AN INTRODUCTION TO THE EULER CHARACTERISTIC

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1. The Euler Characteristic of a Polyhedron

In this talk we will work almost exclusively with the Euler characteristic of a polyhedron. A polyhedron is a solid in three dimensions that consists of straight edges and faces. Two distinct faces intersect in an edge and two distinct edges intersect in a vertex.

Definition 1.1 (The Euler Characteristic). Let P be a polyhedron with V vertices, E edges, and F faces. Then we define the Euler characteristic to be

$$\chi_E(P) = V - E + F.$$

Notice that the Euler characteristic is the alternating sum of the "building blocks" of our polyhedron in each dimension: the vertices are dimension 0 and get a + sign, the edges are dimension 1 and get a − sign, and the faces are dimension 2 and get a + sign.

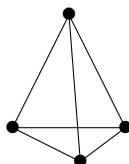The Euler characteristic is best understood by example.

Example 1.2 (Cube). Let C be the cube in 3-dimesions:
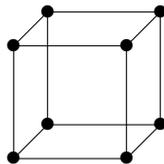
There are 8 dots in the above diagram of the cube, so $V = 8$. Simiarly, $E = 12$ and $F = 6$. So

$$\begin{aligned}
\chi_E(C) &= V - E + F \\
&= 8 - 12 + 6 \\
&= 2.
\end{aligned}$$

Example 1.3 (Tetrahedron). Let T be the tetrahedron in 3-dimensions:


Here $V = 4, E = 6, F = 4$, so

$$\chi_E(T) = 4 - 6 + 4 = 2.$$

From these examples it seems that the Euler characteristic of polyhedra also satisfy the seemingly mysterious relation the $\chi_E = 2$ seen for connected planer graphs. However, the above polyhedra have a very special property: they can be drawn *on the surface of a sphere.* Imagine that one of the above polyhedra were made of an elastic rubber material like a balloon. Then you could image "blowing up" the polyhedron through an imaginary hole at one of the vertices, and the figure would lie on the face of a sphere!

From here it is not hard to see why the Euler characteristic of these examples must be 2, given the theorem for planer graphs. Given a graph on the circle, we could puncture a tiny hole in one of the faces of the graph, grab the sphere through that whole, and "splay" it out so that it lies flat, so that the boundary of our tiny puncture becomes the surrounding boundary of the... **connected planer graph**!

There are, however, polyhedra that cannot be drawn on the surface of a sphere. Consider the following examples.

Example 1.4. Let D be the cube with a central shaft removed:
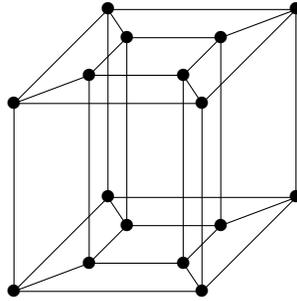
In this case, $V = 16, E = 32, F = 16$. So in total

$$\chi_E(D) = 16 - 32 + 16 = \mathbf{0} \neq 2!$$

If we imagine doing the same procedure of "blowing up" to the polyhedron D above, we would end up with something that looked like a donought, called a *torus*:

(insert image of a torus)

Clearly, for this polyhedron D lying on the surface of a torus, the Euler characteristic is 0. But is this true of all polyhedra that can be drawn on the surface of a torus?

2. A Topological Invariant

In order to understand if it should be true that the Euler characteristic "reads" the information of what kind of surface a certain polyhedron can be drawn on, we must understand what kinds of changes to a graph leave the Euler characteristic unchanged.

Lemma 2.1. *The Euler characteristic of a polyhedron does not change when we subdivide a face or an edge.*

Proof. As an example, consider the following triangle that may be part of the underlying graph of a polyhedron:
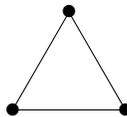
Now, if we subdivide the face of triangle, this involves adding a single vertex, and then 3 edges to connect this vertex to the other 3 vertices of the trianglular face:
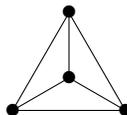
In general, a face might be an n-gon. We still add a single vertex, so the number of vertices V changes by $\Delta V = 1$. This vertex must connect to all outer vertices on the n-gon, of which there are n. This means that E increases by n, so $\Delta E = n$. These new edges divide the face into n compartments (subfaces) where there used to just be 1. Thus F has increased by $\Delta F = n + 1$. So in total the Eular characteristic changes by

$$\Delta \chi_E = \Delta V - \Delta E + \Delta F$$
$$= 1 - n + n - 1$$
$$= 0,$$

so the process of subdividing a face does not change the global Euler characteristic.

Similarly, if we "subdivide" an edge we introduce one new vertex in the middle of the edge, so $\Delta V = 1$. We also have divided one edge into two, so $\Delta E = 1$. Thus in total

$$\Delta \chi_E = \Delta V - \Delta E + \Delta F = 1 - 1 + 0 = 0,$$

and again this process does not affect the global Euler characteristic.

$\square$

Lemma 2.2. *The Euler characteristic of a polyhedron does not change when we delete an edge bordering distinct faces, or a vertex bordering 2 distinct edges.*

Remark. It is necessary to have these qualifications on to which edges and vertices the lemma applies, in order to disqualify the following examples:

vertex meets more
than 2 edges

loop around
a single vertex

edge bounds
same face
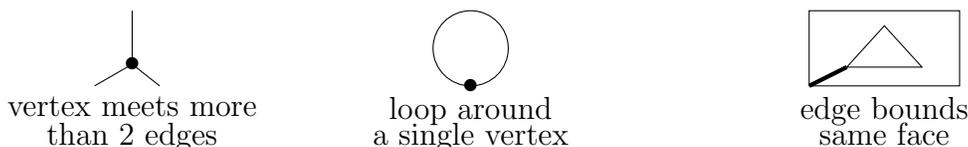
for which the vertex (respectively edge) is essential for the "integrity" and information inherent in the graph.

Proof. As in the previous lemma, we will argue by looking at the local changes to the Euler characteristic caused by our deletion. Upon deleting an edge, clearly $\Delta E = -1$. But, additionally, as long as the edge bordered 2 distinct faces, the number of faces decreases by 1 as well, so $\Delta F = -1$. So in total:

$$\Delta \chi_E = \Delta V - \Delta E + \Delta F = 0 + 1 - 1 = 0.$$

Similarly, if we delete a vertex bordering 2 distinct edges then $\Delta V = -1$, $\Delta E = -1$ and $\Delta F = 0$, so

$$\Delta \chi_E = \Delta V - \Delta E + \Delta F = -1 + 1 + 0 = 0.$$

$\square$

This last lemma suggests that the underlying graph of a polyhedron drawn on some surface S can be "stripped down" to its essentials - that is vertices that do not border distinct edges and edges that do not border distinct faces - without changing the Euler characteristic.

Using both of these lemmas, we can now answer our question posed originally,

Do all polyhedra that can be drawn on the same
surface have the same Euler characteristic?

with the following theorem.

Theorem 2.3. *The Euler characteristic of a polyhedron depends only on the surface on which it can be drawn; in other words, it is constant for polyhedra that can be drawn on the same surface.*

Sketch of Proof. Any graph can be transformed into any other graph by subdividing faces and edges and deleting edges and vertices. By Lemmas 2.1 and 2.2 these do not affect the Euler characteristic.

We could use this theorem to give a definition of the Euler characteristic of a surface as the Euler characteristic of the class of polyhedra that can be drawn on its surface. There is a more sophisticated way to define the Euler characteristic of a topological space, but for the surfaces we are considering, these definitions agree.

Thus the Euler characteristic is something we might call a crude *topological invariant.* Knowing the Euler characteristic of a surface, or polyhedron tells you something about its shape, its topology, irrespective of stretching, skewing, "blowing up", or similar nondestructive deformations.

We have already seen that the Euler characteristic of a sphere is 2 and of a torus is 0. What about other surfaces? It turns out that for a donought with g holes, what we call a g-hole torus, the Euler characteristic has a very nice form.

Theorem 2.4. *Let M_g be a g-hole torus. Then we have*

$$\chi_E(M_g) = 2 - 2g.$$

Remark. Note that in the cases we already know, this theorem agrees with what we have calculated. When $g = 0$, the surface is just the sphere, and so

$$\chi_E(M_0) = 2 - 2 \cdot 0 = 2.$$

And when $g = 1$, the surface is the torus so

$$\chi_E(M_1) = 2 - 2 \cdot 1 = 0.$$

3. Applications of the Euler Characteristic

3.1. **Brussel Sprouts.** In the course of playing Brussels Sprouts, you create a geometric object known as a planer graph.

The game of Brussels Sprouts goes as follows:

(1) Draw some number n of four-pointed crosses. In the game below, $n = 2$.
(2) On each round, players alternate joining crosses by drawing a line from one point of one cross, to another point of another cross, without intersecting any the other lines already drawn. Then adding a dash in the middle of the line, so as to create another cross in the middle (a cross that is already occupied by in two positions).
(3) The game ends when it is no longer possible to add any lines without there being an intersection somewhere.

Here is an example of a game starting with 2 crosses. As you will see, there are 8 rounds and player 2 wins.
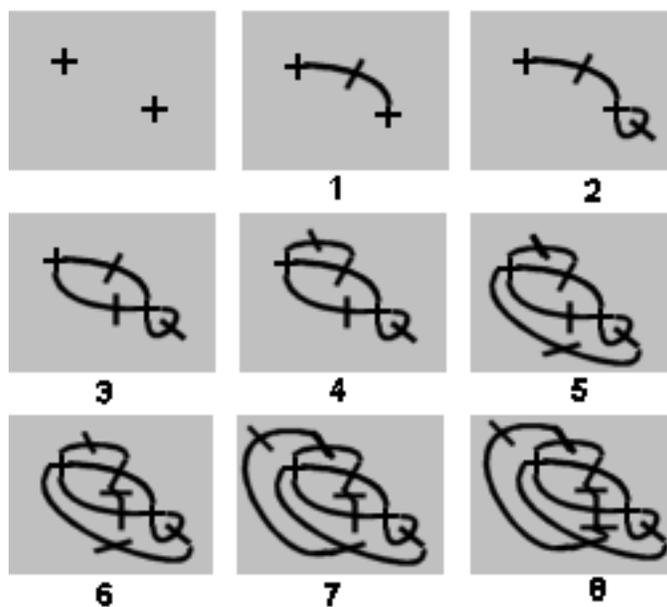
In fact, a game with 2 crosses will *always* be a win for the second player, and you can prove this using the Euler characteristic

3.2. Platonic Solids.

The Platonic solids are polyhedra built out of regular polygons (like squares, pentagons, triangles) such that the number of edges meeting each vertex is constant (the same for each vertex).
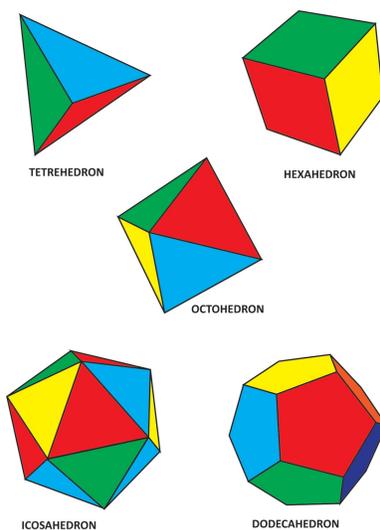

FIG-1 THE PLATONIC SOLIDS

The Euler characteristic allows us to prove that there are only 5 platonic solids, in other words, this list is exhaustive.

3.3. Fixed Point Theory.

Let S be a surface. Then in an appropriate sense that I cannot make less vague here, χ_E equals the sum of the fixed points of S (with appropriate coefficients) under an infinitessimal deformation of S.

Consider the case that S is just a sphere. Then one such "infintessimal deformation" of S is just to draw a line through the sphere, passing through the north and south poles and rotate the sphere about this axis. This map has 2 fixed points: the north and south poles where the line forming the axis of the rotation met the sphere.

We can visualize this idea by thinking of a "hairy" sphere. Then by combing the hairs in some particular way gives us a map of the sphere: send each point to the corresponding point at the end of the ahir. A fixed point would correspond to a hair pointing straight up, ie a cowlick of hair.

The fact that χ_E of a sphere is 2 means that any "combing" of the sphere always has at least one cowlick! This is also known as the hairy-ball theorem.

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