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Umbrella organization: research and outreach

Four programs:

- MIT PRIMES: research on MIT campus for local students
- PRIMES-USA: research for students not in the Boston area
- PRIMES Circle: enrichment program for high school students from urban Boston area public high schools
- PRIMES STEP: new enrichment program for middle school students
• Pair 2-3 high school students with an undergraduate mentor to do directed reading for a semester
• Requirements:
  • Individual groups meet once a week for 2 hours
  • Three whole group meetings
  • A 20 minute talk at a “mini-conference”
  • A 5-7 page paper on a topic related to their reading
Students presenting at the conference

Emily, CRLS
Wilkin, BLS
Natalia, Saugus High
Sheinya, Saugus High
Much of the math the students are taught in school is rote memorization, and bears little resemblance to the mathematical process experienced by mathematicians.

In high school, if you don’t know how to solve a problem, you probably weren’t paying attention in class.

In research mathematics, if you can solve a problem immediately, it means that it is not very interesting, and thus not worth working on.
The reading topic is meant to be a *vehicle* for learning about the mathematical process:

- Reading formal exposition
- Chipping away at a problem you don’t immediately know how to solve
- Communicating mathematics with your peers and mentors
- Formally presenting in oral and written form
What does this actually entail?

- Example subject: combinatorial and geometric game theory
- Let’s play a game:
  - The game starts with $N$ “crosses” on a page (for concreteness, say $N = 3$)
  - Two players, player 1 and player 2, alternate making moves
  - On a move, a player draws a line between two “free ends” that does not cross any existing lines and draws a new “cross” in the middle
  - The game ends when there are no more available moves
For example... (with $N = 2$)
The game of Brussels Sprouts

For example... (with $N = 2$)

So this game was a win for player 2!
Some natural questions that come up:

1. Does the game always end?
2. If so, can we bound how long a game will last?
3. Can we determine who will a game with $N$ starting crosses?
4. Given a finished game, can we tell who won the game?
5. ... etc...
The game of Brussels Sprouts

...but don’t just let me tell you! Let’s try it!

Record your answers for the following games:

\[
\begin{array}{c|ccc}
N = & 2 & 3 & 4 \\
\hline
\text{Number of moves} & \\
\text{Winner} & \\
\end{array}
\]
The game of Brussels Sprouts

<table>
<thead>
<tr>
<th>$N =$</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of moves</td>
<td>8</td>
<td>13</td>
<td>18</td>
</tr>
<tr>
<td>Winner</td>
<td>player 2</td>
<td>player 1</td>
<td>player 2</td>
</tr>
</tbody>
</table>

- An **odd** number of moves means **player 1** wins, and an **even** number of moves means **player 2** wins.
- The number of moves (and hence the winner) seems to be determined by $N$... in fact it looks like

  \[ \text{number of moves} = 5N - 2. \]

- ...but, can we **prove** this?
Definition

A planar graph is a collection of vertices (dots) and edges between vertices such that no two edges cross. The graph is connected if it is possible to follow a sequence of edges from any one vertex to another.
Which of these graphs do you think are planar?
Which of these graphs do you think are planar?

- Yes!
- Yes!
- No

Being planar is equivalent to being able to be drawn on the surface of a sphere (with no overlapping edges).
Besides vertices and edges, we’ll need one more concept: the number of enclosed **faces** of the graph.

- This is counted as if the graph lay on a surface... e.g. there is one more face than the “obvious” ones, the “outside of the graph”

\[
\begin{align*}
V &= \# \text{ Vertices} = 7 \\
E &= \# \text{ Edges} = 9 \\
F &= \# \text{ Faces} = 4
\end{align*}
\]
**Definition**

The *Euler characteristic* of a graph $G$ is the integer

$$\chi(G) = V - E + F.$$  

So, let’s do another game: everyone draw a connected planar graph and calculate its Euler characteristic.
What you’ve discovered is known as “Euler’s Formula”, and is a fundamental first step in the area of topology!

**Theorem (Euler)**

*If \( G \) is a connected planar graph, then \( \chi(G) = 2 \).*

You might guess that this really says something about the “type of surface” the graph can be drawn on.
The Euler Characteristic

\[ \chi = 8 - 12 + 6 = 2. \]

\[ \chi = 16 - 32 + 16 = 0 \neq 2! \]

The surfaces that these can be drawn on are:

sphere

torus
In fact, you can define the Euler characteristic of a (compact, oriented) surfaces in terms of the Euler characteristic of the graphs that lie on its surface.

\[
\begin{align*}
\text{sphere} & \quad \chi = 2 \\
\text{torus} & \quad \chi = 0 \\
2\text{-hole torus} & \quad \chi = -2 \\
3\text{-hole torus} & \quad \chi = -4
\end{align*}
\]

In fact, it is a theorem that if the number of holes is \( g \), then \( \chi = 2 - 2g \). And further this is the only topological invariant – it classifies (compact, oriented) surfaces!
So what does this have to do with Brussels Sprouts?

- At the end of the game, we’ve produced a connected, planar graph
- So if we can find formulas for $V$, $E$, $F$ in terms of the number of moves in the game $m$, then we can use

$$V - E + F = 2$$

to determine $m$. 
Vertices:
- begin with $N$
- add 1 on every move

So at the end of the game there are $N + m$ vertices.
Edges:
- begin with 0
- add 2 on every move

So at the end of the game there are $2m$ edges.
Faces:

- By definition, the game ends where there is exactly 1 “free end” in every face
- And on every move the number of “free ends” stays constant

So at the end of the game there are $4N$ faces.
...so putting it all together:

$$\chi = V - E + F$$

$$= (N + m) - (2m) + (4N)$$

$$= 5N - m$$

$$= 2.$$  

To as we guessed at the beginning, the number of moves is $5N - 2!$
Why this problem?

- Can be explored by playing many games
- The fact that the result does not depend on the particulars of the game makes seeing patterns easier
- It relates to some deep and fundamental mathematics – the Euler characteristic and topology more generally
- After understanding Brussels Sprouts, students can move on to more difficult games where skill is involved in winning – such as Sprouts
If you want further information:

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http://math.mit.edu/research/highschool/primes/circle/index.php