1. Numerical models of neutron stars (50 pts)

In this problem you will model a neutron star using a more realistic equation of state. To prepare for this problem, please peruse the classic paper by Arnett & Bowers (1977), provided on the class web site. (There are more recent papers on this subject, but they are a lot more complicated; Arnett & Bowers does a good job covering the key points and presenting a representative sample of equations of state.)

(a) Show that a power law

\[ P = K_{AB} \rho^{\gamma_{AB}} \]

with \( \gamma_{AB} = 2 \), \( K_{AB} = 10^5 \text{ ba} (\text{gm} - \text{cm}^{-3})^{-\gamma_{AB}} \) is a good “average” of the models A through G in Figure 4 of Arnett & Bowers. (A “barye” [ba] is the cgs unit of pressure; 1 ba = 1 dyne cm\(^{-2}\) = 0.1 Pa.)

You will assume that this expression is valid down to arbitrarily low densities, including \( \log_{10} \rho \lesssim 14.6 \) (where \( \rho \) is in g cm\(^{-3}\)), the smallest value plotted in Figure 4.

**Solution:** Here is a plot of \( P \) versus \( \rho \):

[Graph of \( P = K_{AB} \rho^{\gamma_{AB}} \)]

Comparing with Arnett & Bowers, we see that this EOS sits right on models A through G.
(b) Consider a range of central densities $14 < \log_{10} \rho_c < 16.5$ (where $\rho_c$ is in g cm$^{-3}$), uniformly spaced in $\log_{10} \rho_c$. For each of these, integrate the Tolman-Oppenheimer-Volkoff equation to find $\rho(r)$. Although it is not necessary, you may find it useful to start by nondimensionalizing the equations

$$\frac{dP}{dr} = -\frac{1}{K_{AB}\gamma_{AB}\rho^{\gamma_{AB}-1}} \left( \frac{dM}{dr} \right)_{TOV}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho.$$  

In other words, introduce some fiducial density scale $\rho_0$ and normalize $\rho$ to that; deduce related length, mass, and pressure scales, and scale those variables accordingly.

Hint 1: Some numerical integrators become quite sad if you begin integrating at $r = 0$ — because the enclosed mass there is zero, you find lots of $0/0$ type singularities. Start at some very small radius $\epsilon$, and assume that $\rho$ varies very slowly in that region to put $M(\epsilon) \approx (4\pi/3)\rho_c \epsilon^3$.

Hint 2: If you elect to nondimensionalize, note that $(G\rho)^{-1/2}$ is a time and so $(G\rho_0)^{-1/2}$ can be regarded as a fiducial timescale. Likewise, in any relativistic calculation $c$ is a relevant fiducial velocity.

Solution: Nondimensionalizing is definitely a good idea: it makes the equations simpler, and allows the calculations to be performed on numbers which are all of the same order of magnitude.

As in the “constant density neutron star” problem on the previous problem set, we choose some fiducial values for our parameters. This choice sets a pressure scale, $P_0 = \rho_0 c^2$; a length scale, $r_0 = c/\sqrt{G\rho_0}$; and a mass scale, $m_0 = \rho_0 r_0^3 = c^3/\sqrt{G^3 \rho_0}$.

Let us nondimensionalize all dimensionful quantities with these scales,

$$\xi = r/r_0 , \quad x = \rho/\rho_0 , \quad p = P/P_0 , \quad m = M/m_0 .$$

This gives the system

$$\frac{dp}{d\xi} = -\frac{(m + 4\pi \xi^3 p)(x + p)}{\xi(\xi - 2m)}$$

$$\frac{dm}{d\xi} = 4\pi \xi^2 x .$$

Taking a dimensional polytrope equation of state $P = K \rho^\gamma$ and nondimensionalizing gives a dimensionless polytrope equation of state $p = k x^\gamma$ where $k = K\rho_0^{\gamma-1} c^{-2}$. Let us choose our fiducial density to be our lowest central density, $\rho_0 = 10^{14}$ g cm$^{-3}$. For the Arnett & Bowers equation $K_{AB}$ and $\gamma_{AB}$, this gives a dimensionless $k_{AB} \approx 0.011$.

Using our equation of state, the TOV equation becomes

$$k^{\gamma-1} x^\gamma \frac{dx}{d\xi} = -\frac{(m + 4\pi \xi^3 k x^\gamma)(x + k x^\gamma)}{\xi(\xi - 2m)} .$$

Integrate this with “initial” conditions $x(\epsilon) = \rho_c/\rho_0$, $m(\epsilon) = 4\pi x(\epsilon) c^3/3$. Integrate outward in $\xi$ until the density plummets to zero (or as close to zero as the computer can get).

A characteristic radial density profile is plotted in Fig. 2.

(c) Plot the total mass as a function of $\log_{10} \rho_c$ for your models. What is the maximum mass of a neutron star for this equation of state?

Solution: See Fig. 3. From the numerical models, the maximum mass is about $M \lesssim 1.65M_\odot$. If you use a slightly different fit to the Arnett & Bowers curves, you will get different numbers.

(d) Plot the radius as a function of $\log_{10} \rho_c$ (for masses corresponding to stable neutron stars).

Solution: See Fig. 4.

(e) Plot the mass-radius relationship for your models (for masses corresponding to stable neutron stars).

Solution: See Fig. 5.
Radial density profile of model with $M = 1.65M_\odot$

Figure 2: A radial density profile for a polytropic NS model by approximating Arnett & Bowers. This is the maximum mass model (Prob. 1b).

Total mass as a function of central density

Figure 3: The dependence of total mass with central density of a neutron star (Prob. 1c).
Radius as a function of central density

Figure 4: The dependence of radius with central density of a neutron star (Prob. 1d).

Mass–radius relationship

Figure 5: The mass-radius relationship for a neutron star (Prob. 1e).
(f) Finally, investigate a “maximally stiff” equation of state, for which (at high densities) the sound speed is equal to the speed of light. Please repeat steps 1b and 1c using an equation of state

\[
P = \rho c^2 \quad \text{for} \quad \rho > 10^{14.6} \text{ g cm}^{-3} \quad (5)
\]

\[
P = K \rho^{5/3} \quad \text{for} \quad \rho < 10^{14.0} \text{ g cm}^{-3}, \quad (6)
\]

where \( K \approx 5.5 \times 10^9 \) (cgs) is the appropriate constant for a non-relativistic Fermi gas of neutrons. For densities between \( 10^{14.0} \) and \( 10^{14.6} \) g cm\(^{-3}\), calculate \( \log_{10} P \) as a function of \( \log_{10} \rho \) via linear interpolation between the two expressions given above.

**Solution:** First, let us construct the broken power law which connects the high and low density regimes. Let \( \rho_1 = 10^{14} \text{ g cm}^{-3} \) and \( \rho_2 = 10^{14.6} \text{ g cm}^{-3} \). From the two regimes, we have \( P_1 = K \rho_1^{5/3} \approx 1.2 \times 10^{33} \text{ ba} \), and \( P_2 = \rho_2 c^2 \approx 3.6 \times 10^{35} \text{ ba} \). Then the logarithmic slope in the middle region is

\[
\gamma_m = \frac{\ln P_2 - \ln P_1}{\ln \rho_2 - \ln \rho_1} \approx 4.13.
\]

Then we only need to match the normalization through \( P = K_m \rho^{\gamma_m} \). Solving gives \( K_m = P_2 \rho_1^{\gamma_m} / P_1 \rho_2 \approx 1.6 \times 10^{-25} \text{ ba} (\text{g cm}^{-3})^{\gamma_m} \).

The above can be performed nondimensionally, with \( x_1 = \rho_1 / \rho_0 \), etc. This gives a nondimensional \( k_m \approx 0.013 \).

Since the previous parts assumed a single power law throughout, we must change the formalism slightly for this broken power law. Instead of Eq. (4), we only have

\[
\frac{dp}{dx} = \frac{[m + 4\pi x^3 p(x)](x + p[x(\xi)])}{x(\xi - 2m)}.
\]

Both \( p \) and \( dp/dx \) are piecewise continuous functions, and take the dimensionless density \( x \) as an argument.

See the companion notebook for an example of how to implement this in Mathematica. This generates Fig. 6 Notice the maximum mass in this model is about \( 5.5 M_\odot \).
2. Pulsar spin-down properties (15 pts)

Consider a pulsar with spin period $P = 2\pi/\Omega$ that is losing energy and therefore spinning down.

(a) If the energy loss mechanism is magnetic dipole radiation, then

$$\frac{dE}{dt} = -\frac{B^2 \Omega^4 R^6 \sin^2 \alpha}{6c^3}, \quad (7)$$

where $B$ is the polar magnetic field strength, $R$ is the pulsar radius, and $\alpha$ is the angle between the magnetic and rotational poles. Show that this implies $\dot{\Omega} = -k \Omega^3$ where $k$ is a constant. Also show that in this case, $B \propto \sqrt{P\dot{P}}$.

**Solution:** The energy which is being lost here is rotational kinetic energy, with

$$E_{K, rot} = \frac{1}{2} I \Omega^2,$$

where $\beta$ is a constant of order unity. Plugging this in to Eq. (7), find

$$I \dot{\Omega} \dot{\Omega} = -\frac{B^2 \Omega^4 R^6 \sin^2 \alpha}{6c^3},$$

$$\dot{\Omega} = -\frac{B^2 R^6 \sin^2 \alpha}{6Ic^3} \Omega^3, \quad k = \frac{B^2 R^6 \sin^2 \alpha}{6Ic^3}.$$

Solving for $B$ (dropping other constants now) gives

$$B \propto \sqrt{-\frac{1}{\Omega^2 \dot{\Omega}}},$$

$$B \propto \sqrt{-P^2 \frac{d \ln \Omega}{dt}}.$$

Now since $\Omega \propto P^{-1}$,

$$\frac{d \ln \Omega}{dt} = -\frac{d \ln P}{dt}. \quad (8)$$

Plugging this back in,

$$B \propto \sqrt{+P^2 \frac{d \ln P}{dt}}$$

$$B \propto \sqrt{P\dot{P}}.$$

(b) For the more general case $\dot{\Omega} = -k \Omega^n$, where $n$ is the braking index, show that $n = \dot{\Omega} \Omega / \dot{\Omega}^2$.

**Solution:** Taking an additional time derivative gives

$$\ddot{\Omega} = -kn \Omega^{n-1} \dot{\Omega} = (-k \Omega^n)(n \dot{\Omega} / \Omega)$$

$$\ddot{\Omega} = n \dot{\Omega}^2 / \Omega.$$

Solving for $n$ now gives $n = \dot{\Omega} \Omega / \dot{\Omega}^2$.

(c) Show that if the braking index is $n$, the age of the pulsar may be estimated as

$$\tau \approx \frac{|P / \dot{P}|_{\text{final}}}{n - 1} \left[ 1 - \frac{P_{\text{initial}}^{n-1}}{P_{\text{final}}^{n-1}} \right]. \quad (9)$$
Solution: Assume that \( k \) is constant over the lifetime of the pulsar. Separate \( \dot{\Omega} = -k\Omega^n \) and integrate:

\[
k dt = -\frac{d\Omega}{\Omega^n}
\]

\[
k\tau = -\int_{\Omega_i}^{\Omega_f} \frac{d\Omega}{\Omega^n}
\]

\[
k\tau = \frac{1}{n-1} \left( \frac{1}{\Omega_f^{n-1}} - \frac{1}{\Omega_i^{n-1}} \right),
\]

where \( \tau = t_f - t_i \). From \( \dot{\Omega} = -k\Omega^n \), substitute \( k = -\dot{\Omega}_f/\Omega_f^n \) on the left hand side, giving

\[
-\frac{1}{\Omega_f^{n-1}} \frac{\dot{\Omega}_f}{\Omega_f} \tau = \frac{1}{n-1} \left( \frac{1}{\Omega_f^{n-1}} - \frac{1}{\Omega_i^{n-1}} \right)
\]

\[
-\frac{\dot{\Omega}_f}{\Omega_f} \tau = \frac{1}{n-1} \left( 1 - \frac{\Omega_f^{n-1}}{\Omega_i^{n-1}} \right).
\]

Now notice the combination \(-d\ln \Omega/dt = d\ln P/dt\) on the left hand side. Rearranging,
3. Blackbody radiation from a compact object (15 pts)

Because general relativity is important for compact objects, even seemingly basic quantities such as luminosity, temperature and radius need to be defined carefully, as you will see in this problem.

Consider a spherical blackbody of constant temperature and mass $M$ and an outer surface defined by the radial coordinate $r = R$. Two observers are measuring the blackbody radiation: an observer located at the surface of the sphere, and a very distant observer.

(a) If the observer at the surface of the sphere measures the luminosity of the blackbody to be $L$, show that the observer at infinity measures

$$L_\infty = L \left(1 - \frac{2GM}{Rc^2}\right).$$

An important bit of physics to use here is the gravitational redshift $z_g$ associated with a photon travelling radially in the Schwarzschild spacetime. It is given by

$$1 + z_g = \left(1 - \frac{2GM}{Rc^2}\right)^{-1/2}.$$ (10)

This is derived on p. 387-392 of Choudhuri. A photon that is emitted with energy $E$ at radius $R$ will be measured to have energy $E/(1 + z_g)$ very far away. This redshift also applies to clocks — a time interval $dt$ at radius $R$ is measured to be an interval $dt(1 + z_g)$ by observers far away. In other words, clocks deep in a gravitational potential well run slow. (This is why GPS satellites need to correct for general relativity — clocks in high orbit run demonstrably faster than clocks on the surface of the Earth.)

**Solution:** The luminosity of a source measured at infinity can be written as

$$L_\infty = \frac{dE_\infty}{dt_\infty} = \int_0^\infty d\nu_\infty \frac{dE_{\nu,\infty}}{dt_\infty d\nu_\infty},$$ (11)

and let us define the luminosity measured at a radius $r$ as

$$L_r = \frac{dE_r}{dt_r} = \int_0^\infty d\nu_r \frac{dE_{\nu,r}}{dt_r d\nu_r}.$$ (12)

We would like to write Eq. (11) in terms of quantities appearing in Eq. (12). First we must recognize that photons starting in frequency bin $\nu_r$ end up in a different frequency bin $\nu_\infty$, related by

$$\nu_\infty = \frac{\nu_r}{1 + z_g(r)},$$

that is, the wavelength gets stretched, and the frequency shrinks. Now, since the energy of light is proportional to its frequency, the energy of a photon is not conserved,

$$E_{\nu,\infty}(\nu_\infty) \neq E_{\nu,r}(\nu_r),$$

but rather the energy is also redshifted,

$$E_{\nu,\infty}(\nu_\infty) = \frac{E_{\nu,r}(\nu_r)}{1 + z_g(r)}.$$ (13)

Finally, the time element is also redshifted,

$$d\tau_\infty = [1 + z_g(r)]d\tau_r.$$ (14)

Putting it all together,

$$L_\infty = \int_0^\infty d\nu_\infty \frac{dE_{\nu,\infty}}{dt_\infty d\nu_\infty} = \int_0^\infty d\nu_r [1 + z_g(r)]^{-2} \frac{dE_{\nu,r}}{dt_r d\nu_r} = \frac{L_r}{[1 + z_g(r)]^2}.$$ (15)
Notice that the Jacobian for \( d\nu_\infty /d\nu_c \) cancels.

Finally, evaluating this at \( r = R \) with \( L \equiv L_R \),

\[
L_\infty = L \left( 1 - \frac{2GM}{Rc^2} \right). \tag{13}
\]

(b) Suppose both observers use Wien’s law,

\[
\lambda_{\text{max}} T = 0.28978 \text{ cm K},
\]

to determine the blackbody’s temperature. Here \( \lambda_{\text{max}} \) is the wavelength corresponding to the peak in the blackbody spectrum. Show that

\[
T_\infty = T \sqrt{1 - \frac{2GM}{Rc^2}}.
\]

**Solution:** Both observers determine \( T \) from \( T = b / \lambda_{\text{max}} \) where \( b = 0.28978 \text{ cm K} \) is Wien’s constant. Again apply the gravitational redshift, Eq. (10),

\[
T_\infty = \frac{b}{\lambda_{\text{max},\infty}} = \frac{b}{[1 + z_g(R)] \lambda_{\text{max},R}} = T \sqrt{1 - \frac{2GM}{Rc^2}}, \tag{14}
\]

where \( T \equiv T_R \) is the temperature measured at a coordinate radius \( R \).

(c) Suppose both observers use the Stefan-Boltzmann law to determine the radius of the spherical blackbody. Show that

\[
R_\infty = \frac{R}{\sqrt{1 - \frac{2GM}{Rc^2}}}
\]

Thus, using the Stefan-Boltzmann law without accounting for general relativity will lead to an *overestimate* of the size of a compact blackbody.

**Solution:** From the Stefan-Boltzmann law, an observer at coordinate radius \( r \) infers the object’s radius as

\[
R_r = \sqrt{\frac{L_r}{4\pi\sigma T_r^4}}.
\]

Plugging in Eq. (13) and Eq. (14) into the above, find

\[
R_\infty = \sqrt{\frac{[1 + z_g(r)]^{-2} L_r}{[1 + z_g(r)]^{-4} 4\pi\sigma T_r^4}} = [1 + z_g(r)] R_r,
\]

or, comparing to the surface at \( r = R \),

\[
R_\infty = \frac{R}{\sqrt{1 - \frac{2GM}{Rc^2}}}.
\]
4. Supernova explosion in a binary system (15 pts)

Two stars of mass $m_1$ and $m_2$ are in a circular orbit. Star 1 undergoes a supernova explosion in which mass $\Delta m$ is blown away spherically symmetrically (in the frame of star 1) on a time scale that is very short compared to the orbital period. Show that the condition for the orbit to remain bound is

$$\Delta m < \frac{m_1 + m_2}{2}.$$ 

**Hint:** Compute the total energy of the binary after the explosion in the (new) center of mass frame of the binary. Assume that immediately after the explosion, the binary separation and the orbital velocities of both stars are unchanged. A useful relation is that the total kinetic energy is $(1/2)\mu v_{rel}^2$, where $\mu$ is the reduced mass and $v_{rel}$ is the relative velocity.

**Solution:** Initially the energy of the system can be written

$$E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - G m_1 m_2 \left| \frac{r_1}{r_2 - r_1} \right|.$$ 

It is very useful at this point to go into the center of mass frame, putting $m_1 r_1 + m_2 r_2 = 0$. To this end, we introduce relative coordinates:

$$r = r_2 - r_1, \quad v = v_2 - v_1.$$ 

Note that $r = |r|$ and $v = |v|$ are both constants by the assumption that the pre-supernova orbit is circular, and that $v = \sqrt{G(m_1 + m_2)/r}$. We’ll take advantage of this later in the problem.

In terms of the relative coordinates, we have

$$r_1 = -\frac{m_2}{m_1 + m_2} r, \quad r_2 = \frac{m_1}{m_1 + m_2} r,$$

$$v_1 = -\frac{m_2}{m_1 + m_2} v, \quad v_2 = \frac{m_1}{m_1 + m_2} v.$$ 

These give us

$$E = \frac{1}{2} m_1 \left( \frac{m_2}{m_1 + m_2} \right)^2 v^2 + \frac{1}{2} m_2 \left( \frac{m_1}{m_1 + m_2} \right)^2 v^2 - G m_1 m_2 \frac{r}{r}.$$ 

This expression could be rewritten quite nicely using the reduced mass $\mu$ and total mass $M$, but it is helpful to leave it in this form for what we will do next.

The supernova has several effects. First, $m_1 \to m_1 - \Delta m$. Second, the orbit will no longer have the same center of mass — the system as a whole will be kicked by the explosion, and translate with respect to its original center of mass frame. Finally, the stars’ separation and velocities will both in general change. By assumption, the velocities and separation will be the same as before immediately following the explosion, but they will stay this way. This tells us that, not only does the system begin translating away from its original center of mass location, but the orbit configuration changes — what was circular will become some new eccentric orbit.

The simplest way to analyze the system is to again go into the center of mass frame. Although the center of mass after the explosion is not the same as the center of mass beforehand, it nonetheless exists. Repeating the above analysis but now in terms of new relative coordinates $r'$ and $v'$, we find the energy in this frame to be

$$E' = \frac{1}{2} \frac{(m_1 - \Delta m)m_2}{m_1 - \Delta m + m_2} v'^2 - \frac{G(m_1 - \Delta m)m_2}{r'}.$$ 

As discussed above, $v'$ and $r'$ are not constant. However, $E'$ is a constant, so we can evaluate this quantity for any convenient choice of $v'$ and $r'$. We choose the moment right after the explosion, and put $v' = v, r' = r$. For
the binary to remain bound, we must have $E' < 0$:

$$\frac{1}{2} \frac{(m_1 - \Delta m) m_2}{m_1 - \Delta m + m_2} v^2 - \frac{G(m_1 - \Delta m) m_2}{r} < 0$$

$$\frac{1}{2} \frac{(m_1 - \Delta m) m_2}{m_1 - \Delta m + m_2} \left( \frac{G(m_1 + m_2)}{r} \right) - \frac{G(m_1 - \Delta m) m_2}{r} < 0$$

$$\frac{1}{2} \frac{m_1 + m_2}{m_1 - \Delta m + m_2} < 1$$

$$\frac{m_1 + m_2}{m_1 + m_2} < 2(m_1 - \Delta m + m_2)$$

We finally see that we must have

$$\Delta m < \frac{m_1 + m_2}{2}$$