{ip} = y_{ikf'p'}^2 \) for \( p = p' \). For \( p \neq p' \), \( v_{1f'p'}^{ip} \) and \( v_{1f'p'}^{ip} \) will be equal to 0 because of their non-negative cost in the objective function. This replacement in model (EC.10.9)-(EC.10.22) results in model (EC.10.23)-(EC.10.34).

\[
\min_{x \in \mathcal{X}} \left( \sum_{f \in \mathcal{F}_1 \cup \mathcal{F}_2} c_f x_f + \max_{u \in \mathcal{U}} \left( \min_{y \in \mathcal{Y}(x)} \sum_{i \in \mathcal{I}} \sum_{f \in \mathcal{F}_1} \sum_{p' \in \mathcal{P}_i} (c_{ifp} u_{ip} y_{1f'p'}^1) + \sum_{f \in \mathcal{F}_1} \sum_{f' \in \mathcal{F}_2} \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \sum_{p' \in \mathcal{P}_i} (c_{ikf'p} u_{ip} y_{ikf'p'}^2) \right) \right) \\
\mathcal{X} = \left\{ x \mid x_f \in \{0, 1\}, \quad f \in \mathcal{F}_1 \cup \mathcal{F}_2 \right\} \\
\mathcal{U} = \left\{ u \mid \sum_{p \in \mathcal{P}_i} u_{ip} = 1 \right\} \\
\sum_{i \in \mathcal{I}} \sum_{p \in \mathcal{P}_i} d_{ikp} u_{ip} \leq b_k \quad k \in \mathcal{K} \\
u_{ip} \in \{0, 1\} \quad i \in \mathcal{I}, p \in \mathcal{P}_i \right\} \quad \text{(EC.10.24)} \\
\mathcal{Y}(x) = \left\{ y \mid \sum_{f \in \mathcal{F}_1} y_{1f'p'} = 1 \right\} \\
\quad \quad \quad i \in \mathcal{I}, p' \in \mathcal{P} \\
\quad \quad \quad \text{(EC.10.25)} \\
\quad \quad \quad \text{(EC.10.26)} \\
\quad \quad \quad \text{(EC.10.27)} \\
\quad \quad \quad \text{(EC.10.28)}
\]
\[ y_{ifp'} \leq x_f \quad i \in \mathcal{I} \quad f \in \mathcal{F}_1 \quad p' \in \mathcal{P} \]  
(EC.10.29)

\[ \sum_{f' \in \mathcal{F}_2} y_{ikfp'}^{1} = y_{ifp'}^{1} \quad i \in \mathcal{I} \quad k \in \mathcal{K} \quad f \in \mathcal{F}_1 \quad p' \in \mathcal{P} \]  
(EC.10.30)

\[ y_{ikfp'}^{2} \leq x_{f'} \quad i \in \mathcal{I} \quad k \in \mathcal{K} \quad f \in \mathcal{F}_1 \quad f' \in \mathcal{F}_2 \quad p' \in \mathcal{P} \]  
(EC.10.31)

\[ y_{ifp'}^{1} \in \{0, 1\} \quad i \in \mathcal{I} \quad f \in \mathcal{F}_1 \quad p' \in \mathcal{P} \]  
(EC.10.32)

\[ y_{ikfp'}^{2} \in \{0, 1\} \quad i \in \mathcal{I} \quad k \in \mathcal{K} \quad f \in \mathcal{F}_1 \quad f' \in \mathcal{F}_2 \quad p' \in \mathcal{P} \]  
(EC.10.33)

\[ y_{ikfp'}^{2} \in \{0, 1\} \quad i \in \mathcal{I} \quad k \in \mathcal{K} \quad f \in \mathcal{F}_1 \quad f' \in \mathcal{F}_2 \quad p' \in \mathcal{P} \]  
(EC.10.34)

This model is in the format of model (P3). Therefore, we can pass the inner \( \min_{y \in \mathcal{Y}(x)} \) out of \( \max_{u \in \mathcal{U}} \) that results in model (58)-(66).

**EC.11. Tri-level Benders algorithm for the nurse scheduling problem**

The structure of this tri-level Benders algorithm that we present in this section is proposed by Chen (2013). In this Benders algorithm, we have an outer master problem that fixes a first-stage solution and proposes it to the inner loop. In the inner loop, an inner master problem looks for the worst-case scenario for the uncertainty considering the given first-stage solution. After fixing the uncertainty, we solve a subproblem to find a second-stage solution for the given uncertainty and the fixed first-stage solution. We present the pseudo code of the tri-level Benders algorithm by Algorithm EC.1.

In Line 1, we initialize the values of lower and upper bounds. The inner loop of the algorithm starts in Line 2 and finishes in Line 14 when the optimality gap is small enough or the time limit is reached. In Line 3, we solve the outer master problem (EC.11.1)-(EC.11.7) that provides a first-stage solution for the rest of the algorithm. We have provided the details on the outer master problem after this explanation for the pseudo code of Algorithm 4. We update the lower bound in Line 4 by setting it equal the optimal objective value of the recent outer master problem. Then, the inner loop of the algorithm starts in Line 5 and finishes in Line 11. In this loop, we first add an optimality cut (EC.11.10) to the inner master problem (EC.11.9)-(EC.11.15) using the most recent second-stage solution \( \hat{y}_d \) that is obtained in the subproblem. In the first iteration of the inner loop, we consider a trivial second-stage solution \( \hat{y}_d = M \) \( d \in \mathcal{D} \). This solution means that we hire \( M \) nurses on all days over the planning horizon. In Line 7, we solve the inner master problem (EC.11.9)-(EC.11.15) to find a new worst-case scenario for the realization of the uncertainty. In Line 8, we update the upper bound by \( UB := \max\{UB, UB_{new}\} \) where \( UB \) is the current upper bound in the algorithm and \( UB_{new} \) is the sum of current first-stage cost obtained in the outer master problem and the worst-case second stage cost that is the optimal objective value of recent inner master problem. In Line 9,
Algorithm EC.1. Tri-level Benders algorithm

1: Initialize $UB=\infty$ and $LB=-\infty$.

2: repeat

3: Solve the outer master problem (EC.11.1)-(EC.11.7) to find a first-stage solution.

4: Update the lower bound.

5: repeat

6: Add an optimality cut (EC.11.10) to the inner master problem (EC.11.9)-(EC.11.15) for the current second-stage solution.

7: Solve the inner master problem (EC.11.9)-(EC.11.15) to find a new worst-case scenario.

8: Update the upper bound if necessary.

9: Modify the right-hand side of constraint (EC.11.17) in the subproblem (EC.11.16)-(EC.11.18) based on the new worst-case scenario.

10: Solve the subproblem (EC.11.16)-(EC.11.18) to find a new second-stage solution for the given first-stage solution and the given worst-case scenario.

11: until (the objective value of the subproblem is equal to the objective value of the inner master problem or the time limit $AlgTimeLimit$ is reached)

12: Remove all optimality cuts from the inner master problem.

13: Add the optimality cut (EC.11.2) to the outer master problem.

14: until $100(UB - LB)/LB \leq \delta_{acc}$ or time limit $AlgTimeLimit$ is reached

we update the right-hand side value of Constraint (EC.11.17) in the subproblem using the current worst-case scenario obtained in the inner master problem (EC.11.9)-(EC.11.15). Then we solve the subproblem (EC.11.16)-(EC.11.18) for the current first-stage solution and the current worst-case scenario. In Line 11, the inner loop stops if the objective value of the subproblem is equal to the objective value of the inner master problem or if the time limit is reached. In Line 12, we remove all optimality cuts added to the inner master problem because the inner master problem is solved locally for each first-stage solution fixed in the outer master problem. We then add an optimality cut (EC.11.2) to the outer master problem. In Line 14, the algorithm stops if the optimality gap is less than $\delta_{acc}$ or if the time limit is reached.

In the following, we present the outer master problem, inner master problem, and the subproblem of the tri-level benders algorithm. We suppose that all variables, sets, and parameters presented in Section 5 are defined in the same way.

**Outer Master Problem**

We introduce the following notation for the outer master problem.

**Sets:**
\( \mathcal{J} \): Index set for the first-stage solutions that are evaluated by the inner master problem.

\( \mathcal{I}_d \): Index set for the number of first-stage nurses that we may hire for day \( d \). We compute it by \( \mathcal{I}_d = \{0, 1, \ldots, S_{d}^{\text{max}}\} \).

Parameters:

- \( \alpha_j \): The second-stage cost for first-stage solution \( j \in \mathcal{J} \).
- \( S_{d}^{\text{max}} \): The maximum number of patients that can be in the ward on day \( d \).
- \( \hat{x}_{jd} \): The number of first-stage nurses on day \( d \) in solution \( j \in \mathcal{J} \).

Variables:

- \( v_{di} \): 1 if we hire \( i \) nurses in the first stage, 0 otherwise.
- \( \theta_{\text{outer}} \): A lower bound on the second-stage cost in the outer master problem.

The outer master problem of the tri-level Benders algorithm reads as follows.

\[
\text{Outer-MP} \quad \min_{x, \theta_{\text{outer}}} c_1^T x + \theta_{\text{outer}} \tag{EC.11.1}
\]

\[
\theta_{\text{outer}} \geq \alpha_j - c_2 \sum_{d \in \mathcal{D}} \sum_{i \in \mathcal{I}_d; i > \hat{x}_{jd}} (i - \hat{x}_{jd}) v_{di} \quad j \in \mathcal{J} \tag{EC.11.2}
\]

\[
x_d = \sum_{i \in \mathcal{I}_d} iv_{di} \quad d \in \mathcal{D} \tag{EC.11.3}
\]

\[
\sum_{i \in \mathcal{I}_d} v_{di} = 1 \quad d \in \mathcal{D} \tag{EC.11.4}
\]

\[
\theta_{\text{outer}} \geq 0 \tag{EC.11.5}
\]

\[
v_{di} \in \{0, 1\} \quad d \in \mathcal{D}, i \in \mathcal{I}_d \tag{EC.11.6}
\]

\[
\delta x_d + \delta M \geq \rho S_{d}^{\text{max}} \quad d \in \mathcal{D} \tag{EC.11.7}
\]

In (EC.11.1)-(EC.11.6), the objective function (EC.11.1) minimizes the sum of the first-stage cost \( c_1^T x \) and \( \theta_{\text{outer}} \) that provides a lower bound on the second-stage. Optimality cut (EC.11.2) approximates the second-stage cost and tightens \( \theta_{\text{outer}} \) as the algorithm adds more cuts to the outer master problem. This cut ensures that \( \theta_{\text{outer}} \) will be equal to the second-stage cost \( \alpha_j \) if solution \( j \in \mathcal{J} \) is selected again, and for all other first-stage solutions that are not visited yet, \( \theta_{\text{outer}} \) is a lower bound on the second-stage cost. In this constraint, for a fixed day \( d \in \mathcal{D} \), if the number of first-stage nurses that the model selects is \( a \) units less than the number of hired first-stage nurses in a previously visited solution \( j \) (i.e., \( \sum_{i \in \mathcal{I}_d; i > \hat{x}_{jd}} (i - \hat{x}_{jd}) v_{di} = a \)), then the second-stage cost for the new solution on this day is most \( c_2 \times a \) less than the second-stage cost \( \alpha_j \). This constraint is an improved version of the following big-M constraint.

\[
\theta_{\text{outer}} \geq \alpha_j - \sum_{d \in \mathcal{D}} M^* \left[ 1 - \sum_{i \in \mathcal{I}_d; i \leq \hat{x}_{jd}} v_{di} \right] \quad j \in \mathcal{J} \tag{EC.11.8}
\]
In Constraint (EC.11.8), $M'$ is a big-M value. This constraint becomes inactive when $\sum_{i \in I_d} v_{di} = 0$. Since, constraint (EC.11.2) is stronger than (EC.11.8), we consider the former constraint in the outer master problem. (EC.11.3) links variables $v_{di}$ to variables $x_d$. (EC.11.4) implies that for each day $d \in D$, exactly one of $v_{di}$ must be equal to 1. Constraint (EC.11.6) is the set of all feasibility cuts that prevents from infeasibility in the subproblem.

**Inner Master Problem**

We introduce the following notation for the inner master problem.

**Sets:**

$J'$ : Index set for the second-stage solutions that are generated by the subproblem.

**Parameters:**

$\hat{y}_{jd}$ : The number of second-stage nurses on day $d$ in the second-stage solution $j \in J'$.

**Variables:**

$\theta_{inner}$: A upper bound on the second-stage cost for the current first-stage solution.

\[
\begin{align*}
\text{Inner-MP} & \quad \max_{w,u,\theta_{inner}} \theta_{inner} \quad \text{ (EC.11.9)} \\
\theta_{inner} & \leq \sum_{d \in D} \sum_{s \in S_d} c_{2d} \hat{y}_{jd} w_{ds} \quad j \in J' \quad \text{ (EC.11.10)} \\
\sum_{s \in S_d} w_{ds} & = 1 \quad d \in D \quad \text{ (EC.11.11)} \\
\sum_{t \in T} \sum_{p \in P_t} u_{tp} & = \sum_{s \in S_d} s w_{ds} \quad d \in D \quad \text{ (EC.11.12)} \\
\sum_{p \in P_t} u_{tp} & = 1 \quad t \in T \quad \text{ (EC.11.13)} \\
\quad u_{tp} & \in \{0,1\} \quad t \in T, p \in P_t \quad \text{ (EC.11.14)} \\
\quad w_{ds} & \in \{0,1\} \quad d \in D, s \in S_d \quad \text{ (EC.11.15)}
\end{align*}
\]

The objective function (EC.11.9) minimize the upper bound on the second-stage cost. Constraint (EC.11.10) is the optimality cut of the inner master problem and approximate the second-stage cost. This approximation becomes more accurate as more optimality cuts are generated and added to the inner master problem. Constraints (EC.11.11)-(EC.11.12) define variables $w_{ds}$ and link them to variables $u_{tp}$. Constraint (EC.11.13) implies that for each patient, exactly one of the possible scenarios realizes.

**Subproblem**

We define the subproblem as follows. In this model, We introduce parameters $\hat{u}_{tp}$ and $\hat{x}_d$ as follows.
The value of $u_{tp}$ in the recent inner master problem.

The value of $x_d$ in the recent outer master problem.

\[
\begin{align*}
\text{SP} \quad & \min y \sum_{d \in D} c_{2y_d} \\
& \delta \hat{x}_d + \delta y_d \geq \rho \sum_{t \in T} \sum_{p \in P_{td}} \hat{u}_{tp} \quad d \in D \\
& 0 \leq y_d \leq M_d, \text{integer} \quad d \in D
\end{align*}
\] (EC.11.16)

(EC.11.17)

(EC.11.18)

The objective function (EC.11.16) minimizes the second-stage cost. Constraint (EC.11.17) is the demand constraint for a first-stage solution fixed by the outer master problem and an uncertainty scenario given by the inner master problem.

EC.12. Experiments on the tuning of $\varepsilon$ for nurse planning instances

In the first experiment, we investigate the impact of parameter $\varepsilon$ in the proposed stopping conditions. In Table EC.1, we report the computational results of the proposed Benders algorithm for $\varepsilon = \{0.5, 5, 50, 500\}$. In Table EC.1, each row gives the average results for 50 instances with different values of the incentive factor. Under “Data Info.”, “L”, “OR”, and “Sur.” respectively give the number of weeks in the planning horizon, the number of operating room, and the number of surgeries over the planning horizon. “Time (sec)” gives the total computational time of algorithms in seconds. Also “Ite.” gives the number of iterations that algorithms repeat their main loops. Furthermore, “LB” and “UB” indicate the best lower and upper bounds in the last iteration of the algorithms and “Gap.” computes the gap between these bounds.

In Table EC.1, we observe that the algorithm obtains the best average optimality gaps for $\varepsilon = 5$. The average optimality gaps deteriorate for very large or small values of $\varepsilon$. For large values of $\varepsilon = 50$ and 500, the stopping conditions terminate the master problem and the subproblem more rarely. In this case, the algorithm spends a lot of computational time to prove the optimality of the problems that is futile especially in initial iterations. As a result, as we can see in Table EC.1, the algorithm repeats fewer iterations without enough interactions between the master problem and the subproblem. In the other extreme case, when the $\varepsilon$ is very small ($\varepsilon = 0.5$), the algorithm spends less time to improve the quality of lower and upper bounds in the master problem and the subproblem and terminates them more frequently. Therefore, the algorithm repeats more iterations compared to the case of $\varepsilon = 5$ and results in slightly higher optimality gaps.
Table EC.1. Computational results of the Benders algorithm for different values of $\varepsilon$ in the proposed stopping conditions.

<table>
<thead>
<tr>
<th>Data Info.</th>
<th>$\varepsilon = 0.5$</th>
<th>$\varepsilon = 5$</th>
<th>$\varepsilon = 50$</th>
<th>$\varepsilon = 500$</th>
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<td></td>
<td>Time (sec)</td>
<td>Ite.</td>
<td>LB</td>
<td>UB</td>
</tr>
<tr>
<td>L OR Sur.</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>2 1 39</td>
<td>1 29 336 336 0.00</td>
<td>2 29 336 336 0.00</td>
<td>1 29 336 336 0.00</td>
<td>1 29 336 336 0.00</td>
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<tr>
<td>2 79</td>
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<td>12 49 650 650 0.00</td>
<td>12 49 650 650 0.00</td>
<td>12 49 650 650 0.00</td>
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<tr>
<td>3 119</td>
<td>340 65 958 958 0.00</td>
<td>341 65 958 958 0.00</td>
<td>343 65 958 958 0.00</td>
<td>349 65 958 958 0.00</td>
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<tr>
<td>4 157</td>
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<td>1567 91 1254 1254 0.00</td>
<td>2377 87 1254 1254 0.02</td>
<td>2542 87 1254 1254 0.04</td>
</tr>
<tr>
<td>5 202</td>
<td>8415 114 1577 1581 0.25</td>
<td>8512 113 1577 1581 0.24</td>
<td>10905 104 1576 1583 0.43</td>
<td>11211 101 1576 1588 0.71</td>
</tr>
<tr>
<td>Average</td>
<td>2070 89 955 957 0.05</td>
<td>2087 69 955 956 0.05</td>
<td>2728 67 955 956 0.09</td>
<td>2823 66 955 957 0.15</td>
</tr>
<tr>
<td>3 1 59</td>
<td>28 44 672 672 0.00</td>
<td>28 44 672 672 0.00</td>
<td>27 44 672 672 0.00</td>
<td>27 44 672 672 0.00</td>
</tr>
<tr>
<td>2 121</td>
<td>7882 102 1323 1326 0.23</td>
<td>7235 101 1323 1326 0.23</td>
<td>8407 73 1322 1327 0.36</td>
<td>8588 72 1322 1327 0.37</td>
</tr>
<tr>
<td>3 182</td>
<td>14400 171 1914 1968 2.80</td>
<td>14400 171 1913 1965 2.68</td>
<td>14400 98 1903 1973 3.65</td>
<td>14400 70 1876 2048 9.22</td>
</tr>
<tr>
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<td>14400 279 2505 2619 4.52</td>
<td>14400 258 2506 2618 4.46</td>
<td>14400 143 2494 2627 5.29</td>
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</tr>
<tr>
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<td>14400 412 3122 3297 5.56</td>
<td>14400 370 3122 3294 5.48</td>
<td>14400 187 3112 3308 6.26</td>
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<tr>
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<td>886 84 1031 1031 0.00</td>
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<tr>
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<td>14400 300 3708 3999 7.81</td>
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<tr>
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<td>14400 576 4551 4978 9.39</td>
<td>14400 565 4573 4985 8.98</td>
<td>14400 317 4547 4994 9.82</td>
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</tr>
<tr>
<td>Average</td>
<td>11695 382 2830 3019 5.23</td>
<td>11697 363 2835 3019 5.11</td>
<td>11745 195 2824 3025 5.65</td>
<td>11723 79 2766 3070 9.16</td>
</tr>
</tbody>
</table>
EC.13. Tri-level algorithm for the supply chain problem

The structure of the tri-level algorithm for the supply chain problem is the same as the one already explained for the nurse planning problem in Appendix EC.11. In this appendix, we present the models for the outer master problem, the inner master problem, and the subproblem of the tri-level algorithm for this application. The pseudo code for the algorithm will be the same as the one in Algorithm in Appendix EC.11.

**Outer Master Problem**

We introduce the following notation for the outer master problem.

**Sets:**

- $\mathcal{J}$: Index set for the first-stage solutions that are evaluated by the inner master problem.

**Parameters:**

- $\alpha_j$: The second-stage cost for first-stage solution $j \in \mathcal{J}$.
- $\hat{x}_{jf}$: 1 if facility $f \in \mathcal{F}_1 \cup \mathcal{F}_2$ is open in solution $j \in \mathcal{J}$, 0 otherwise.

**Variables:**

- $x_f$: 1 if we decide to open facility $f \in \mathcal{F}_1 \cup \mathcal{F}_2$, 0 otherwise.
- $\theta_{outer}$: A lower bound on the second-stage cost in the outer master problem.

The outer master problem of the tri-level Benders algorithm reads as:

$$\text{Outer-MP} \quad \min_{x, \theta_{outer}} \sum_{f \in \mathcal{F}_1 \cup \mathcal{F}_2} c_f x_f + \theta_{outer}$$  \hspace{1cm} (EC.13.1)

$$\theta_{outer} \geq \alpha_j - M \left[ \sum_{f \in \mathcal{F}_1 \cup \mathcal{F}_2: \hat{x}_{jf} = 0} x_f \right] \quad j \in \mathcal{J}$$  \hspace{1cm} (EC.13.2)

$$x_f \in \{0,1\} \quad f \in \mathcal{F}_1 \cup \mathcal{F}_2$$  \hspace{1cm} (EC.13.3)

In (EC.13.1)-(EC.13.3), the objective function (EC.11.1) minimizes the sum of the first-stage opening cost and $\theta_{outer}$ that provides a lower bound on the second-stage. Optimality cut (EC.13.2) approximates the second-stage cost and tightens $\theta_{outer}$ as the algorithm adds more cuts to the outer master problem. This cut implies that $\theta_{outer}$ will be equal to the second-stage cost $\alpha_j$ if solution $j \in \mathcal{J}$ is selected again. Otherwise, $\theta_{outer}$ provides a lower bound on the second-stage cost. In this constraint, $M$ represents a very large number that we can set to the trivial value $\alpha_j$.

**Inner Master Problem**

We introduce the following notation for the inner master problem.
Sets:

\[ \mathcal{J}' : \text{Index set for the second-stage solutions that are generated by the subproblem.} \]

Parameters:

\[ \hat{y}_{jif}^1 : 1 \text{ if, in the second-stage solution } j \in \mathcal{J}', \text{ first-layer facility } f \text{ supplies the demand of customer } i. \]

\[ \hat{y}_{jikff'}^2 : 1 \text{ if, in the second-stage solution } j \in \mathcal{J}', \text{ customer } i \text{'s demand for product } k \text{ is transported from second-layer facility } f' \text{ to first-layer facility } f; 0 \text{ otherwise.} \]

Variables:

\[ \theta_{\text{inner}} : \text{A upper bound on the second-stage cost for the current first-stage solution.} \]

\[ u_{ip} : 1 \text{ if local scenario } p \text{ realizes for customer } i, 0 \text{ otherwise.} \]

Inner-MP

\[
\begin{align*}
\max_{\theta_{\text{inner}}} & \quad \theta_{\text{inner}} \\
\text{s.t.} & \quad \theta_{\text{inner}} \leq \left( \sum_{i \in \mathcal{I}} \sum_{f \in \mathcal{F}_1} \sum_{p \in \mathcal{P}_i} (c_{ifp} \hat{y}_{jif}^1 u_{ip}) + \right. \\
& \quad + \sum_{j \in \mathcal{J}'} \sum_{f \in \mathcal{F}_1} \sum_{f' \in \mathcal{F}_2} \sum_{i \in \mathcal{K}} \sum_{k \in \mathcal{K}_p} (c_{ikff'} \hat{y}_{jikff'}^2 u_{ip}) \right) j \in \mathcal{J}' \\
& \quad \sum_{p \in \mathcal{P}_i} u_{ip} = 1 i \in \mathcal{I} \\
& \quad \sum_{i \in \mathcal{I}} \sum_{p \in \mathcal{P}_i} d_{ikp} u_{ip} \leq b_k k \in \mathcal{K} \\
& \quad u_{ip} \in \{0, 1\} i \in \mathcal{I} p \in \mathcal{P}_i
\end{align*}
\]  

\[(EC.13.4)\]

\[ \text{The objective function (EC.13.4) minimize the upper bound on the second-stage cost. Constraint (EC.13.5) is the optimality cut of the inner master problem and approximate the second-stage cost. This approximation becomes more accurate as more optimality cuts are generated and added to the inner master problem. Constraints (EC.13.6)-(EC.13.8) define the uncertainty set.} \]

Subproblem

We introduce parameters \( \hat{u}_{ip} \) and \( \hat{x}_f \) as follows.

\[ \hat{u}_{ip} : \text{The value of } u_{ip} \text{ in the recent inner master problem.} \]

\[ \hat{x}_f : \text{The value of } x_f \text{ in the recent outer master problem.} \]

\[
\begin{align*}
\text{SP} & \quad \min_y \left( \sum_{i \in \mathcal{I}} \sum_{f \in \mathcal{F}_1} \sum_{p \in \mathcal{P}_i} (c_{ifp} \hat{u}_{ip} \hat{y}_{iif}^1) \\
& \quad + \sum_{f \in \mathcal{F}_1} \sum_{f' \in \mathcal{F}_2} \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}_p} (c_{ikff'} \hat{u}_{ip} \hat{y}_{iikff'}^2) \right) \\
& \quad (EC.13.9)
\end{align*}
\]
\[
\sum_{f \in \mathcal{F}_1} y^1_{if} = 1 \quad i \in \mathcal{I} 
\] (EC.13.10)

\[
y^1_{if} \leq \hat{x}_f \quad i \in \mathcal{I} \quad f \in \mathcal{F}_1 
\] (EC.13.11)

\[
\sum_{f' \in \mathcal{F}_2} y^2_{ikff'} = y^1_{if} \quad i \in \mathcal{I} \quad f \in \mathcal{F}_1 
\] (EC.13.12)

\[
y^2_{ikff'} \leq \hat{x}_f \quad i \in \mathcal{I} \quad k \in \mathcal{K} \quad f \in \mathcal{F}_1 \quad f' \in \mathcal{F}_2 
\] (EC.13.13)

\[
y^1_{if} \in \{0, 1\} \quad i \in \mathcal{I} \quad f \in \mathcal{F}_1 
\] (EC.13.14)

\[
y^2_{ikff'} \in \{0, 1\} \quad i \in \mathcal{I} \quad k \in \mathcal{K} \quad f \in \mathcal{F}_1 \quad f' \in \mathcal{F}_2 
\] (EC.13.15)

The objective function (EC.13.9) minimizes the second-stage cost. Constraints (EC.13.10)-(EC.13.15) are the second-stage constraint (52)-(57) for a first-stage solution fixed by the outer master problem and an uncertainty scenario given by the inner master problem.

An noticeable point about the given tri-level formulation is that the optimality cut (EC.13.2) is similar to the optimality cut (EC.11.8) for the nurse planning problem, because both constraint will be redundant when the the value multiplied by M is equal or larger than 1. In the nurse planning problem, we could consider constraint (EC.11.2) as an enhanced version (EC.11.8) that could provide a valid lower bound approximation on the second-stage cost of neighbourhood solutions. However, in the supply chain problem, the structure of the problem is such that we cannot improve the original optimality cut (EC.13.2). This is why the tri-level algorithm provides poor optimality gaps in Table 5.

**EC.14. Generalization to finite sets of fractional parameters**

The main reason that we consider integrality constraints on \( U \) variables is that it makes the representation and the reformulation of the problem much easier. Here we argue that, even when the uncertainty set \( U \) includes a finite number of fractional points, we can use integer variables to represent it. Let us suppose that \( U \) includes \( m \) fractional points \( \hat{u}_1, ..., \hat{u}_m \). To represent \( U \) using integer variables we introduce binary variables \( u'_1, ..., u'_m \) where \( u'_i \) is 1 if the realized uncertainty is point \( u_i \), 0 otherwise. Set \( U \) can be described as:

\[
U = \left\{ u \mid u = \sum_{i=1}^{m} \hat{u}_i u'_i, \quad \sum_{i=1}^{m} u'_i = 1, \quad u'_i \in \{0, 1\} \quad i \in \{1, ..., m\} \right\} 
\] (EC.14.1)

Then in the second-stage constraints of the two-stage robust model, we can substitute variables \( u \) using \( u = \sum_{i=1}^{m} \hat{u}_i u'_i \) and define the new uncertainty set \( U' \) as:
In the new uncertainty set all variables are integer. One may criticize this uncertainty set by saying that it is trivial as we have one binary variable for each fractional point. Indeed, in the case of an exponential number of points, $U'$ would require the same exponential number of binary variables. The answer is that it is possible to reduce the number of such variables significantly using additional information about the relation between the fractional points. For example, in the supply chain problem provided in Section 6.2, the uncertainty set includes $|P_1| \times \ldots \times |P_{|\mathcal{I}|}|$ fractional demand points. As the demand realizations for customers are independent (local scenarios) then they can be represented by $|P_1| + \ldots + |P_{|\mathcal{I}|}|$ binary variables, and that is much less than $|P_1| \times \ldots \times |P_{|\mathcal{I}|}|$. 

$$U' = \left\{ u' \mid \sum_{i=1}^{m} u'_i = 1, \right\}$$  \hspace{1cm} \text{(EC.14.4)}

$$u'_i \in \{0,1\} \quad i \in \{1,\ldots,m\}$$ \hspace{1cm} \text{(EC.14.5)}