1. Introduction and the case of GL₁

The Local Langlands Conjecture posits the existence of a bijection between certain classes of irreducible representations of a reductive group $G$, such as $GL_n(K)$, and certain $n$-dimensional representations of an additive extension of the Weil group $W_K$ (closely related to the Galois group). Here we focus solely on the case $G = GL_n$ and $K$ is a $p$-adic field (such as $\mathbb{Q}_p$).

The Local Langlands conjecture is inspired by the result for $GL_1(K)$, namely local class field theory. Local class field theory gives an isomorphism of topological groups

$$\text{Art}_K : K^\times \xrightarrow{\sim} W_K^{ab}$$

where $W_K^{ab}$ is the abelianization of the Weil group $W_K$ (see section 2.1 below). The image of a uniformizer $\varpi \in K$ in $W_K$ is a geometric Frobenius element $\Phi_K$, inverse to arithmetic Frobenius $\sigma_K$ explained in section 2.1 below.

We may translate this into the language of the Local Langlands Correspondence for $GL_1$ as follows. Define $G_1(K)$ to be the set of continuous homomorphisms $\rho : W_K \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ has the usual classical topology. Similarly define $\mathcal{A}_1(K)$ to be the set of complex irreducible admissible representations $(\pi, V)$ of $GL_1(K) = K^\times$, i.e. such that $V_C$ is finite dimensional for every $C$ an open compact subgroup and $\text{Stab}_G(v) \subseteq G$ is open for all $v \in V$. This admissibility condition forces $V$ to be finite-dimensional, hence one-dimensional. Thus this class $\mathcal{A}_1(K)$ is equivalently described as the set of continuous homomorphisms $\pi : K^\times \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is given the discrete topology. The Local Langlands Conjecture for $GL_1$ asserts that these sets $\mathcal{A}_1(K)$ and $G_1(K)$ are in bijection. This is simply a restatement of local class field theory. The map $\rho : W_K \to \mathbb{C}^\times$ factors through $W_K \to W_K^{ab} \to \mathbb{C}^\times$ as $\mathbb{C}^\times$ is abelian. Thus, using the Artin map $\text{Art}_K$ defined above, all that remains to check this bijection is to verify that the map $\rho$ is continuous if and only if the map $\pi$ is continuous. But $\rho$ is continuous if and only if it is continuous on the inertia subgroup $I_K \subseteq W_K$, which is compact and totally disconnected. Thus its image in $\mathbb{C}^\times$ is finite. Hence $\rho : W_K \to \mathbb{C}^\times$ is continuous with respect to the usual topology on $\mathbb{C}^\times$ if and only if it is continuous with respect to the discrete topology.

In the remainder of the paper we will explore the more general sets $\mathcal{G}_n(K)$ of equivalence classes of Frobenius-semisimple $n$-dimensional complex Weil-Deligne representations of $W_K$, and $\mathcal{A}_n(K)$ the set of equivalence classes of irreducible admissible complex representations of
GL_n(K). These sets are in bijection via a function rec_n: A_n(K) → G_n(K) satisfying several compatibility conditions. In the GL_1 case, this is given by

\[ \text{rec}_1: A_1(K) → G_1(K), \]

such that rec_1(\pi) = \pi \circ \text{Art}^{-1}_K.

We will begin in Section 2 with a discussion of the “Galois side”, which involves a discussion of Weil-Deligne representations and the local L- and ϵ-factors associated to such representations. Then we will turn to the “GL_n side” in Section 3 and, after recalling briefly key results in the representation theory of reductive groups over a p-adic field, we will define the L- and ϵ-factors for these representations. Finally, we will put these together in Section 4 to precisely state the Local Langlands Conjecture for GL_n over a p-adic field. This paper follows [Tat79] and [Wed08] closely.

2. The Galois Side

2.1. The Weil Group. Let K be a p-adic field. Fix \overline{K} an algebraic closure of K and let \text{Gal}(\overline{K}/K) denote Gal(\overline{K}/K). Let \varpi be a uniformizer and let k denote the residue field k = \mathcal{O}_K/\varpi\mathcal{O}_K of order q, with algebraic closure \overline{k} = \mathcal{O}_{\overline{K}}/\varpi\mathcal{O}_{\overline{K}} (where \varpi is a prime of \overline{K} lying over \varpi), and \text{Gal}(\overline{k}/k) = \widehat{\mathbb{Z}}. It can be easily shown that the map \text{Gal}(\overline{K}/K) → \text{Gal}(\overline{k}/k) is surjective, and we have the short exact sequence

\[ 0 → I_K → G_K → \widehat{\mathbb{Z}} → 0 \]

where I_K is called the inertia subgroup. If we call K^{ur} the maximum unramified extension of K, then I_K = \text{Gal}(\overline{K}/K^{ur}) and Gal(\overline{k}/k) = \text{Gal}(K^{ur}/K). The group Gal(\overline{k}/k) = \widehat{\mathbb{Z}} is topologically generated by the arithmetic Frobenius \sigma_K : x → x^q. This has subgroup \mathbb{Z} ≅ \langle \sigma_K \rangle ⊆ \widehat{\mathbb{Z}} the free group generated by the arithmetic Frobenius. Taking the preimage of this subgroup inside G_K we get the Weil group W_K fitting into the short exact sequence:

\[ 0 → I_K → G_K → \widehat{\mathbb{Z}} → 0 \]

\[ 0 → I_K → W_K → \mathbb{Z} → 0 \]

For a more general definition of the Weil group see [Tat79, Section 1]. We endow W_K with the topology of a locally compact group such that the projection to \mathbb{Z} is continuous with respect to the discrete topology on \mathbb{Z} and such that the topology on I_K ⊆ W_K is the induced profinite topology from G_K. (Note that this topology on W_K is distinct from that induced by W_K ⊆ G_K however this inclusion map is still continuous.)

We call the map \nu: W_K → \mathbb{Z} the valuation homomorphism, which gives rise to a norm on W_K via

\[ \|w\| = q^{-\nu(w)}. \]

Under the Artin map W_K → \text{W}_K^{ab} ≃ K^\times, this agrees with the usual valuation on K^\times.

2.2. Local L-functions and ε-factors. In this section we will define the local L- and \epsilon-factors associated to a representation V of a Weil group, in the case K a non-Archimedean field. This can be done explicitly when V is 1-dimensional, i.e. the representation is given by a quasi-character \chi: K^\times → \mathbb{C}^\times. In the nonabelian case, there is merely an existence theorem for the local \epsilon-factors, but no explicit formula. For this reason, we begin with the abelian case.
2.2.1. Local abelian $L$-functions and $\epsilon$-factors. Choose \( \varpi \) a uniformizer of \( K \). Then for \( \chi : K^\times \rightarrow \mathbb{C}^\times \) a quasi-character, set \( L(\chi) \in \mathbb{C}^\times \) as

\[
L(\chi) = \begin{cases} 
(1 - \chi(\varpi))^{-1} & \text{\( \chi \) unramified}, \\
1 & \text{\( \chi \) ramified}.
\end{cases}
\]

In addition, setting \( \omega_s = \| \cdot \|^s \) for \( s \in \mathbb{C} \), we could consider \( L(\chi \omega_s) \), which is a meromorphic function of \( s \) without any zeros.

Now, we define the $\epsilon$-factor for \( \chi \) as the quantity satisfying a certain functional equation in terms of $L$-functions. First choose a Haar measure \( dx \) for \( K \) (i.e. an additively left-invariant measure on \( K \)) and let \( d^*x = dx \|x\|^{-1} \) denote the Haar measure on \( K^\times \). Further choose a nontrivial additive character \( \psi \) of \( K \). Then we may form the “Fourier transform” of a function \( f \) as

\[
\hat{f}(y) = \int f(x) \psi(xy) dx.
\]

There exists a unique factor \( \epsilon(\chi, \psi, dx) \in \mathbb{C}^\times \) satisfying

\[
\frac{\int \hat{f}(x) \omega_1 \chi^{-1}(x) d^*x}{L(\omega_1 \chi)} = \epsilon(\chi, \psi, dx) \int \frac{f(x) \chi(x) d^*x}{L(\chi)},
\]

where \( f \) is continuous and \( f(x), \hat{f}(x) \) are \( O(e^{-\|x\|}) \) so that both sides make sense. As is implicit in the notation, a main result of Tate’s thesis is that this determination of \( \epsilon(\chi, \psi, dx) \) does not depend upon the choice of test function \( f \) beyond the requirement that the above integrals make sense [Tat79, Section 3].

**Proposition 2.1.** Let \( \chi : K^\times \rightarrow \mathbb{C}^\times \) be a quasi-character. For \( n(\psi) \) the largest integer \( n \) such that \( \psi(\pi^{-n}\mathcal{O}_K) = 1 \), and \( a(\chi) \) the exponent of the conductor of \( \chi \), let \( c \in K^\times \) have valuation \( n(\psi) + a(\chi) \). Then

\[
\epsilon(\chi, \psi, dx) = \begin{cases} 
\chi^{\frac{c}{|c|}} \int_{\mathcal{O}_K} dx & \text{\( \chi \) unramified}, \\
\chi^{-1} \mathcal{O}_K^* \chi^{-1}(x) \psi(x) dx & \text{\( \chi \) ramified}.
\end{cases}
\]

**Remark.** Knowing that \( \epsilon(\chi, \psi, dx) \) is independent of the choice of \( f \) means that this computation can be carried out without too much trouble by choosing \( f \) to be the characteristic function of a compact neighborhood of the identity, i.e. \( 1 + \varpi^r \mathcal{O}_K \) for some \( r \). Then

\[
\hat{f}(y) = \int_{1 + \varpi^r \mathcal{O}_K} \psi(xy) dx \\
= \psi(y) \int_{\varpi^r \mathcal{O}_K} \psi(xy) dx \\
= \begin{cases} 
\psi(y) \cdot \text{vol}(\varpi^r \mathcal{O}_K) & \psi(xy) \text{ trivial on } \varpi^r \mathcal{O}_K \\
0 & \psi(xy) \text{ nontrivial},
\end{cases}
\]

which will be another characteristic function for \( y \) depending upon \( n(\psi) \).

Note that although \( \epsilon(\chi, \psi, dx) \) is dependent upon the choice of Haar measure \( dx \) and additive character \( \psi \), we can see that

\[
\begin{align*}
(1) & \quad \epsilon(\chi, \psi, r dx) = r \epsilon(\chi, \psi, dx) & r > 0, \\
(2) & \quad \epsilon(\chi, \psi(ax), r dx) = \chi(a)\|a\|^{-1} \epsilon(\chi, \psi(x), dx) & a \in K^\times.
\end{align*}
\]
2.2.2. Local nonabelian $L$-functions and $\epsilon$-factors. Now we consider the case of nonabelian $L$-functions. That is $L$-functions associated to the representation $W_K \to \text{GL}(V)$ for some $(n > 1)$-dimensional $V$. The definition of such a function is due to Artin; in fact, the $L$-function is an inductive function of representations.

**Definition 2.2.** Let $R(W_K)$ denote the Grothendieck group of virtual representations of $W_K$.

1. A function $\lambda = \lambda_K : R(W_K) \to X$ for $X$ an abelian group is additive if $\lambda(V') \cdot \lambda(V'') = \lambda(V)$ for
   $$0 \to V' \to V \to V'' \to 0$$
a short exact sequence of representations.

2. We say that a family of functions $\lambda_F$ for $F/K$ a finite separable extension, is inductive if the $\lambda_F$ are additive and commute with induction $\text{Ind}_{F'/F}$ for $F'/F/K$ finite separable extensions:

$$R(W_F) \xrightarrow{\text{Ind}_{F'/F}} R(W_{F'})$$

$$\lambda_F \downarrow \quad \downarrow \lambda_{F'}$$

$$X \quad X$$

Let $R^0(W_K) \subseteq R(W_K)$ be the subset of degree 0 virtual representations — that is of the form $[V] - [V']$ where $\text{dim}(V) = \text{dim}(V')$. Then we say that $\lambda$ is inductive in degree 0 if it is satisfies the above hypotheses with respect to $R^0(W_K)$.

Let $\Phi \in W_K$ be a choice of geometric Frobenius, i.e. $\Phi \in \text{Gal}(\bar{k}/k)$ is inverse to the arithmetic Frobenius element. Then, we can explicitly give the $L$-function of the representation $V$ as

$$L(V) = \det(1 - \Phi|V^{I_K}),$$

where $V^{I_K}$ denotes the inertial invariants of $V$.

We can easily verify this is consistent with our previous description of the $L$-function for an abelian representation. If $\chi : W_K^{ab} \to \mathbb{C}^\times$, $\chi \circ \text{Art}_K : K^\times \to \mathbb{C}^\times$, is an unramified character, then $I_K \subseteq W_K$ acts trivially and $V^{I_K} = V$. Thus

$$\det(1 - \chi(\Phi)) = \det(1 - \chi \circ \text{Art}_K(\text{Art}_K^{-1}(\Phi))) = \det(1 - (\chi \circ \text{Art}_K)\varpi).$$

If $\chi$ is ramified, then $V^{I_K} = 0$, and so $L(V) = 1$.

There is no explicit general formula for local $\epsilon$-factors; however there is the following existence theorem of Langlands and Deligne:

**Theorem 2.3.** There is a unique family of functions $\epsilon_F$, for $F/K$ a finite separable extension, on representations $(\rho, V)$ of the $W_F$, depending upon a choice of Haar measure $dx$ and additive character $\psi$, such that $\epsilon_F(V, \psi, dx) \in \mathbb{C}^\times$ and

1. If $V$ is 1-dimensional, then $\epsilon_F(V, \psi, dx)$ agrees with the definition for characters
2. $\epsilon_F(V, \psi, dx)$ is additive in exact sequences of representations of $W_F$
3. For $F/K$ a finite separable extension and $(\rho, V)$ a representation of $W_F$, the function $\epsilon_F(V, \psi \circ \text{Tr}_{F/K}, dx)$ is inductive in degree 0.
We write \( \epsilon = \epsilon_K \) for simplicity. The inductivity in degree 0 combined with (\( \square \)) gives the following properties of \( \epsilon(V, \psi, dx) \):
\[
\epsilon(V, \psi, r dx) = r^{\dim V} \epsilon(V, \psi, dx) \quad \text{for } r > 0
\]
\[
\epsilon(V, \psi(ax), r dx) = (\det V)(a)^{\dim V} \epsilon(V, \psi, dx) \quad \text{for } a \in K^\times
\]
\[
\epsilon(V \omega_s, \psi, dx) = q^{-s(\alpha(\psi) \cdot \dim(V) + \alpha(V))} \epsilon(V, \psi, dx),
\]
where \( a(V) \) is the Artin conductor of the representation \( V \).

2.3. Weil-Deligne Representations. In defining the bijections \( \text{rec}_n \) we must consider representations not simply of the Weil group \( W_K \) but of an additive extension of \( W_K \) we call the Weil-Deligne group \( W'_K \).

**Definition 2.4.** Let \( E \) be a field of characteristic 0. The \( E \)-points of the Weil-Deligne group \( W'_K(E) \) are \( E \times W_K \) where \( E \) has the additive group structure and \( W_K \) acts on \( E \) by
\[
w \cdot a \cdot w^{-1} = \|w\| \cdot a \quad \forall w \in W_K.
\]
This has the obvious group law \( (a_1, w_1)(a_2, w_2) = (a_1 + \|w_1\|a_2, w_1w_2) \) for all \( a_1, a_2 \in E \) and \( w_1, w_2 \in W_K \).

Any algebraic finite-dimensional representation \( (\rho, V) \) of the additive group of \( E \) must be of the form \( \rho : a \mapsto \exp(aN) \) for \( N \) a nilpotent element of \( GL(V) \). Thus any finite-dimensional algebraic representation of \( W'_K(E) \) must be given by \( (\rho, N) \) for \( \rho \) a representation of \( W_K \) and \( N \) a nilpotent endomorphism such that \( \rho(w)N\rho(w)^{-1} = \|w\|N \) for all \( w \in W_K \). This motivates the following definition:

**Definition 2.5.** A Weil-Deligne representation of the group \( W'_K \) over a field \( E \) of characteristic 0 is a pair \((\rho', N)\) given by

1. A finite dimensional vector space \( V \) over \( E \) and a homomorphism \( \rho : W_K \to GL(V) \) which is continuous, i.e., whose kernel contains an open subgroup of \( I_K \).
2. A nilpotent endomorphism \( N \in GL(V) \) such that \( \rho(w)N\rho(w)^{-1} = \|w\|N \) for all \( w \in W_K \).

Then for any \((a, w) \in E \times W_K\), we let \( \rho'(a, w) = \exp(aN) \cdot \rho(w) \).

As indicated in the introduction, we also want to restrict our attention to Frobenius-semisimple representations. That is, the image of the Frobenius element \( \Phi \) must be a semisimple automorphism of \( V \). We may arrange this as any element of \( GL(V) \) can be written as \( s \cdot u \) where \( s \) is semisimple and \( u \) is unipotent.

**Proposition 2.6.** Let \( \rho' = (\rho, N) \) be a Weil-Deligne representation. Then there exists a unique automorphism \( u \in GL(V) \) that commutes with \( N \) such that \( \exp(aN) \rho(w) u^{-v(w)} \) is semisimple for all \( w \in W_K - I_K \), \( a \in E \), and \( v \) the valuation map introduced in Section 2.1. We call this \( \rho'_{ss} = (\rho u^{-v}, N) \). Further we say that \( \rho' \) is Frobenius-semisimple (written \( \Phi \)-semisimple) if and only if \( \rho'_{ss} = \rho' \), i.e., the associated \( u = 1 \).

In fact, if \( \rho' \) is \( \Phi \)-semisimple this implies that \( \rho(W_K) \) (note: not all of \( \rho'(W'_K) \) is semisimple. Indeed, because \( \rho(I_K) \) is finite, \( \langle \rho(\Phi) \rangle \) is of finite index in \( \rho(W_K) \). And in general, whenever \( \rho \) is a representation of a group \( G \) in characteristic zero, which is semisimple on a finite-index subgroup \( H \subseteq G \), then \( \rho \) is semisimple on all of \( G \). (This is perhaps best-known in the special case when \( G \) is finite and \( H = 0 \): “Maschke’s Theorem”.) As an example consider the following \( \Phi \)-semisimple representation.
Example 2.7. For any $n \in \mathbb{Z}_{\geq 1}$, define the special representation $\text{Sp}(n)$ to be the representation $(\rho, N)$ of $W_K$ over $\mathbb{C}$ for

$$V = \mathbb{C}e_0 \oplus \cdots \mathbb{C}e_{n-1}$$

with the action given by

$$\rho(w)e_i = \|w\|^i e_i, \quad \text{and} \quad Ne_i = e_{i+1} \quad 0 \leq i < n-1 \quad Ne_{n-1} = 0.$$

Note that this representation is not semisimple as it is indecomposable but not irreducible.

We will return to this example in the next section and compute the $L$-function and $\epsilon$-factors associated to $\text{Sp}(n)$. This representation is especially important for the following reason: by the commutation relation between $\rho(W_K)$ and $N$, ker $N$ is a stable subspace under $\rho$ and hence $\rho'$ is irreducible if and only if $N = 0$ and $\rho$ is irreducible. The $\Phi$-semisimple indecomposable representations of $W_K'$ all resemble the situation in Example 2.7, i.e. they are of the form $\rho' \otimes \text{Sp}(n)$ for $\rho'$ an irreducible representation (i.e. $\rho' = (\rho, 0)$ for $\rho$ irreducible).

Alternatively, instead of looking at Weil-Deligne representations, we could equivalently look at homomorphisms

$$\varphi : W_K \times \text{SL}_2(\mathbb{C}) \to \text{GL}_n(\mathbb{C}).$$

The following Theorem [GR10, Section 2.1] guarantees this equivalence:

Theorem 2.8. There is a bijection between equivalence classes of Weil-Deligne representations $(\rho, N, V)$ of $W_K'$ and representations $\varphi : W_K \times \text{SL}_2(\mathbb{C}) \to \text{GL}_n(\mathbb{C})$ for which $\varphi$ is trivial on an open subgroup of $I_K$, $\varphi(\Phi)$ is semisimple, and $\varphi|_{\text{SL}_2(\mathbb{C})}$ is algebraic.

Proof. First we prove the following lemma which is a refinement of the Jacobson-Morozov theorem for $\mathfrak{gl}_n$:

Lemma 2.9. Let $N \in \text{GL}_n(\mathbb{C})$ be a nilpotent element coming from a Weil-Deligne representation; to this we can associate a unique $\mathfrak{sl}_2$-triple $(e, f, h)$ with

$$e = N \in \mathfrak{gl}_n^{\rho(I_K)}(q^{-1}), \quad h \in \mathfrak{gl}_n^{\rho(W_K)} = \mathfrak{gl}_n^{\rho(I_K)}(1), \quad f \in \mathfrak{gl}_n^{\rho(I_K)}(q)$$

(where $V(q)$ denotes the $q$-eigenspace for the action of $\rho(\Phi)$ on $V$).

Proof. For any $N$, it is always possible to find an $\mathfrak{sl}_2$-triple $(e, f_0, h_0)$ where $e = N$. (This is the ordinary Jacobson-Morozov theorem.) We may of course choose $f_0, h_0 \in \mathfrak{gl}_n^{\rho(I)}$ as $N \in \mathfrak{gl}_n^{\rho(I)}$. So for simplicity, call this $\mathfrak{gl}_n^{\rho(I)} = \mathfrak{g}$. We have a decomposition of $\mathfrak{g}$ into eigenspaces of the Frobenius action

$$\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}(\lambda),$$

which is actually a grading such that $[\mathfrak{g}(\lambda_1), \mathfrak{g}(\lambda_2)] \subseteq \mathfrak{g}(\lambda_1 \cdot \lambda_2)$. By the commutation relation for $N$ in the Weil-Deligne representation, we know that $e \in \mathfrak{g}(q^{-1})$. By the $\mathfrak{sl}_2$ commutation relations,

$$[e, h_0] = 2e,$$

thus decomposing $h_0 = \sum_{\lambda} h_0(\lambda)$, we have

$$[e, h_0(\lambda)] = \begin{cases} 2e & : \lambda = 1 \\ 0 & : \text{else} \end{cases}$$
So we let $h = h_0(1)$. Now similarly for $f_0$, we know

$$[f_0, e] = h_0, \quad \Rightarrow \quad [f_0(q), e] = h,$$

for $f_0(q)$ in the $q$-eigenspace for Frobenius. However, we do not know that $[h, f_0(q)] = -2f_0(q)$. So we choose a Cartan subalgebra containing $h$ and decompose $g$ into root spaces $g_\alpha$. Then an $\mathfrak{sl}_2$-triple is given by $(g_\alpha, g_{-\alpha}, [g_\alpha, g_{-\alpha}])$. We know that $e \in g_\alpha$ for some $\alpha$, but $f_0(q)$ may be a direct sum of elements of several root spaces. Let $f = f_0(q)(-\alpha)$ be the component of the $-\alpha$ root space. Then clearly $(e, f, h)$ form an $\mathfrak{sl}_2$-triple; the claim is that $f \in g(q)$. This again follows from the representation theory of $\mathfrak{sl}_2$. We know that $[h, f] = -2f$ — so it must be in a negative root space. On the negative root spaces, the raising operator $e$ is injective. Thus as $[f(q), e] = h = [f, e]$, it must be that $f = f(q)$.

Now we must show uniqueness up to conjugation. Suppose the exists another such $\mathfrak{sl}_2$-triple $(e, f', h')$. Then $h - h' \in \mathfrak{g}(1)$ and further

$$[e, h - h'] = 0, \quad h - h' = [e, f - f'].$$

Viewing $\mathfrak{g}(1)$ as a representation of $\mathfrak{sl}_2 = \langle e, f, h \rangle$, we may take the weight space decomposition

$$\mathfrak{g}(1) = \bigoplus_\lambda \mathfrak{g}(1)_\lambda,$$

and let $u$ be the subspace generated by highest weight vectors of positive weight. Then $[e, h - h'] = 0$ implies that $h - h'$ is the direct sum of highest weight vectors, and $h - h' = [e, f - f']$ implies that in fact $h - h' \in u$ as none of its highest-weight components can be of weight 0 or else it would not be in the image of the raising operator $e$. The space $u$ is nilpotent as the weight increases under successive application of the Lie bracket. And further $[h, u] = u$; thus by the Baker-Campbell-Hausdorff formula (which converges for this nilpotent space), this infinitesimal statement translates into $\text{Ad}(\exp u)(h) = h + u$. But $h' \in h + u$, so $h' = \text{Ad}(U_0)(h)$ for some $U_0 = \exp(u_0)$ and hence $h'$ and $h$ are conjugate. Further it is clear by the fact that $e$ kills highest weight vectors that $\text{Ad}(U_0)(e) = e$. Thus it remains only to show that $\text{Ad}(U_0)(f) = f'$, but this follows from the fact that $[e, f' - \text{Ad}(U_0)(f)] = 0$ but taking the bracket with $e$ is injective on negative root spaces.

Now we can define a representation

$$\varphi: W_K \times \text{SL}_2(\mathbb{C}) \to \text{GL}_n(\mathbb{C})$$

by lifting the representation of $\mathfrak{sl}_2$ given by our triple to a representation of $\text{SL}_2$, and for $w \in W_K$ letting:

$$\varphi(w) = \exp \left( \frac{-v(w)}{2} \log q \cdot h \right) \cdot \rho(w).$$

In the other direction, given such a representation $\varphi$, we define

$$\rho(w) = \varphi(w) \cdot \varphi \left( \begin{array}{cc} q^{v(w)/2} & 0 \\ 0 & q^{-v(w)/2} \end{array} \right), \quad N = d\varphi \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right).$$

2.4. Local $L$- and $\epsilon$-functions for Weil-Deligne Representations. For $(\rho, N, V)$ a representation of $W_K$, let $V_N$ denote $\ker(N)$. We may define a local $L$-function as

$$L(V, s) = \det(1 - q^{-s} \Phi | V_N^{I_K})^{-1}.$$
We define the Artin conductor of this Weil-Deligne representation to be $a(V) = a(\rho) + \dim V^\Gamma_K - \dim V^\Gamma_N$. Using this, we can define the $\epsilon$-function as

$$\epsilon(V) = \epsilon(\rho) \det(-\Phi|V^\Gamma_K/V^\Gamma_N), \quad \epsilon(V, s) = q^{-s-a(V)} \epsilon(V).$$

Note that for the sake of notational simplicity we have omitted the $\psi$ and $dx$ from the expression for $\epsilon$; however, the function still depends upon these choices which are implicit.

Now we return to the example of $\text{Sp}(n)$ and calculate the $L$- and $\epsilon$-functions associated to this Weil-Deligne representation.

**Example 2.10.** Clearly, by the description in Example 2.7 above, $V_N = \mathbb{C}e_n$ and inertia $I_K$ acts trivially. Thus $V^\Gamma_N = \mathbb{C}e_n$. And further the geometric Frobenius element acts as $\Phi e_i = q^{-i} e_i$. So by (3) the $L$-factor is given by

$$L(V, s) = \frac{1}{1 - q^{-s-m+1}}.$$

To compute the $\epsilon$-factor choose first an additive character $\psi$ such that $n(\psi) = 0$ and a Haar measure $dx$ on $K$. We choose the self-dual measure $dx$, such that $\text{vol}(\mathcal{O}_K) = q^{d/2}$ for $d$ the valuation of the absolute different ideal $D$ of $K$. Then Proposition 2.1 easily gives that for an unramified character $\chi$,$$
$$

$$\epsilon(\chi, \psi, dx) = q^{n(\psi)(1-s)} \text{vol}_dx(\mathcal{O}_K).$$

And as a representation of the Weil group,$$
\rho \simeq 1 \oplus \| \oplus \cdots \oplus \|^{n-1},$$
i.e. the direct sum of unramified characters. As we have chosen $\psi$ such that $n(\psi) = 0$, and by Theorem 2.3 $\epsilon$-factors are multiplicative in exact sequences, we have that

$$\epsilon(\rho, \psi, dx) = q^{-md/2}.$$

Now $V^\Gamma_K/V^\Gamma_N = V/\langle e_{m-1} \rangle$. And $\det(\Phi|V^\Gamma/V^\Gamma_N)$ is simply the product of the eigenvalues, e.g. $\prod_{i=1}^{m-2} q^{-i}$. Thus

$$\epsilon(\rho', \psi, dx) = q^{-md/2} \cdot q^{(m-2)(m-1)/2}.$$

### 3. The $\text{GL}_n$ Side

Now we describe the “$\text{GL}_n$-side” of the Local Langlands Correspondence. To describe this we will first very briefly review key facts from the representation theory of reductive groups over $p$-adic fields.

#### 3.1. A Brief Review of the Representation Theory of $p$-adic Groups

Let $G$ be a connected reductive group over a $p$-adic field $K$. We fix a maximal open compact subgroup $C_0 \subseteq G$.

**Example 3.1.** In the case of $G = \text{GL}_n(K)$, the open compact subgroups $C_i = 1 + \omega^i M_n(\mathcal{O}_K)$ form a fundamental system of open neighborhoods of the identity and we may take $C_0 = \text{GL}_n(\mathcal{O}_K)$.

Then we have the following key definition:

**Definition 3.2.** Let $\pi : G \to \text{GL}(V)$ be a complex representation.

1. We say that $(\pi, V)$ is smooth if $\text{Stab}_G(v) \subseteq G$ is open for all $v \in V$. 

(2) We say that \((\pi, V)\) is admissible if it is smooth and \(V^C\) is finite dimensional for all \(C\) an open compact subgroup of \(G\).

Recall that we defined \(A_n(K)\) to be the set of equivalence classes of irreducible admissible representations of \(GL_n(K)\).

If we want to talk about the “dual” of a smooth representation \(\pi : G \to GL(V)\), the notion of a \(C\)-linear dual \(V^*\) is insufficient if \(\dim V = \infty\) as \(V^*\) will not be smooth. So we make the following definition:

**Definition 3.3.** For \(\pi : G \to GL(V)\) a smooth representation, define the contragredient \(V^\vee\) of \(V\) by

\[ V^\vee = \{ \lambda \in V^* : \text{Stab}_G(\lambda) \subseteq G \text{ is open} \}. \]

Then as \((V^\vee)^C = (V^C)^*\) for every \(C \subseteq G\) compact and open, we have the following facts:

1. \(\pi\) is admissible if and only if \(\pi^\vee\) is admissible and in this case \(V \to (V^\vee)^\vee\) is an isomorphism of \(G\)-modules.
2. \(\pi\) is irreducible if and only if \(\pi^\vee\) is irreducible.

**Remark.** For \(G = GL_n\) and \(\pi\) irreducible admissible, we can explicitly realize \(\pi^\vee\) as the representation \(g \mapsto \pi((g^{-1})^t)\).

In our definition of \(L\)- and \(\epsilon\)-factors associated to irreducible admissible representations \(\pi\) of \(GL_n\) over a \(p\)-adic field, we will rely upon the classification of such representations in terms of parabolic induction from Levi subgroups. This is the so-called Bernstein-Zelevinsky classification.

For any ordered partition \(\underline{n} = (n_1, \cdots, n_r)\) such that \(\sum_{i=1}^r n_i = n\), let \(L_{\underline{n}} = GL_{n_1} \times \cdots \times GL_{n_r}\) denote the Levi subgroup of block diagonal matrices in \(GL_n\) associated to this partition, and \(P_{\underline{n}} \subseteq GL_n\) the subgroup of block upper-triangular matrices with respect to this partition. The Levi decomposition of a parabolic subgroup gives \(P_{\underline{n}} = L_{\underline{n}} U_{\underline{n}}\) for \(U_{\underline{n}}\) the unipotent radical of \(P_{\underline{n}}\).

Given admissible representations \((\pi_i, V_i)\) of \(GL_{n_i}(K)\), the representation \((\pi_1 \otimes \cdots \otimes \pi_r, V_1 \otimes \cdots \otimes V_r)\) is an admissible representation of \(L_{\underline{n}}\). Recall that after first extending this to a representation of \(P_{\underline{n}}\) via restriction to \(L_{\underline{n}}\), the normalized induction gives a representation

\[ \iota^G_P(\pi_1 \otimes \cdots \otimes \pi_r) = \text{Ind}^G_P \left( \left( (\pi_1 \otimes \cdots \otimes \pi_r) \otimes \delta_{\underline{n}}^{1/2} \right) \right), \]

where \(\delta_{\underline{n}}\) is the modulus character given by

\[ \delta_{\underline{n}}(m) = \| \text{det } (\text{Ad}_{U_{\underline{n}}}(m)) \|, \]

for \(\text{Ad}_{U_{\underline{n}}}(m)\) the adjoint action of \(m \in P_{\underline{n}}\) on the unipotent radical \(U_{\underline{n}}\). In the literature, this is represented by the somewhat misleading notation

\[ \pi_1 \times \cdots \times \pi_r := \iota^G_P(\pi_1 \otimes \cdots \otimes \pi_r), \]

so for the sake of consistency we use it here.

**Definition 3.4.** We say that \(\pi\), an irreducible admissible representation of \(GL_n(K)\), is **supercuspidal** if there does not exist a proper partition \(\underline{n}\) such that \(\pi\) is contained in a subquotient of \(\pi_1 \times \cdots \times \pi_r\). Let \(A^\circ_n(K)\) denote the subset of equivalence classes of supercuspidal representations of \(GL_n(K)\).
Definition 3.5. Let \( \pi : \text{GL}_n(K) \to \text{GL}(V) \) be a smooth representation. For each \( v \in V, \lambda \in V^\vee \), define the matrix coefficient
\[
c_{\pi,v,\lambda} = c_{e,\lambda} : G \to \mathbb{C} \quad \text{by} \quad g \mapsto \lambda(\pi(g)v).
\]

Remark. It is a theorem of Harish-Chandra that an irreducible admissible representation \( \pi \) is supercuspidal if and only if all matrix coefficients of \( \pi \) have compact support modulo the center of \( \text{GL}_n(K) \) [BZ76, Theorem 3.21].

For any admissible representation \( \pi \), we can twist \( \pi \) by the character \( \omega_s = \| \det(\cdot) \|^s \) for any \( s \in \mathbb{C} \). We denote this new representation \( \pi(s) \). This preserves supercuspidality, so we can define a partial ordering on the set of supercuspidal representations by \( \pi \preceq \pi' \) if and only if there exists an integer \( n \geq 0 \) such that \( \pi' = \pi(n) \).

Definition 3.6. A finite chain with respect to this partial ordering on \( \mathcal{A}_n^0(K) \) is a list of representations
\[
\Delta(\pi, m) = [\pi, \pi(1), \ldots, \pi(m - 1)],
\]
for some \( \pi \) supercuspidal. The integer \( m \) is called the length of \( \Delta \) and \( mn \) is called its degree.

Two chains \( \Delta_1 \) and \( \Delta_2 \) are said to be linked if one is not a subset of the other and their union is a chain. We say that \( \Delta_1 \) precedes \( \Delta_2 \) if they are linked and the smallest element of \( \Delta_1 \) is smaller.

Denote by \( \pi(\Delta) = \pi \times \pi(1) \times \cdots \times \pi(m - 1) \) the normalized induction of \( \pi \otimes \pi(1) \otimes \cdots \otimes \pi(m - 1) \) associated to the chain \( \Delta \). We use this language to state the Bernstein-Zelevinsky classification as in [Wed08]:

Theorem 3.7. (Bernstein-Zelevinsky classification of representations of \( \text{GL}_n \) over a \( p \)-adic field, [BZ77], [Zel80]).

1. For any finite chain \( \Delta \subset \mathcal{A}_n^0(K) \) of length \( m \), \( \pi(\Delta) \) has finite length (in particular length \( 2^{m-1} \)) and has a unique irreducible quotient denoted \( Q(\Delta) \) and a unique irreducible subrepresentation \( Z(\Delta) \).
2. Let \( \Delta_i \subset \mathcal{A}_n^0(K) \) be finite chains such that for \( i < j \), \( \Delta_i \) does not precede \( \Delta_j \). Then \( Q(\Delta_1) \times \cdots \times Q(\Delta_r) \) has a unique irreducible quotient \( Q(\Delta_1, \ldots, \Delta_r) \), and \( Z(\Delta_1) \times \cdots \times Z(\Delta_r) \) has a unique irreducible representation \( Z(\Delta_1, \ldots, \Delta_r) \).
3. Every smooth irreducible representation \( \pi \) of \( \text{GL}_n(K) \) is isomorphic to a representation of the form \( Q(\Delta_1, \ldots, \Delta_r) \) for a unique collection of chains \( \Delta_1, \ldots, \Delta_r \) up to permutation. It is also isomorphic to a \( Z(\Delta_1', \ldots, \Delta_r') \) for a unique set up to permutation.
4. \( Q(\Delta_1) \times \cdots \times Q(\Delta_r) \) is irreducible if and only if no two of the chains are linked.

Example 3.8. Let \( \Delta \) be the finite chain
\[
\Delta = (\| \cdot \|^{(1-n)/2}, \| \cdot \|^{(3-n)/2}, \ldots, \| \cdot \|^{(n-1)/2}).
\]
The associated representation of the diagonal torus \( T \subset \text{GL}_n(K) \) is \( \delta_B^{1/2} \), where \( B \) is the Borel subgroup of upper triangular matrices. Thus
\[
\pi(\Delta) = \iota_B^{\text{GL}_n}(\| \cdot \|^{(1-n)/2} \otimes \| \cdot \|^{(3-n)/2} \otimes \cdots \otimes \| \cdot \|^{(n-1)/2}) = \iota_B^{\text{GL}_n}(1) = C_c^\infty(\text{GL}_n(K)/B(K)).
\]

That is the space of locally constant compactly supported functions on \( B(K) \backslash \text{GL}_n(K) \) with a \( \text{GL}_n(K) \)-action corresponding to the natural action on this flag variety. In this case, \( Z(\Delta) = 1 \) is the trivial representation of constant functions, and the irreducible quotient \( Q(\Delta) \) is the Steinberg representation denoted by \( \text{St}(n) \).
Denote by \( U_n(K) \subset \text{GL}_n(K) \) the subgroup of unipotent upper-triangular matrices of \( \text{GL}_n \) over \( K \) (not to be confused with the unitary group!). Let \( \psi \) be an additive character on \( K \). Then \( U_n(K) \) has a 1-dimensional representation
\[
\theta_\psi(u_{ij}) = \psi(u_{i2} + \cdots + u_{n-1,n}),
\]
which evaluates \( \psi \) on the sum of the above-diagonal entries.

**Definition 3.9.** For \( \pi \) a smooth irreducible representation, we say that \( \pi \) is generic if \( \pi|_{U_n(K)} \) contains \( \theta_\psi \) as a quotient, i.e.
\[
\text{Hom}_{U_n(K)}(\pi|_{U_n(K)}, \theta_\psi) \neq 0.
\]
One can show that the property of being generic does not depend upon the choice of additive character \( \psi \).

If \( (\pi, V) \) is generic, then we can associate to it a Whittaker model:

Let \( \lambda \in \text{Hom}_{U_n(K)}(\pi|_{U_n(K)}, \theta_\psi) \), \( \lambda \neq 0 \), and define
\[
V \rightarrow \{ f : \text{GL}_n(K) \rightarrow \mathbb{C} : f(ug) = \theta_\psi(u)f(g), \text{ for } g \in \text{GL}_n(K), u \in U_n(K) \},
\]
\[
v \mapsto (g \mapsto \lambda(\pi(g)(v))).
\]
We call the image \( \mathcal{W}(\pi, \psi) \) of this injective map of \( \text{GL}_n(K) \)-modules the Whittaker model of \( \pi \) with respect to \( \psi \).

### 3.2. \( L \) - and \( \epsilon \)-factors for complex representations of \( \text{GL}_n(K) \).

We will define in this section the \( L \)- and \( \epsilon \)-factors for a pair of representations \((\pi, \pi')\). This added structure is necessary as the Local Langlands Correspondence defines some compatibility conditions between pairs of representations on the \( \text{GL}_n \) side and tensor products of Weil-Deligne representations on the Galois side.

#### 3.2.1. The generic case.
In defining the \( L \)- and \( \epsilon \)-factors we will follow the terminology of [JPSS81]. Fix an additive character \( \psi \). Let \((\pi, \pi')\) be generic representations of \( \text{GL}_n(K) \) and \( \text{GL}_{n'}(K) \) respectively.

#### 3.2.2. The case \( n = n' \).
Let \( C_c^\infty(K^n) \) denote the set of locally constant compactly supported functions \( \phi : K^n \rightarrow \mathbb{C} \). For any choice of \( \phi \in C_c^\infty(K^n) \), and \( W \in \mathcal{W}(\pi, \psi) \) and \( W' \in \mathcal{W}(\pi', \hat{\psi}) \) elements of the Whittaker models of \( \pi \) and \( \pi' \) respectively, define
\[
Z(W, W', \phi, s) = \int_{U_n(K) \backslash \text{GL}_n(K)} W(g)W'(g)\phi(g_n)|\det(g)|^sdg,
\]
where \( g_n \) denotes the last row of \( g \) and \( dg \) is a \( \text{GL}_n(K) \) invariant measure. For \( \text{Re}(s) \) sufficiently large this is absolutely convergent and a rational function of \( q^{-s} \).

Taking the set of all such \( Z(W, W', \phi, s) \) as \( W, W', \phi \) are varied generates a fractional ideal in \( \mathbb{C}[q^s, q^{-s}] \). One can show that there exists a unique generator of the form \( P(q^{-s})^{-1} \) with \( P(0) = 1 \), where \( P \in \mathbb{C}[x] \) is a polynomial, is the \( L \)-function \( L(\pi \times \pi', s) \) of the pair \((\pi, \pi')\).

Then \( \epsilon(\pi \times \pi', s, \psi) \) can be defined as satisfying:
\[
\frac{Z(\hat{W}, \hat{W}', 1-s, \hat{\phi})}{L(\pi^\vee \times \pi'^\vee, 1-s)} = \omega_{\pi'}(-1)^n\epsilon(\pi \times \pi', s, \psi)\frac{Z(W, W', s, \phi)}{L(\pi \times \pi', s)},
\]
where \( \hat{\phi} \) is the Fourier transform of \( \phi \) with respect to \( \psi \), \( \omega_{\pi'} \) is the central of \( \pi' \), \( \hat{W}(g) = W(w_nq^{-1}) \in \mathcal{W}(\pi^\vee, \psi) \), and \( w_n \in \text{GL}_n(K) \) corresponds to the Weyl permutation sending \( i \) to \( n + 1 - i \).
3.2.3. The case $n \neq n'$. By requiring that $L(\pi \times \pi', s) = L(\pi' \times \pi, s)$ and $\epsilon(\pi \times \pi', s, \psi) = \epsilon(\pi' \times \pi, s, \psi)$, it suffices to only consider the case $n' < n$. In this case, for $j = 0, \ldots, n-n'-1$, we associate the matrix $J_j(g, x) \in \GL_n(K)$,

$$J(g, x) = \begin{pmatrix}
g & 0 & 0 \\
x & I_j & 0 \\
0 & 0 & I_{n-n'-j}
\end{pmatrix},$$

which is a function of $g \in \GL_{n'}(K)$ and $x \in M_{j,n'}(K)$.

Now we define

$$Z(W, W', j, s) = \int_{U_{n'}(K) \setminus \GL_{n'}(K)} \int_{M_{j,n'}(K)} W(J(g, x)) W'(g) |\det(g)|^{s-(n-n')/2} \, dx \, dg,$$

and for $\text{Re}(s)$ sufficiently large $L(\pi \times \pi', s)$ is the unique element of the ideal generated by all such $Z(W, W', j, s)$ of the form $P(q^{-s})^{-1}$ with $P(0) = 1$. Similarly to above we can define

$$Z(w_{n,n'}, W, W', n-n'-1-j, 1-s) = \omega_{\pi'}(-1)^{n-1} \epsilon(\pi \times \pi', s, \psi) \frac{Z(W, W', j, s)}{L(\pi \times \pi', s)},$$

where $w_{n,n'}$ is the matrix

$$w_{n,n'} = \begin{pmatrix} I_{n'} & 0 \\ 0 & w_{n-n'} \end{pmatrix} \in \GL_n(K).$$

3.2.4. In general. Now we use the Bernstein-Zelevinsky classification for $\GL_n$ to determine $L$- and $\epsilon$-factors for pairs of arbitrary irreducible admissible representations $(\pi, \pi')$.

We have the following inductive rules:

1. $L(\pi \times \pi', s) = L(\pi' \times \pi, s)$ and $\epsilon(\pi \times \pi', s, \psi) = \epsilon(\pi' \times \pi, s, \psi)$

2. If $\pi = Q(\Delta_1, \ldots, \Delta_r)$, then

$$L(\pi \times \pi') = \prod_{i=1}^r L(Q(\Delta_i) \times \pi', s)$$

$$\epsilon(\pi \times \pi', s, \psi) = \prod_{i=1}^r \epsilon(Q(\Delta_i) \times \pi', s, \psi).$$

3. If $\pi = Q(\Delta)$, where $\Delta = [\sigma, \ldots, \sigma(r-1)]$ and $\pi' = Q(\Delta')$ for $\Delta' = [\sigma', \ldots, \sigma'(r'-1)]$ with $r' \geq r$, then

$$L(\pi \times \pi', s) = \prod_{i=1}^r L(\sigma \times \sigma', s + r + r' - 1)$$

$$\epsilon(\pi \times \pi', s, \psi) = \prod_{i=1}^r \left( \prod_{j=0}^{r+r'-2i} \epsilon(\sigma \times \sigma', s + i + j - 1, \psi) \right) \times \left( \prod_{j=0}^{r+r'-2i-1} \frac{L(\sigma' \times \sigma', 1-s-i-j)}{L(\sigma \times \sigma', s + i + j + 1)} \right).$$

Remark. Given the definition of $\epsilon$- and $L$-factors for a pair, we easily get the definition for a single representation: taking 1 to be the trivial character, we can define

$$L(\pi, s) = L(\pi \times 1, s), \quad \epsilon(\pi, s, \psi) = \epsilon(\pi \times 1, s, \psi).$$
Remark. For \( \pi \) any irreducible admissible representation of \( \text{GL}_n(K) \),
\[
\epsilon(\pi \times \pi^\vee, s = 1/2, \psi) = \omega_\pi(-1)^{n-1},
\]
for \( \omega_\pi \) the central character of \( \pi \). This is a Theorem of Bushnell, Henniart, and Kutzko [BHK98].

Remark. The fact that the \( L \)- and \( \epsilon \)-factors are defined to be inductive allows one to reduce to assuming \( \pi \) supercuspidal, as explained in Section 4.2.2.

4. The Local Langlands Correspondence for \( \text{GL}_n \)

4.1. The Correspondence. Now, given the above work we can state the Local Langlands Correspondence for \( \text{GL}_n \) over a \( p \)-adic field \( K \).

**Theorem 4.1** (Local Langlands Correspondence for \( \text{GL}_n \) over \( p \)-adic fields). There exists a unique collection of bijections
\[
\text{rec}_{n,K} = \text{rec}_n : \mathcal{A}_n(K) \to \mathcal{G}_n(K)
\]
between the set of equivalence classes of irreducible admissible complex representations of \( \text{GL}_n(K) \) and the set of equivalence classes of Frobenius-semisimple \( n \)-dimensional Weil-Deligne representations of the Weil group \( W_K \) such that:

1. The bijection \( \text{rec}_n \) is compatible with local class field theory in the sense that:
   (a) For \( \pi \in \mathcal{A}_1(K) \),
   \[
   \text{rec}_1(\pi) = \pi \circ \text{Art}_{K}^{-1}.
   \]
   (b) For \( \pi \in \mathcal{A}_n(K) \) and \( \chi \in \mathcal{A}_1(K) \),
   \[
   \text{rec}_n(\pi \chi) = \text{rec}_n(\pi) \otimes \text{rec}_1(\chi).
   \]
   (c) For \( \pi \in \mathcal{A}_n(K) \) with central character \( \omega_\pi \),
   \[
   \det \circ \text{rec}_n(\pi) = \text{rec}_1(\omega_\pi).
   \]

2. For pairs \((\pi, \pi')\) with \( \pi \in \mathcal{A}_n(K) \) and \( \pi' \in \mathcal{A}_n'(K) \), the \( L \)-factors and \( \epsilon \)-factors are consistent in the following sense:
   \[
   L(\pi \times \pi', s) = L(\text{rec}_n(\pi) \otimes \text{rec}_{n'}(\pi'), s)
   \]
   \[
   \epsilon(\pi \times \pi', s) = \epsilon(\text{rec}_n(\pi) \otimes \text{rec}_{n'}(\pi'), s).
   \]

3. For \( \pi \in \mathcal{A}_n(K) \),
   \[
   \text{rec}_n(\pi^\vee) = \text{rec}_n(\pi)^\vee.
   \]

Remark. As opposed to the \( \epsilon \)-factors themselves, the bijections \( \text{rec}_n \) do not depend upon the choice of additive character \( \psi \) on \( K \), as implied by the notation for the \( \epsilon \)-factor.
4.2. The Correspondence in Some Cases.

4.2.1. The Unramified Case. In this simple case, we can say explicitly how the Local Langlands Correspondence should be constructed.

**Definition 4.2.** An irreducible admissible representation \((\pi, V)\) of \(\text{GL}_n(K)\) is called unramified if it has conductor 0, i.e. if it has a fixed vector under the compact subgroup \(C_0 = \text{GL}_n(O_K)\).

In the case of a character \(\chi: K^\times \to \mathbb{C}^\times\), \(\chi\) is unramified if and only if \(\chi(O_K^\times) = \{1\}\). Every unramified character is of the form \(\omega_s = \|\cdot\|^s\) for a unique \(s \in \mathbb{C}/(2\pi i (\log q)^{-1})\mathbb{Z}\). Considering supercuspidal representations, the only unramified ones are unramified characters. In general, in fact, knowing this simple case of unramified characters will allow us to build up all unramified representations of \(\text{GL}_n(K)\), from the following Theorem:

**Theorem 4.3.** Using the notation of Theorem 3.7

1. Let \((\chi_1, \cdots, \chi_n)\) be a family of characters such that for \(i < j\), \(\chi_i^{-1} \chi_j \neq \|\cdot\|\). We may thus view this family as a collection of 1-chains \(\Delta_i = \chi_i\) such that \(\Delta_i\) does not precede \(\Delta_j\) for \(i < j\). Then the irreducible quotient \(Q(\chi_1, \cdots, \chi_n)\) of \(\chi_1 \times \cdots \times \chi_n\) is an unramified representation of \(\text{GL}_n(K)\).

2. Conversely, for every unramified representation \((\pi, V)\) of \(\text{GL}_n(K)\), there exists a family of unramified characters \((\chi_1, \cdots, \chi_n)\) of \(K^\times\) such that

\[
\pi \simeq Q(\chi_1, \cdots, \chi_n).
\]

**Remark.** The second direction is a Theorem of Casselman [Cas80].

Similarly, in the context of Weil-Deligne representations:

**Definition 4.4.** A Weil-Deligne representation \(\rho' = (\rho, N)\) is unramified if the image of inertia is trivial, i.e. \(\rho(I_K) = 1\), and \(N = 0\).

Both the class of \(n\)-dimensional Frobenius semisimple unramified Weil-Deligne representations and the class of unramified irreducible admissible representations of \(\text{GL}_n(K)\) are identified with the set of \(S_n\)-orbits of the diagonal torus \((\mathbb{C}^\times)^n \subseteq \text{GL}_n(\mathbb{C})\).

On the Galois side, the representation is determined by the conjugacy class of the image of the geometric Frobenius \(\rho(\Phi) \in \text{GL}_n(\mathbb{C})\). By assumption this is a semi-simple element, thus diagonalizable. Hence the representation \(\rho'\) is determined by the set of eigenvalues, i.e. a \(S_n\)-orbit of the diagonal torus \((\mathbb{C}^\times)^n\).

On the \(\text{GL}_n\)-side, any admissible unramified \(\pi\) is associated to a family of characters \((\chi_1, \cdots, \chi_n)\) which give a homomorphism

\[
T/T_C \to \mathbb{C}^\times, \quad (a_1, \cdots, a_n) \mapsto \chi_1(a_1) \cdots \chi_n(a_n) \in \mathbb{C}^\times,
\]

for \(T = (K^\times)^n\) and \(T_C = (O_K^\times)^n\). The set of such homomorphisms \(\text{Hom}(T/T_C, \mathbb{C}^\times)\) up to \(S_n\)-action is just an \(S^n\)-orbit of the diagonal torus \((\mathbb{C}^\times)^n\), as \(T/T_c = \mathbb{Z}^n\) by the valuation map.

Thus these sets are in bijection. The constraints of Theorem 4.1 are satisfied by the following construction:

1. An unramified character \(\chi\) of \(\text{GL}_1(K) = K^\times\) corresponds to the unramified character \(\text{rec}_1(\chi)\) of \(W_K^{ab}\) under the Artin map.
2. The unramified representation \( \pi = Q(\chi_1, \cdots, \chi_n) \) corresponds to the unramified Weil-Deligne representation

\[
\text{rec}_1(\chi_1) \oplus \cdots \oplus \text{rec}_1(\chi_n).
\]

Remark. From an automorphic perspective, the unramified situation is in fact the “generic case”. If \( \pi \) is an automorphic irreducible admissible representation of \( \text{GL}_n(\mathbb{A}_F) \), then we have the restricted tensor product decomposition

\[
\pi = \bigotimes_v \pi_v,
\]

where \( \pi_v \) is the local representation at the place \( v \) of a number field \( F \). All but finitely many of these local \( \pi_v \) are unramified.

4.2.2. Reduction to the Supercuspidal Case. In fact, it suffices to show that there exists a unique family of bijections \( \text{rec}_n \) satisfying the compatibility requirements of Theorem 4.1 between isomorphism classes of supercuspidal representations of \( \text{GL}_n(K) \) and isomorphism classes of irreducible Weil-Deligne representations (which we recall are the same as irreducible representations of the Weil group).

Remark. Uniqueness actually follows from compatibility of \( \epsilon \)-factors in pairs and inductivity. This is a theorem of Henniart [Hen93].

We have already defined \( \mathcal{A}_n^0(K) \) to be the set of supercuspidal representations. Define \( \mathcal{G}_n^0(K) \) to be the set of irreducible Weil-Deligne representations. Given bijections

\[
\text{rec}_n : \mathcal{A}_n^0(K) \to \mathcal{G}_n^0(K),
\]

it is possible to define a Weil-Deligne representation for an arbitrary admissible irreducible representation \( \pi \in \mathcal{A}_n(K) \). By the Bernstein-Zelevinsky classification in Theorem 3.7 we know \( \pi \simeq Q(\Delta_1, \cdots, \Delta_r) \) with \( \Delta_i = [\pi_{i_1}, \cdots, \pi_{i_k}(m_i - 1)] \) for the \( \pi_i \) supercuspidal. One can show that the corresponding Galois representation \( \text{rec}_{n_1m_1+\cdots+n_rm_r}(\pi) \) is given by

\[
\rho' = \text{rec}_{n_1m_1+\cdots+n_rm_r}(\pi) = \bigoplus_{i=1}^r \text{rec}_{n_i}(\pi_i) \otimes \text{Sp}(m_i).
\]

This follows from the fact that any indecomposable Weil-Deligne representation can be written as \( \rho' \otimes \text{Sp}(m) \) for some irreducible \( \rho' \) and some \( m_i \) and inductivity of \( L- \) and \( \epsilon \)-factors.

From the Weil-Deligne representation \( \rho' \) we may recover the underlying Weil group representation as follows. For simplicity, let \( \rho'_i = \text{rec}_{n_i}(\pi_i) = (\rho_i, 0) \). Then the underlying Weil group representation of \( \rho'_i \otimes \text{Sp}(m_i) \) is

\[
\rho_i \oplus \rho_i(1) \oplus \cdots \oplus \rho_i(m_i - 1),
\]

where \( \rho(\tau) \) denotes the twist of \( \rho \) by the unramified character \( \omega_\tau = \| \cdot \|^\tau \).

We may also recover the conjugacy class of the nilpotent endomorphism \( N_i \) of \( \rho'_i \otimes \text{Sp}(m_i) \). Note that \( N \) acts trivially on \( \rho'_i \) and the action on \( \text{Sp}(m_i) \) is a single Jordan block. Thus in Jordan normal form, \( N_i \) corresponds to \( n_i \) Jordan blocks of size \( m_i \).
4.2.3. Local Langlands for GL$_2$. It is possible to explicitly construct the Local Langlands Correspondence for GL$_2$ over a p-adic field. Here we will give only a cursory treatment to give the reader intuition for the correspondence. This is inspired by the informal notes of Buzzard [Buz07] and Snowden [Sno10]. For simplicity, we will assume that $p \neq 2$. First, we can classify irreducible admissible representations of GL$_2(K)$. Let us define the following representations.

Definition 4.5. For any two characters $\chi_1, \chi_2 : K^\times \to \mathbb{C}^\times$ such that neither precedes the other, we can define the principal series representation $PS(\chi_1, \chi_2) = t_{P_{1,1}}^{GL_2}(\chi_1 \otimes \chi_2)$. That is $f : GL_2(K) \to \mathbb{C}^\times$ locally constant such that

$$f\left(\begin{pmatrix} a & \ast \\ 0 & b \end{pmatrix} g\right) = \chi_1(a)\|a\|^{1/2} \cdot \chi_2(b)\|b\|^{-1/2} \cdot f(g), \quad a, b \in K^\times, \ast \in K.$$ 

Let $F/K$ be a quadratic field and $\chi$ a nontrivial character of $F^\times$ such that $\sigma \chi \neq \chi$ for $\sigma \in \text{Gal}(F,K)$, the nontrivial element. It is then possible to define the automorphic induction $AI(\chi, F/K)$ representation of GL$_2(K)$, but too difficult to go into here. For details see [Lan80].

Proposition 4.6. Let $\pi$ be an irreducible, admissible representation of GL$_2(K)$. Then $\pi$ falls into one of the following classes:

1. One-dimensional representation corresponding to a character $g \mapsto \chi(\text{det}(g))$,
2. Principal series representation $PS(\chi_1, \chi_2)$ corresponding to characters $\chi_1, \chi_2 : K^\times \to \mathbb{C}^\times$ with $\chi_1^{-1} \chi_2 \neq \| \cdot \|$,
3. Twists of the Steinberg representation $St \otimes (\chi \circ \text{det})$,
4. Supercuspidal representations associated to automorphic induction $AI(\chi, F/K)$ of a character $\chi$ for some extension $F/K$.

Similarly we may classify all 2-dimensional Weil-Deligne representations.

Proposition 4.7. Let $\rho', V = (\rho, N), V$ be a 2-dimensional complex Weil-Deligne representation. Then it falls into one of the following cases:

1. The Weil group representation $\rho \simeq \chi_1 \oplus \chi_2$ and $N = 0$.
2. Irreducible Weil group representations with $N = 0$
3. $\rho' \simeq \text{Sp} \otimes (\chi \circ \text{det})$ for some character $\chi$.

Using this we can give a rough dictionary for the Local Langlands Correspondence in the case of GL$_2(K)$:

1. For $\pi = \chi \circ \text{det}$ a 1-dimensional representation, the associated Weil-Deligne representation $\rho' = (\rho, N)$ is given by

$$\rho = \left(\chi \| \cdot \|^{1/2} \circ \text{Art}_K^{-1}\right) \oplus \left(\chi \| \cdot \|^{-1/2} \circ \text{Art}_K^{-1}\right), \quad N = 0.$$ 

2. For $\pi = PS(\chi_1, \chi_2)$, the associated Weil-Deligne representation is

$$\rho = \left(\chi_1 \circ \text{Art}_K^1\right) \oplus \left(\chi_1 \circ \text{Art}_K^1\right), \quad N = 0.$$ 

3. For $\pi = St \otimes \chi$, the associated Weil-Deligne representation is $\text{Sp} \otimes \chi$. Explicitly this is

$$\rho = \chi \oplus \chi(1), \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. $$
(4) For $\pi = AI(\chi, F/K)$, the associated Weil-Deligne representation is the 2-dimensional Weil group representation $\text{Ind}_{W_K}^{W}(\chi \circ \text{Art}^{-1}_K)$ and $N = 0$.

Remark. The fact that the Local Langlands Correspondence exists allows us to understand the complicated process of creating the automorphic induction representation of $\chi$ for $F/K$ as the analog of induction on the Galois side, and hence its name.

4.3. Some Comments on the Necessity of Weil-Deligne Representations. Given the abelian case of $\text{GL}_1$, namely local class field theory, one might wonder at the necessity of considering the class of Weil-Deligne representations instead of simply representations of the Weil group $W_K$.

One obvious answer is that this construction “works”, as in the Local Langlands Correspondence gives a bijection between the set of equivalence classes of irreducible admissible representations of $GL_n(K)$ and the set of equivalence classes of Frobenius-semisimple $n$-dimensional Weil-Deligne representations, not simply $W_K$-representations. Consider the following example:

Example 4.8. Under the Local Langlands Correspondence, $\text{St}(n)$ corresponds to a homomorphism

$$\varphi : W_K \times \text{SL}_2(\mathbb{C}) \to \text{GL}_n(\mathbb{C})$$

such that $\varphi(W_K) = 1$ and $\varphi(\text{SL}_2(\mathbb{C}))$ is the unique $n$-dimensional irreducible representation of $\text{SL}_2(\mathbb{C})$. Note that restricted to $W_K$, this coincides with the trivial representation.

However, as demonstrated in the previous section, if we restrict our attention to super-cuspidal representations of $GL_n(K)$, then these correspond to the class of irreducible Weil group representations. So a more deep answer is that considering Weil-Deligne representations (e.g. all irreducible admissible representations of $GL_n(K)$ not just the supercuspidal ones) is more relevant to other questions in number theory. Namely, the more fundamental $\lambda$-adic representations that arise as the étale cohomology of algebraic varieties can be understood more naturally in terms of complex representations of $W_K$ rather than $W_K$.

The reason is as follows: any complex representation of $W_K$ must send some neighborhood of the identity $1 \in GL_n(\mathbb{C})$ as $GL_n(\mathbb{C})$ has no small subgroups. However, this is not the case if we are considering representations whose image is $GL_{E_\lambda}(V_\lambda)$ for $E_\lambda$ a finite extension of $\mathbb{Q}_\ell$ and $V_\lambda$ is a finite dimensional vector space over $E_\lambda$. Another way of stating this problem is that $\rho(I_K)$ must be finite if $\rho$ is a complex representation of $W_K$ (some open subgroup of $I_K$ is in the kernel of $\rho$), but $\rho_\lambda(I_K)$ need not be finite if $\rho_\lambda$ is a $\lambda$-adic representation:

Example 4.9. Consider an elliptic curve $E$ and a prime $p$ for which $E$ has multiplicative reduction. If we let $\rho_{E, p}$ denote the $\ell$-adic Galois representation of $E$, then

$$\rho(I_p) \subseteq \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix},$$

of finite index (given by the power of $\ell$ dividing the $p$-adic valuation of $j(E)$).

However, including the action of $N$ accounts for this “no small subgroup” problem, as indicated in the following theorem of Grothendieck:

Theorem 4.10. For every $(\rho_\lambda, V_\lambda)$ $\lambda$-adic representation of $W_K$, there exists an open subgroup $J$ of $I_K$ and a nilpotent endomorphism $N$ of $V_\lambda$ such that

$$\rho_\lambda(\sigma) = \exp(t_\ell(\sigma)N), \quad t_\ell : I_K \to \mathbb{Q}_\ell, \sigma \in J.$$
This is used to prove the following theorem of Deligne that indicates the significance of Weil-Deligne representations:

**Theorem 4.11.** There is a bijection between the set of $\lambda$-adic representations $(\rho_\lambda, V_\lambda)$ of $W_K$ and the set of Weil-Deligne representations $(\rho, N, V)$ of $W'_K$ over $E_\lambda$. Given $t_\ell : I_K \to \mathbb{Q}_\ell$ a nonzero homomorphism, the above bijection is given by

$$
\rho_\lambda(\Phi^n \sigma) = \rho(\Phi^n \sigma) \exp(t_\ell(\sigma) N), \quad \sigma \in I_K, n \in \mathbb{Z}.
$$

This bijection between isomorphism classes of $\lambda$-adic representations and Weil-Deligne representations is independent of the choice of $\Phi$ and $t_\ell$.

**References**


