Some Linear Algebra Problems

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Most of these problems were written for my students in Math 23a/b at Harvard in 2011/2012 and 2012/2013.

1. Consider a parallelogram spanned by vectors $\vec{v}$ and $\vec{w}$.

(a) Prove the “parallelogram law,” which says that the sum of the squares of the lengths of the diagonals of the parallelogram is equal to $2(|\vec{v}|^2 + |\vec{w}|^2)$.

(b) Use vectors to prove that the diagonals of a rhombus are perpendicular.

2. T/F:

(a) For all $\vec{v} \in \mathbb{R}^3$, the set of vectors $\vec{u} \in \mathbb{R}^3$ such that $\vec{u} \times \vec{v} = \vec{0}$ forms a subspace of $\mathbb{R}^3$.

(b) The set of all invertible $n \times n$ real matrices forms a subspace of $\mathbb{R}^{n^2}$.

3. A linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ sends the first standard basis vector $\vec{e}_1$ to the vector $\vec{a}_1$. Similarly $T(\vec{e}_2) = \vec{a}_2$ and $T(\vec{e}_3) = \vec{a}_3$. Furthermore $\vec{a}_1 + \vec{a}_2 + \vec{a}_3 = 0$. Does there exist a unique $S = T^{-1}$ such that $S \circ T = T \circ S = \mathbb{I}$. Provide a proof.

4. T/F: The linear transformation $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by rotation through angle $\theta$ about the origin has at least one real eigenvalue.

5. T/F: If $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has $n$ linearly independent eigenvectors, then $T$ is invertible.

6. T/F: Given an orthonormal basis $\{\vec{v}_1, ..., \vec{v}_n\}$ for a vector space $V$, if $\vec{w} \in V$ is in terms of the standard basis, then $\vec{w} = c_1 \vec{v}_1 + ... + c_n \vec{v}_n$ where $c_i = w \cdot \vec{v}_i$.

7. T/F: If $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are two linear transformations such that $A = C^{-1} \circ B \circ C$ for some invertible $C$, then $\dim \ker(A) = \dim \ker(B)$

8. T/F: Each of the follow forms a vector space over $\mathbb{R}$ (evaluate them seperately):

(a) $D[\pi, 2\pi]$, the space of discontinuous real-valued functions

\[
d : [\pi, 2\pi] \rightarrow \mathbb{R}
\]

under the composition law $(d_1 + d_2)(x) = d_1(x) + d_2(x)$ and scaling law $r(d_1(x)) = r \ast d_1(x) \ \forall r \in \mathbb{R}$

(b) The space of invertible linear transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ under the composition law $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ and obvious scaling law by elements of $\mathbb{R}$
(c) The set of symmetric matrices $A \in \text{Mat}(3 \times 3)$ with $\text{trace}(A) = 0$.

(d) Functions from a set $S = \{1, 2, ..., n\}$ to $\mathbb{R}$ under the obvious composition and scaling rules.

(e) The set of real polynomials of degree less than or equal to 3 with a root at $+3$.

9. Does row reduction preserve the kernel and image of a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$? If yes, why? If no, what does it preserve?

10. Given that a linear transformation $T$ acting on a real vector space $V$ has at least one eigenvalue, what is the codomain of $T$? If $T$ has only one eigenvalue what can you say? If $T$ has an eigenbasis without distinct eigenvalues what can you say about geometry of $T$? Say $T : \mathbb{R}^3 \to \mathbb{R}^3$, $|T| \neq 0$, but $\text{trace}(T) = 0$. If $T$ does not have distinct eigenvalues but has an eigenbasis, what can you say about the eigenvalues of $T$?

11. Let $V$ be an $n$-dimensional complex vector space. Consider two linear transformation $A : V \to V$ and $B : V \to V$. Prove that if $A$ and $B$ do no commute (i.e. if $A \circ B \neq B \circ A$) then there cannot exist a basis $\{v_i\}$ for $V$ which is simultaneously an eigenbasis for both $A$ and $B$. (Hint: proceed by contradiction.)

12. This problem will lead you through a proof and application of the Buckingham $\pi$ Theorem, one of the most fundamental results in dimensional analysis. Given a system of $n$ physical variables $u_i$ (say the gravitational constant $g$, mass of an object $m$, the length of a string $l$, etc.) in $k$ independent physical dimensions $v_i$ (for example $T$ time, $M$ mass, $L$ length, etc),

(a) Show that the space of fundamental and derived “units” forms a vector space over $\mathbb{Q}$. (Hint: if $L, T, M$ are fundamental units, then $M \ast L \ast T^{-2}$ is a derived unit, as is $(M \ast L \ast T^{-2})^{3/2}$. How might we represent these derived units in terms of fundamental units? What might we call the “fundamental units”? Note that multiplying a derived/fundamental unit by a scaler is considered an equivalent unit.)

(b) Now consider a matrix $M$ which we will call the dimension matrix for obvious reasons. Each column of $M$ tells how to form the $n$ variables $u_i$ out of the $k$ physical dimensions $v_i$, i.e. the $(s,t)^{th}$ entry of $M$ is the power of the unit $v_s$ in the constant $u_t$.

i. What are the dimensions of $M$?

ii. Describe in words what the result of a matrix multiplication $M \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is.

iii. Apply a theorem to find a formula for the number of independent *dimensionless* parameters $\pi_i$ (combinations of the original $n$ physical variables) in terms of $n$ and some characteristic of the matrix $M$. What are the bounds on the maximum and minimum number of independent dimensionless parameters?

(c) A standard application of dimensional analysis is to determine a relation for the period of a pendulum. The obvious list of physical quantities here are
<table>
<thead>
<tr>
<th>Description</th>
<th>Variable</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>length of string</td>
<td>( l )</td>
<td>( L )</td>
</tr>
<tr>
<td>arc of displacement</td>
<td>( s )</td>
<td>( L )</td>
</tr>
<tr>
<td>gravitational constant</td>
<td>( g )</td>
<td>( L , T^{-2} )</td>
</tr>
<tr>
<td>mass at end of string</td>
<td>( m )</td>
<td>( M )</td>
</tr>
<tr>
<td>period of swing</td>
<td>( \tau )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

Where \( L \) is length, \( M \) is mass, and \( T \) is time. Use the methods developed in part (b) and in class to find an expression for the period \( \tau \) of the pendulum in terms of a \(*\)dimensionless constant\(*\) times some relation in the above physical variables.

13. Suppose \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) is a linear transformation such that for three linearly independent \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) we have

\[
A \vec{v}_1 = \lambda \vec{v}_1, \quad A \vec{v}_2 = \lambda \vec{v}_2, \quad A \vec{v}_3 = \lambda \vec{v}_3
\]

for some \( \lambda \in \mathbb{R} \).

(a) In the eigenbasis, what does \( A \) look like? Justify your answer.
(b) in the standard basis, what does \( A \) look like? Justify your answer.
(c) Give a condition on \( \lambda \) that determines when \( A \) is an isomorphism.

14. Let \( T \) be the linear transformation given by

\[
T : \mathbb{R}^4 \to \mathbb{R}^3, \quad T \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3x + y \\ 2z \\ y + w \end{bmatrix}
\]

Furthermore, let \( B_1 \) be an alternative basis for \( \mathbb{R}^4 \) given by

\[
B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 5 \end{bmatrix} \right\}
\]

And let \( B_2 \) be an alternative basis for \( \mathbb{R}^3 \) given by

\[
B_2 = \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \right\}
\]

(a) Find the matrix representation for \( T \) with respect to the standard basis. Is this unique? Find a basis for the image and kernel of \( T \).
(b) Determine the matrix representation for \( T \) with respect to \( B_1 \) in the domain and \( B_2 \) in the codomain in two ways:
   i. Determine how \( T \) acts on the basis vectors \( B_1 \) in terms of \( B_2 \)
ii. Write down an expression in terms of change of basis matrices and evaluate it.

(c) Does $T$ admit an eigenbasis?

15. Let $V$ be a finite dimensional vector space. Prove that any ordered spanning set of vectors can be reduced to a basis by removing vectors from the set, and any ordered linearly independent set of vectors can be expanded to a basis by adding vectors in $V$.

16. T/F: If $A$ is a $3 \times 3$ matrix with an eigenbasis but only 2 distinct eigenvalues, then for any $\vec{w} \in \mathbb{R}^3$, $\{\vec{w}, A\vec{w}, A^2\vec{w}\}$ are linearly dependent.

17. T/F: The operation $\times$ (the cross product) is a linear operator from $(\mathbb{R}^3, \mathbb{R}^3) \rightarrow \mathbb{R}^3$.

18. Approximations

(a) True or False: for every $n \times n$ real matrix $A$, if $A$ has an eigenvalue, then there is an explicit formula for this eigenvalue in terms of the coefficients of $A$ (i.e. something along the lines of the quadratic formula)

(b) As you may have discovered, it is something of a pain to determine the eigenvalues of an $n \times n$ matrix of dimension greater than 3 using the formulas we derived in class; as $n$ becomes even larger, you can imagine that this becomes unwieldy even for a computer. As such there are many eigenvalue-computing algorithms that approximate the eigenvalues of a matrix.

Assume that $A$ is diagonalizable (i.e. an eigenbasis exists) with $\lambda_1$ the unique eigenvalue of greatest magnitude. For a random (you can assume “nice”, but see part D) vector $b_0$, we define the recursive relation

$$b_{k+1} = \frac{Ab_k}{|Ab_k|}$$

i. First, find an expression for $b_{k+1}$ in terms of the original $b_0$.

ii. As $k \rightarrow \infty$, what can you hypothesis about $b_{k+1}$? Use what you know about $A$ to prove your claim. In the process, if you haven’t approximated $\lambda_1$ and an associated unit eigenvector, you might want to reconsider.

iii. Bonus: What is the (approximate) rate of convergence of the above algorithm? How does this compare to other algorithms we have seen in this class. (Note: your answer will depend on some characteristic of $A$).

iv. Where could the above algorithm go wrong (think about $b_0$).

19. The Trace
(a) Working from the definition of matrix multiplication, prove that $\text{tr}(AB) = \text{tr}(BA)$.

(b) Use this to prove that the trace of a matrix $A$ is preserved under change of basis. This is very good because it says that the trace is an algebraic invariant of the operator $A$ that is independent of coordinate system.

(c) Use the above analysis to conclude that if $A$ has an eigenbasis with eigenvalues $\{\lambda_i\}$, then

$$\text{tr}(A) = \sum_i \lambda_i$$

20. Symmetric Matrices

(a) For $A$ a symmetric matrix which admits an eigenbasis with unique eigenvalues. Prove that for $v_i$ and $v_j$ eigenvectors,

$$v_i \cdot v_j = 0 \text{ for } i \neq j$$

This says that the eigenvectors are pairwise orthogonal.