1. **Hydrogen burning is the main thing [10 pts]**

Consider two hypothetical stars of the same mass and (constant) luminosity. The stars are originally pure hydrogen. In Star A, fusion proceeds until the entire star is converted into $^{56}$Fe. In Star B, fusion does not proceed all the way to $^{56}$Fe. Instead, fusion is halted after all the hydrogen is converted into $^4$He. How much longer is the lifetime of Star A, compared to Star B?

You will find it useful to know the masses of the relevant nuclei:

\[
\begin{align*}
  m_{H} &= 1.0078250 \text{ amu} \\
  m_{^4\text{He}} &= 4.0026032 \text{ amu} \\
  m_{^{56}\text{Fe}} &= 55.9349421 \text{ amu}
\end{align*}
\]

where the amu is approximately $m_p$ (it is defined as one-twelfth of the mass of an unbound carbon-12 nucleus in its ground state).

**Solution:** We begin by calculating the fraction, $\epsilon$, of a proton’s mass-energy that is released when hydrogen is fused into $^4$He and $^{56}$Fe. Then, the nuclear energy released in the conversion of 4 protons to $^4$He ($56$ protons to $^{56}$Fe) corresponds to a fraction $\epsilon_{^4\text{He}}$ ($\epsilon_{^{56}\text{Fe}}$) of the initial mass energy of the protons.

\[
\begin{align*}
  \epsilon_{^4\text{He}} &= \frac{4m_{H} - m_{^4\text{He}}}{4m_{H}} = 0.0071 \\
  \epsilon_{^{56}\text{Fe}} &= \frac{56m_{H} - m_{^{56}\text{Fe}}}{56m_{H}} = 0.0089
\end{align*}
\]

Since stars A and B have identical masses and constant luminosities, their nuclear burning lifetimes $\tau = \epsilon M c^2 / L$ are related by

\[
\frac{\tau_A}{\tau_B} = \frac{\epsilon_{^{56}\text{Fe}}}{\epsilon_{^4\text{He}}} \approx 1.25
\]

The lifetime of star A is 25% longer than that of star B.

2. **Electron degeneracy pressure for an arbitrary degree of relativistic motion [20 pts]** (based on Choudhuri 5.2)

The degeneracy pressure of an electron gas is given by Equation (5.5) of Choudhuri,

\[
P = \frac{8\pi}{3h^3} \int_0^{\rho_p} \frac{p^4 c^2}{\sqrt{p^2 c^2 + m_e^2 c^4}} dp.
\]

(a) Work out this integral by substituting $p = m_e c \sinh \theta$ and show that the general expression for the electron degeneracy pressure is equal to

\[
P = \frac{\pi m_e^4 c^5}{3h^3} f(x),
\]

where

\[
f(x) = x(2x^2 - 3) \sqrt{x^2 + 1} + 3 \sinh^{-1} x
\]
and \( x \equiv p_F/m_e c \).

**Solution:** With the recommended change of variables, and using \( dp = m_e c \cos \theta d\theta \),

\[
P = \frac{8 \pi}{3h^3} \int_0^{\theta_F} m_e^5 \sinh^4 \theta \cosh \theta d\theta,
\]

where \( \theta_F = \sinh^{-1}(p_F/m_e c) = \sinh^{-1} x \). Use the identity \( \cosh^2 \theta - \sinh^2 \theta = 1 \) to cancel \( \cosh \theta \) in the numerator with the denominator. Note that we already have the prefactor right,

\[
P = \frac{\pi m_e^5}{3h^3} f(x), \quad f(x) \equiv 8 \int_0^{\sinh^{-1} x} \sinh^4 \theta d\theta
\]

This can be integrated by writing \( \sinh \theta = \frac{1}{2}(e^{\theta} - e^{-\theta}) \) and expanding,

\[
8 \sinh^4 \theta = \frac{1}{2} e^{4\theta} - 2 e^{2\theta} + 3 - 2 e^{-2\theta} + \frac{1}{2} e^{-4\theta}
= 3 - 4 \sinh(2\theta) + \sinh(4\theta).
\]

Integrating this is straightforward:

\[
f(x) = \left[ 3\theta - 2 \sinh(2\theta) + \frac{1}{4} \sinh(4\theta) \right]^{\sinh^{-1} x}_0
= 3 \sinh^{-1} x - 2 \sinh(2 \sinh^{-1} x) + \frac{1}{4} \sinh(4 \sinh^{-1} x).
\]

One can complete the evaluation by using the identity (see [http://dlmf.nist.gov/4.37.E16](http://dlmf.nist.gov/4.37.E16)) \( \sinh^{-1} x = \ln(x + \sqrt{1 + x^2}) \) and expanding the sinh functions as exponentials, or by using the multiple angle formulas (see [http://dlmf.nist.gov/4.35.iii](http://dlmf.nist.gov/4.35.iii)) \( \sinh 2\theta = 2 \sinh \theta \cosh \theta \) and \( \sinh 4\theta = 4 \sinh^3 \theta \cosh \theta + 4 \sinh \theta \cosh^3 \theta \) along with \( \cosh(\sinh^{-1} x) = \sqrt{1 + x^2} \). Either way, the function evaluates as

\[
f(x) = 3 \sinh^{-1} x + x \sqrt{1 + x^2} (2x^2 - 3).
\]

(b) Evaluate \( f(x) \) numerically for various values of \( x \) and use these numerical values to make a plot of \( \log P \) versus \( \log(\rho/\mu_c) \). Indicate regions of the plot corresponding to the two limiting equations \( P \propto \rho^{5/3} \) for nonrelativistic motion, and \( P \propto \rho^{5/3} \) for ultrarelativistic motion.

**Solution:** The asymptotic forms of \( f(x) \) are

\[
f(x) = \begin{cases} 
\frac{2}{3} x^5 + \mathcal{O}(x^6), & x \ll 1 \\
2x^4 + \mathcal{O}(x^5), & x \gg 1
\end{cases}
\]

Recall the definition of \( p_F \),

\[
p_F = \left( \frac{3h^3 \pi}{8} \right)^{1/3} = \left( \frac{3h^3 \rho}{8 \pi \mu_e m_p} \right)^{1/3},
\]

so we may write \( x = (\rho/\rho_0)^{1/3}, \) with

\[
\rho_0 = \frac{8 \pi \mu_e m_p m_e^3 c^3}{3h^3},
\]

and define

\[
P_0 = \frac{\pi m_e^5}{3h^3} = \frac{1}{8} \rho_0 c^2 \mu_e \mu_p.
\]

Then one can write \( \frac{P}{P_0} = f((\rho/\rho_0)^{1/3}) \). Then a log-log plot of the function \( f \) is a horizontally scaled log-log plot of \( P/P_0 \) vs. \( \rho/\rho_0 \). Such a plot appears in Fig. [1].
3. Polytropic relation between core density and temperature [20 pts]

Previously you derived some useful analytic relations for a polytropic stellar model. Here you will use them to derive the relation between core density ($\rho_c$) and core temperature ($T_c$) that a star of a given mass $M$ is expected to obey. As discussed in class, this relationship constrains the core’s trajectory to a particular locus in the space of $\log \rho_c$ and $\log T_c$.

(a) Show that the central pressure can be written in terms of $M$ and $\rho_c$ as

$$P_c = (4\pi)^{1/3} GM^{2/3} \rho_c^{4/3} F(n),$$

where $F$ is a function of the polytropic index $n$ that you should specify; it will involve $\xi_1$, the coordinate of the surface, as well as the function $\phi_n$, and/or its derivative(s). Show further that for $n$ ranging from 1 to 3.5 (encompassing most of the realistic range of pressure/density profiles), the function $F$ is nearly equal to 0.2 (within $\approx 30\%$).

**Solution:** Recall from the previous problem set the relations

$$P_c = \frac{GM^2}{R^2} \left[ 4\pi(n + 1) \left| \frac{d\phi_n}{d\xi} \right|_{\xi_1}^2 \right]^{-1}$$

and

$$\langle \rho \rangle = \frac{3M}{4\pi R^3} = \frac{3\rho_c}{\xi_1} \left| \frac{d\phi_n}{d\xi} \right|_{\xi_1}.$$  

Using Eq. [3], the radius $R$ may be eliminated in favor of the mass $M$, central density $\rho_c$, the scaled maximum radius $\xi_1$, and the slope of the temperature profile at the surface $|d\phi_n/d\xi|_{\xi_1}$. The radius may be written

$$R = \left( \frac{M\xi_1}{4\pi \rho_c} \left| \frac{d\phi_n}{d\xi} \right|_{\xi_1}^{-1} \right)^{1/3}.$$
Plug this in to Eq. (2) to find

\[ P_c = GM^2 \left( \frac{4\pi \rho_c}{M} \frac{d\phi_n}{d\xi} \right)^{4/3} \left[ \frac{4\pi (n + 1)}{4} \left( \frac{d\phi_n}{d\xi} \right)^2 \right]^{-1} \]

\[ P_c = (4\pi)^{1/3} GM^{2/3} \rho_c^{4/3} F(n), \quad \frac{1}{F(n)} = (n + 1) \xi_1^{4/3} \left( \frac{d\phi_n}{d\xi} \right)^{2/3} \xi_1 \]

A plot of the function \( F(n) \) appears in Fig. 2. Note that the function is monotonic and it ranges from 0.233 to 0.145 over the \( n \) range from 1 to 3.5.

(b) Show that if the ideal gas equation of state is applicable, then one expects

\[ \log \rho_c = 3 \log T_c - 2 \log M + \text{constant}. \]

**Solution:** Equate \( P_c \) from Eq. (1) with the ideal gas equation of state,

\[ (4\pi)^{1/3} GM^{2/3} \rho_c^{4/3} F(n) = \frac{\rho_c}{\mu m_p} k_b T_c. \]

Taking the log and solving for \( \log \rho_c \),

\[ \frac{1}{3} \log \rho_c + \frac{2}{3} \log M + \log \left( (4\pi)^{1/3} GF(n) \right) = \log T_c + \log \left( \frac{k_b}{\mu m_p} \right), \]

\[ \log \rho_c = 3 \log T_c - 2 \log M + 3 \log \left( \frac{k_b}{(4\pi)^{1/3} GF(n) \mu m_p} \right), \]

where the final term is a slowly varying function of \( n \), and so is almost constant.

(c) Show that if nonrelativistic degeneracy pressure is dominant, then one expects

\[ \log \rho_c = 2 \log M + \text{constant}. \]

**Solution:** Equate \( P_c \) from Eq. (1) with the nonrelativistic degenerate equation of state,

\[ (4\pi)^{1/3} GM^{2/3} \rho_c^{4/3} F(n) = K_{NR} \rho_c^{5/3}. \]

Taking the log and solving for \( \log \rho_c \),

\[ \frac{2}{3} \log M + \log \left( (4\pi)^{1/3} GF(n) \right) = \frac{1}{3} \log \rho_c + \log K_{NR} \]
\[
\log \rho_c = 2 \log M + 3 \log \left( \frac{(4\pi)^{1/3} GF(n)}{K_{NR}} \right),
\]

where the final term is a slowly varying function of \( n \), and so is almost constant.

4. Extremes of the main sequence [20 pts]

Near the "lower" end of the main sequence is a star with \( M = 0.072 \, M_\odot \), \( \log_{10} T_{\text{eff}} = 3.23 \) and \( \log_{10}(L/L_\odot) = -4.3 \). Near the "upper" end is a star with \( M = 85 \, M_\odot \), \( \log_{10} T_{\text{eff}} = 4.705 \) and \( \log_{10}(L/L_\odot) = +6.006 \).

(a) Estimate the hydrogen-burning lifetime of each star. For the 85 \( M_\odot \) star, assume that only the innermost 10% of the hydrogen is available for burning. For the 0.072 \( M_\odot \) star, assume the interior is entirely convective and all of its hydrogen becomes available for burning.

\[ t \approx 1.5 \times 10^{14} \text{ yr}, \quad t_U \approx 8.7 \times 10^5 \text{ yr}. \]

(b) Estimate the radius of each star, and the radius ratio.

\[ R_L \approx 5.7 \times 10^9 \text{ cm} \approx 0.08 \, R_\odot, \quad R_U \approx 9.1 \times 10^{11} \text{ cm} \approx 13 \, R_\odot. \]

The radius ratio may be found with less calculation by taking the ratio of the Stefan-Boltzmann law for the lower mass and higher mass object,

\[ \left( \frac{R_U}{R_L} \right)^2 = \frac{L_U}{L_L} \left( \frac{T_L}{T_U} \right)^4 = 10^{+6.006-(-4.3)+4(3.23-4.705)} \implies \frac{R_U}{R_L} \approx 160. \]

(c) Compare the Eddington luminosity of each star to its actual luminosity. For the low-mass star, use \( \kappa = 0.01 \text{ cm}^2 \text{ g}^{-1} \). For the high-mass star, assume the opacity is dominated by electron scattering. For which star is radiation pressure significant?

\[ L_{\text{Edd},L} \approx 3.6 \times 10^{38} \text{ erg s}^{-1} \approx 9.4 \times 10^4 \, L_\odot \]
\[ L_{\text{Edd},U} \approx 1.1 \times 10^{40} \text{ erg s}^{-1} \approx 2.8 \times 10^6 \, L_\odot. \]

Comparing to their physical luminosities,

\[ \frac{L_L}{L_{\text{Edd},L}} \approx 5 \times 10^{-10}, \quad \frac{L_U}{L_{\text{Edd},U}} \approx 0.36. \]

Radiation pressure is important for the high mass star, since its luminosity is within an order of magnitude of the Eddington luminosity.