1. **Stability against convection [10 pts]**

   (a) In lecture, we derived the condition
   
   \[
   \left| \frac{dT}{dr} \right| < \frac{T}{P} \left( 1 - \frac{1}{\gamma_a} \right) \left| \frac{dP}{dr} \right|
   \]
   
   for stability against convection. Using the appropriate equation(s) of stellar structure and noting the sign of the radial gradients, show that this can be recast as a condition on the luminosity profile:
   
   \[
   L(r) < \left( 1 - \frac{1}{\gamma_a} \right)\frac{64\pi\sigma_{SB}T^4GM(r)}{3\kappaRP}
   \]

   **Solution:** We’ve derived the condition
   
   \[
   \frac{\rho}{\gamma P} \frac{dP}{dr} - \frac{d\rho}{dr} > 0
   \]
   
   for stability against convection. Using the ideal gas law \( P = \rho kT/\mu m_p \), we can calculate \( d\rho/dr \), and find
   
   \[
   \frac{dp}{dr} = \frac{\rho}{P} \frac{dP}{dr} - \frac{\rho}{T} \frac{dT}{dr}.
   \]
   
   Substituting into the condition for stability and simplifying, we obtain:
   
   \[
   \frac{dT}{dr} > \left( 1 - \frac{1}{\gamma} \right) \frac{T}{P} \frac{dP}{dr}
   \]
   
   Using the equation of radiative transport,
   
   \[
   \frac{dT}{dr} = -\frac{3\kappa R\rho L(r)}{16\pi acT^4r^2},
   \]
   
   and solving the inequality for \( L(r) \):
   
   \[
   L(r) < \frac{-16\pi acT^4r^2}{3\kappa R\rho} \frac{dP}{dr} \left( 1 - \frac{1}{\gamma} \right)
   \]
   
   And substituting in the equation for hydrostatic equilibrium, we get
   
   \[
   L(r) < \left( 1 - \frac{1}{\gamma} \right)\frac{16\pi acT^4GM(r)}{3\kappa RP}
   \]

   \[
   L(r) < \left( 1 - \frac{1}{\gamma} \right)\frac{64\pi\sigma_{SB}T^4GM(r)}{3\kappa RP}
   \]

   (b) Show that to avoid convection in a stellar region where the equation of state is that of an ideal monatomic gas, the luminosity at a given radius must be limited by
   
   \[
   L(r) < 1.22 \times 10^{-18} \frac{\mu T^3}{\kappa R\rho} M(r)
   \]
where \( \mu \) is the mean molecular weight, \( T(r) \), \( \kappa_R \) is the Rosseland mean opacity, and \( M(r) \) is the mass enclosed at radius \( r \). All quantities are measured in the appropriate cgs units.

**Solution:**
For an ideal monotomic gas, \( \gamma = \frac{5}{3} \) and \( P = \rho kT / \mu m_p \). Plugging in these expressions, we arrive at the desired result (in cgs units):

\[
L(r) < 1.22 \times 10^{-18} \frac{\mu T^3}{\kappa_R \rho} M(r)
\]

2. **Polytropes: Analytic calculations [25 pts]**

Polytropes are simple models of self-gravitating bodies, based on the assumption \( P = K \rho^{1+1/n} \). This assumption leads to a single differential equation for the density profile that can be nondimensionalized to give the Lane-Emden equation,

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left[ \xi^2 \frac{d\phi_n}{d\xi} \right] = -\phi_n
\]

where \( \phi_n \) is related to the star's density by \( \rho = \rho_c \phi_n \), and \( \xi \) is related to radius by \( r = \lambda_n \xi \) (\( \lambda_n \) is given below). Some objects can be approximated as polytropes with an appropriate choice for \( n \). For example \( n = 1 \) describes a brown dwarf or giant planet pretty well; \( n = 3/2 \) describes a white dwarf; \( n = 3 \) does OK describing the Sun.

In this problem, you will work out some useful mathematical properties of this model.

(a) Show that the total mass of a polytropic star is

\[
M = 4\pi \rho_c \lambda_n^3 \xi_1^2 \left[ \frac{d\phi_n}{d\xi} \right]_{\xi = \xi_1}
\]

The factor \( \lambda_n \) is defined as

\[
\lambda_n = \left[ (n + 1) \frac{K \rho_c^{(1-n)/n}}{4\pi G} \right]^{1/2}
\]

(you may assume this form), and \( \xi_1 \) specifies the outer radius of the star: \( \phi_n(\xi_1) = 0 \).

**Solution:**

\[
M = \int_0^R 4\pi r^2 \rho dr = \int_0^{\xi_1} 4\pi (\lambda_n \xi)^2 (\rho_c \phi^n) (\lambda_n d\xi) = 4\pi \lambda_n^3 \rho_c \int_0^{\xi_1} \xi^2 \phi^n d\xi
\]

Using the Lane-Emden equation to replace \( \phi^n \),

\[
M = 4\pi \lambda_n^3 \rho_c \int_0^{\xi_1} \xi^2 \left( -\frac{1}{\xi^2} \frac{d}{d\xi} \left[ \xi^2 \frac{d\phi_n}{d\xi} \right] \right) d\xi = -4\pi \lambda_n^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi} \left[ \xi^2 \frac{d\phi_n}{d\xi} \right] d\xi = -4\pi \lambda_n^3 \rho_c \left[ \xi^2 \frac{d\phi_n}{d\xi} \right]_{\xi = 0}^{\xi = \xi_1} = -4\pi \lambda_n^3 \rho_c \xi_1^2 \left[ \frac{d\phi_n}{d\xi} \right]_{\xi = \xi_1}
\]

\[
M = 4\pi \lambda_n^3 \rho_c \xi_1^2 \left[ \frac{d\phi_n}{d\xi} \right]_{\xi = \xi_1}
\]
The last line obtains since \[
\frac{d\phi}{d\xi} \bigg|_{\xi=\xi_1} < 0.
\]

(b) Show that the ratio of the mean density to the central density is
\[
\frac{\langle \rho \rangle}{\rho_c} = \frac{3}{\xi_1} \left| \frac{d\phi_n}{d\xi} \right|_{\xi=\xi_1}.
\]

**Solution:** Using our result from part a), the mean density of the star is,
\[
\langle \rho \rangle = \frac{M}{\frac{4}{3}\pi R^3} = \frac{4\pi \lambda_n^3 \rho_c \xi_1^2}{\frac{2}{3}\pi (\lambda_n \xi_1)^3} = \frac{3\rho_c}{\xi_1} \left| \frac{d\phi_n}{d\xi} \right|_{\xi=\xi_1}.
\]

The desired result follows directly.

(c) Show that the central pressure is
\[
P_c = \frac{GM^2}{R^4} \left[ 4\pi(n+1) \left| \frac{d\phi_n}{d\xi} \right|_{\xi=\xi_1}^2 \right]^{-1}.
\]

Notice that this justifies the scaling of central pressure with mass and radius we found using a crude order of magnitude estimate in an earlier lecture.

**Solution:** Starting from the polytropic equation of state, we have \( P_c = K \rho_c^{1+1/n} \). Solving for \( K \) using equation 2a, we find,
\[
K = \frac{4\pi G \lambda_n^2}{(n+1)\rho_c^{(1-n)/n}}.
\]

Thus,
\[
P_c = \frac{K \rho_c^{1+1/n}}{\frac{4\pi G \lambda_n^2 \rho_c^2}{(n+1)}} = \frac{4\pi G \lambda_n^2 (\rho)^2}{(n+1) \left( \frac{\rho}{\rho_c} \right)^2} = \frac{4\pi G \lambda_n^2 \frac{M}{\frac{4}{3}\pi R^3}^2}{(n+1) \left( \frac{3}{\xi_1} \left| \frac{d\phi_n}{d\xi} \right|_{\xi=\xi_1} \right)^2} = \frac{GM^2 (\lambda_n \xi_1)^2}{4\pi(n+1)R^6 \left( \left| \frac{d\phi_n}{d\xi} \right|_{\xi=\xi_1} \right)^2} = \frac{GM^2}{R^4} \left[ 4\pi(n+1) \left| \frac{d\phi_n}{d\xi} \right|_{\xi=\xi_1}^2 \right]^{-1}.
\]

(d) For a polytrope that also obeys the ideal gas equation of state, show \( \phi_n = T/T_c \), where \( T_c \) is the central temperature.

**Solution:** From the ideal gas equation of state, \( P = \frac{n}{\mu m_p} kT \), we have
\[
\frac{T}{T_c} = \left( \frac{P}{P_c} \right) \left( \frac{\rho_c}{\rho} \right) \\
= \left( \frac{\rho}{\rho_c} \right)^{1+1/n} \left( \frac{\rho_c}{\rho} \right) \quad \text{(Polytropic EOS)} \\
= \left( \frac{\rho}{\rho_c} \right)^{1/n} \\
= (\phi^n)^{1/n} \\
= \phi
\]
3. Polytropes: Numerical calculations [35 pts]

In this problem you will calculate the density structure of various polytropes, including a model of the Sun. Numerically integrate the Lane-Emden equation to find \( \phi_n(\xi) \) for polytropic indices of \( n = 1.0, 1.5, 2.0, 2.5, 3.0, \) and \( 3.5 \). One possible approach is to break up the second-order differential equation into two first-order equations,

\[
\frac{d\phi_n}{d\xi} = u, \quad \frac{du}{d\xi} = -\phi_n - \frac{2u}{\xi}.
\]

Then use a 4th-order Runge-Kutta integration scheme to find \( \phi_n(\xi) \). The boundary conditions at the center are \( u(0) = 0 \) and \( \phi_n(0) = 1 \). The surface of the star is defined by \( \phi_n(\xi_1) = 0 \).

(a) Show that near the center of the star, \( \phi_n(\xi) = 1 - \frac{1}{6} \xi^2 + \frac{n}{120} \xi^4 - \cdots \)

To do this, first show that the polynomial expansions of \( \phi_n(\xi) \) contain only even terms in \( \xi \). Then substitute such a polynomial into the Lane-Emden equation and find the first three coefficients.

**Solution:** Write \( \phi \) as a power series about the origin,

\[
\phi(\xi) = \sum_{i=0}^{\infty} a_i \xi^i.
\]

We know \( \phi(0) = 1 \), so \( a_0 = 1 \). Furthermore, we know \( \phi'(0) = 0 \), so \( a_1 = 0 \). Note that the left hand side of the Lane-Emden equation reduce \( \phi(\xi) \) by two powers of \( \xi \). Therefore, \( a_1 = 0 \) also implies \( a_3 = 0 \). With some thought, we see there can be no odd powers of \( \xi \) in the series. The power series is then

\[
\phi(\xi) = 1 + a_2 \xi^2 + a_4 \xi^4 + a_6 \xi^6 + \ldots
\]

Let’s proceed by pluggin this into the Lane-Emden equation.

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\phi}{d\xi} \right) = \frac{1}{\xi^2} \frac{d}{d\xi} \left[ \xi^2(2a_2\xi^2 + 4a_4\xi^4 + 6a_6\xi^6 + \ldots) \right] = \frac{1}{\xi^2} \frac{d}{d\xi} \left( 2a_2\xi^3 + 4a_4\xi^5 + 6a_6\xi^7 + \ldots \right) = \frac{1}{\xi^2} (6a_2\xi^2 + 20a_4\xi^4 + 42a_6\xi^6 + \ldots) = 6a_2 + 20a_4\xi^2 + 42a_6\xi^4 + \ldots = -\phi^n = -(1 + na_2\xi^2 + \ldots).
\]

By equating powers of \( \xi \), we can read off \( a_2 = -1/6 \) and \( a_4 = -na_2/20 = n/120 \). So \( \phi \) becomes

\[
\phi_n(\xi) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + \ldots
\]

(b) Plot the dimensionless temperature \( \phi_n(\xi) \) and the dimensionless density \( \phi_n^p(\xi) \) for all 6 values of \( n \). It would be best to overlay all the temperature plots on a single set of axes, and all the density plots on another.

**Solution:** For an example of how to integrate the Lane-Emden equation numerically in *Mathematica*, see the [notebook on the web site](#). The plots of density and temperature are reproduced here in Fig. [1].

(c) Compute for each model the dimensionless potential energy \( \Omega \equiv E_{\text{grav}}/(-GM^2/R) \) and the dimensionless moment of inertia \( k \equiv I/MR^2 \). Tabulate \( \xi_1, -(d\phi_n/d\xi)|_{\xi_1}, \Omega, \) and \( k \) for each of the 6 polytropic models.
Figure 1: Dimensionless temperature and density for different polytropic indices. Note that the horizontal scales differ. (Prob. 3b).
**Solution:** We must write $\Omega$ in terms of integrals of $\phi_n$ to be able to compute it. Starting from the definition,

$$\Omega = \left( \frac{R}{-GM^2} \right) \int_0^R \rho \left( \frac{-GMr}{r} \right) 4\pi r^2 dr = \frac{4\pi \xi_1^3 \rho_c}{M^2} \int_0^{\xi_1} \phi_n^M \xi d\xi$$

where $M_r$ is the mass interior to radius $r$ and the second equality is just rewriting the integral in terms of $\xi$, with $M_\xi = M_{r=\lambda_\xi}$. Next, calculate $M_\xi$,

$$M_\xi = \int_0^{\lambda_\xi} \rho 4\pi r^2 dr = 4\pi \lambda_\xi^3 \rho_c \int_0^{\xi_1} \phi_n^M \xi^2 d\xi \equiv 4\pi \lambda_\xi^3 m_\xi,$$

where we have defined $m_\xi \equiv \int_0^\xi \phi_n^M \xi^2 d\xi = -\xi^2 \frac{d\phi_n}{d\xi}$.

Now the total mass is $M_r = M_{\xi=\xi_1} = 4\pi \lambda_\xi^3 \rho_c m_\xi$. Plugging this in to Eq. (3c), we find

$$\Omega = \frac{\xi_1}{m_\xi} \int_0^{\xi_1} \phi_n^M m_\xi d\xi.$$

This can be numerically integrated from the solution. This is performed in the notebook on the web site.

Now, for the moment of inertia, calculate

$$k = \frac{1}{MR^2} \int_0^R \frac{2}{3} r^2 4\pi r^2 \rho dr = \frac{2}{3m_\xi \xi_1^4} \int_0^{\xi_1} \xi^4 \phi_n^M d\xi.$$

The first equality is taking the definition of $k$ and integrating the moments of inertia of a series of concentric shells having moments $dI = \frac{2}{3} r^2 dm$ with $dm = 4\pi r^2 \rho dr$. The second equality is plugging in $R = \lambda_\xi \xi_1$, $M = 4\pi \lambda_\xi^3 \rho_c m_\xi$, and changing variables in the integral to $\xi$. This too is computed in the notebook.

All of the values are tabulated in Table 1.

<table>
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<th>$n$</th>
<th>$\xi_1$</th>
<th>$-\frac{d\phi_n}{d\xi}$</th>
<th>$\Omega$</th>
<th>$k$</th>
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<td>1.20</td>
<td>0.112</td>
</tr>
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<td>1.50</td>
<td>0.075</td>
</tr>
<tr>
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<td>9.52</td>
<td>0.020</td>
<td>2.00</td>
<td>0.046</td>
</tr>
</tbody>
</table>

Table 1: Values for polytropic models with index $n$: scaled radius $\xi_1$, temperature slope at surface $-(d\phi_n/d\xi)|_{\xi_1}$, dimensionless binding energy $\Omega$, and dimensionless moment of inertia $k$ (Prob. 3c)

Next, you will use an $n = 3$ polytrope as a model of the Sun. One purpose of this exercise is to practice re-dimensionalizing your dimensionless solution. Another is to perform some order-of-magnitude checks on the applicability of this model.

Set the dimensional scales using the “known” values for the central density, temperature, and hydrogen mass fraction: $\rho_c = 158$ g cm$^{-3}$, $T_c = 15.7 \times 10^6$ K, $X = 0.6$.

(c) How do the total mass and radius of the model star compare to the actual Sun’s mass and radius?

**Solution:** One needs to calculate $\lambda$ for the sun. Recall the definition,

$$\lambda_n^2 = \frac{(n + 1) K \rho_c^{\frac{1}{n-1}}}{4\pi G} = \frac{(n + 1) P_c}{4\pi G \rho_c^2}$$
where the second equality comes from the polytropic equation of state \( P = K \rho^\gamma = K \rho^{\frac{4}{\gamma} + 1} \). Now use the ideal gas law, \( P = \frac{\rho kT}{\mu m_p} \). Remember that

\[
\frac{1}{\mu} \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z \approx 2X + \frac{3}{4}(1 - X)
\]

where \( X, Y, Z \) are respectively the hydrogen, helium, and metal mass fraction. We assume that the metallicity is low, i.e. that all of the mass which is not hydrogen is helium. Plugging in these numbers gives a value for \( \lambda_3 \approx 7.97 \times 10^9 \text{ cm} \approx 0.115 R_\odot \).

Then \( R_\star = \lambda_3 \xi_1 \approx 0.79 R_\odot \), and \( M_\star = 4\pi \rho_c \lambda_3^2 \xi_1^3 \left| \frac{d\phi}{d\xi} \right|_{\xi=\xi_1} \approx 1.0 M_\odot \). An \( n = 3 \) polytrope with the sun’s central properties predicts the solar mass and radius to within \( \sim 20\% \).

(d) Plot the following quantities as a function of \( r/R_\odot \) (with \( R_\odot \) being the radius of the model star): (i) \( \log_{10} T \) with \( T \) in Kelvin; (ii) \( \log_{10} \rho \) with \( \rho \) in g cm\(^{-3}\).

Solution: The dimensionful quantities are simply \( \rho = \rho_c \phi_n^3 \) and \( T = T_c \phi_n^3 \). These are calculated in the notebook and are plotted in Fig. 2.

(e) Compute the implied nuclear luminosity of the polytropic model. Take the nuclear energy generation rate per unit volume to be

\[
\epsilon_V = (2.46 \times 10^6) \rho^2 X^2 T_6^{-2/3} \exp(-33.81 T_6^{-1/3}) \text{ erg s}^{-1} \text{ cm}^{-3},
\]

where \( \rho \) is in g cm\(^{-3}\), \( T_6 \) is the temperature in units of \( 10^6 \text{ K} \), and \( X = 0.6 \) is the hydrogen mass fraction. First, write the calculation as the product of a dimensioned constant and a dimensionless integral involving \( \phi_n \) and \( \xi \). (For the \( T_c \) inside the integral you can use \( 15.7 \times 10^6 \text{ K} \).) Show the value of your constant, and the form of the dimensionless integral. Then, evaluate the nuclear luminosity in erg s\(^{-1}\). Compare to the actual luminosity of \( 3.839 \times 10^{33} \text{ erg s}^{-1} \).

Solution: Plugging \( \rho_c, T_c \) and \( X \) into \( \epsilon_V \) gives

\[
\epsilon = (3.53 \times 10^9 \text{ erg s}^{-1} \text{ cm}^{-3}) \phi_3^{16/3} \exp(-13.5/\phi_3^{1/3}).
\]

This may be integrated over the volume to calculate the total luminosity,

\[
L = \int_0^R \epsilon 4\pi r^2 dr = 4\pi \lambda_3^3 \int_0^{\xi_1} \epsilon \xi^2 d\xi.
\]

Numerically integrating the luminosity (see the notebook online) and plugging in the value of \( \lambda \) from part a) \( (\lambda \approx 7.66 \times 10^9 \text{ cm}) \) gives

\[
L_{\odot,\text{poly}} \approx (2.25 \times 10^{40} \text{ erg s}^{-1}) \int_0^{\xi_1} \phi_3^{16/3} \exp(-13.5/\phi_3^{1/3}) \xi^2 d\xi \approx 6.5 \times 10^{33} \text{ erg s}^{-1}.
\]

This is about a factor of two greater than the measured solar luminosity of \( L_{\odot,\text{meas}} \approx 3.84 \times 10^{33} \text{ erg s}^{-1} \).
Figure 2: Dimensionful temperature and density for a polytropic model of the sun with $n = 3$. (Prob. 3d).
4. Overcoming the Coulomb barrier [15 pts]

In this problem you will show that classical mechanics predicts that hydrogen fusion cannot happen in the Sun.

(a) Suppose two protons approach each other with equal speeds. What is the minimum speed needed to overcome the Coulomb barrier and collide, neglecting quantum effects? Take the radius of a proton to be \( 1 \text{ fermi} = 10^{-13} \text{ cm} \).

**Solution:** Here we take the interaction potential to be the Coulomb potential \( V(r) = Z_1 Z_2 e^2 / r \) down to the point of contact at about 2 fm. Take a pair of interacting particles to both have velocity \( v \) at infinity – then their combined energy is \( E = m_p v^2 \). To overcome the Coulomb barrier, this energy must at least be equal to the potential at the radius of 2 fm. That is,

\[
V(2 \text{ fm}) = m_p v^2 \implies v_{cl} = \sqrt{\frac{1}{m_p} V(2 \text{ fm})} = \sqrt{\frac{Z_1 Z_2 e^2}{m_p 2 \text{ fm}}}.
\]

For two protons, \( Z_1 = Z_2 = +1 \). Plugging in numbers, we find, classically, the individual velocities must be \( v_{cl} = 8 \times 10^6 \text{ m/s} \) or 4\% of the speed of light.

(b) Assuming the proton speeds obey a Maxwell-Boltzmann distribution

\[
p(v) = \sqrt{\frac{2}{\pi}} \left( \frac{m_p}{kT} \right)^{3/2} v^2 \exp\left(-\frac{m_p v^2}{2kT} \right)
\]

with \( T = 15.7 \times 10^6 \text{ K} \) (the central temperature of the Sun), what is the most probable speed? How does it compare to your answer to Prob. 4a?

**Solution:** First we give a reminder of how the Maxwell-Boltzmann distribution can be derived. In thermal equilibrium, the density of a given microstate is proportional to the Boltzmann factor of that state, which depends on the energy. Here, the energy is \( E = \frac{1}{2} m_p \vec{v} \cdot \vec{v} \), with \( \vec{v} \) the velocity. The phase-space density is therefore proportional to

\[
p(\vec{v}) d\vec{v} \propto \exp\left(-\frac{1}{2} m_p \vec{v} \cdot \vec{v} / 2kT \right) d\vec{v} \implies p(v) dv \propto v^2 \exp\left(-\frac{m_p v^2}{2kT} \right) dv,
\]

where \( v = |\vec{v}| \) and where the second equality comes from the assumption of isotropy and the volume element in 3-dimensional velocity space. One can integrate over \( v \) from 0 to \( \infty \) to find the normalization constant; one finds

\[
p(v) dv = \sqrt{\frac{2}{\pi}} \left( \frac{m_p}{kT} \right)^{3/2} v^2 \exp\left(-\frac{m_p v^2}{2kT} \right) dv.
\]

The most probable velocity, defined by the maximum value of \( p(v) \), is found by solving \( p'(v_{\text{peak}}) = 0 \). This gives \( v_{\text{peak}} = \sqrt{2kT/m_p} \).

For a temperature of \( 15.7 \times 10^6 \text{ K} \), the most probable velocity is \( v_{\text{peak}} \approx 5 \times 10^5 \text{ m/s} \). Comparing this to the classical velocity from Prob. 4a, we see that \( v_{cl}/v_{\text{peak}} \approx 16 \), which is not very close.

(c) You might wonder whether a small minority of protons in the tail of the M-B distribution could fuse. Give an order of magnitude estimate for the number of protons in the Sun, and for the number of those protons that are energetic enough to fuse. You may find it useful to know that for large \( u_0 \),

\[
\frac{4}{\sqrt{\pi}} \int_{u_0}^{\infty} u^2 e^{-u^2} du \approx \frac{2}{\sqrt{\pi}} u_0 e^{-u_0^2},
\]

**Solution:** If we treat the sun as being composed completely of hydrogen, then the number of protons is simply \( N_p = M_{\odot} / m_p \approx 10^{57} \).

The number of protons faster than the classical velocity needed to fuse is the total number of protons times the probability that a proton has the requisite velocity,

\[
N_{\text{cl. fuse}} = N_p \int_{v_{cl}}^{\infty} p(v) dv = N_p \int_{v_{cl}}^{\infty} p(v) dv
\]
Plug in Eq. (4b) and nondimensionalize, setting \( u = v/v_{\text{peak}} = v\sqrt{m_p/2kT} \).

\[ P(v > v_{cl}) = \frac{4}{\sqrt{\pi}} \int_{u_{cl}}^{\infty} u^2 e^{-u^2} du, \]

where \( u_{cl} \equiv v_{cl}/v_{\text{peak}} \). We must turn this integral into one which we know how to do in terms of special functions. One trick is to write it as a derivative of a Gaussian integral,

\[ P(v > v_{cl}) = -\frac{d}{d\lambda} \left( \frac{2}{\sqrt{\pi}} \int_{u_{cl}}^{\infty} e^{-\lambda^2 u^2} du \right)_{\lambda=1}. \]

One can now look up the definition of the complementary error function

\[ \text{erfc}(z) = 1 - \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2} dt. \]

Change variables again to \( t = \lambda u \) to get

\[ P(v > v_{cl}) = -\frac{d}{d\lambda} \left( \frac{1}{\lambda^{\frac{3}{2}}} \text{erfc}(\lambda u_{cl}) \right)_{\lambda=1} = \frac{1}{\lambda^2} \text{erfc}(\lambda u_{cl}) + \frac{2}{\lambda^{\frac{3}{2}}} e^{-\lambda^2 u_{cl}^2} u_{cl} \bigg|_{\lambda=1} = \text{erfc}(u_{cl}) + \frac{2}{\sqrt{\pi}} e^{-u_{cl}^2} u_{cl}. \]

One must now numerically evaluate this with \( u_{cl} = v_{cl}/v_{\text{peak}} \approx 16 \), which is problematic since \( u_{cl} \gg 1 \), so \( P(v > v_{cl}) \) is going to be very, very small. This can be made simpler by using an asymptotic expansion for \( \text{erfc} \) (see e.g. Abromowitz and Stegun)

\[ \text{erfc}(z) = \frac{e^{-z^2}}{z \sqrt{\pi}} \left( 1 + \mathcal{O}(z^{-2}) \right). \]

We see that the second term in \( P(v > v_{cl}) \) is more important than the \( \text{erfc} \) term. Taking the log,

\[ \ln P(v > v_{cl}) \approx \ln \frac{2}{\sqrt{\pi}} + \ln u_{cl} - u_{cl}^2 \approx -263. \]

or \( P(v > v_{cl}) \approx 10^{-114} \).

Comparing this with the number of protons in the sun,

\[ N_{\text{cl}, \text{fuse}} = N_p P(v > v_{cl}) \approx 10^{57} \times 10^{-114} \approx 10^{-57}. \]

There are no protons in the sun moving fast enough to classically overcome the Coulomb barrier in a collision with an equally energetic proton.
5. Tunneling through the Coulomb barrier [10 pts]

Now we compute the quantum-mechanical probability for two nuclei to tunnel through the Coulomb barrier. Let the two nuclei have charges \( Z_1, Z_2 \) and atomic masses \( A_1, A_2 \). Assume the interaction potential between the two nuclei is \( Z_1 Z_2 e^2 / r \), i.e., ignore the nuclear force until the nuclei are essentially touching (at a separation of a few fermi).

Calculate the tunneling probability using the WKB approximation,

\[
\text{Trans. Prob.} \approx \exp \left[ -2 \int_{r_{\min}}^{r_{\max}} \sqrt{\frac{2 \mu (V - E)}{\hbar^2}} \, dr \right],
\]

where \( \mu \) is the reduced mass (not the mean molecular weight) and \( r_{\max} \) is the classical turning point, and you may approximate \( r_{\min} \approx 0 \). You should find that the probability varies as \( \exp(-bE^{-1/2}) \) where \( b \) is a constant.

**Solution:** The classical turning point is found where

\[
E = V(r_{\max}) \implies r_{\max} = \frac{Z_1 Z_2 e^2}{E}.
\]

Let us non-dimensionalize the integral in terms of this length by choosing the dimensionless length coordinate \( \xi = r/r_{\max} \). Plugging this and the potential into the integral gives

\[
\int_0^{r_{\max}} \sqrt{\frac{2 \mu}{\hbar^2}} (V - E) \, dr = \frac{\sqrt{2 \mu}}{\hbar} \int_0^{r_{\max}} \sqrt{\frac{Z_1 Z_2 e^2}{r_{\max} \xi} - E} r_{\max} \, d\xi
\]

\[
= \frac{\sqrt{2 \mu}}{\hbar} r_{\max} \int_0^{1} \sqrt{E \left( \frac{1}{\xi} - 1 \right)} \, d\xi
\]

\[
= \frac{\sqrt{2 \mu} Z_1 Z_2 e^2}{\hbar \sqrt{E}} \int_0^{1} \sqrt{\left( \frac{1}{\xi} - 1 \right)} \, d\xi.
\]

By non-dimensionalizing the integral, we found the energy dependence without actually doing the integral. One can then integrate to find \( \int_0^1 \sqrt{\xi^{-1} - 1} \, d\xi = \pi/2 \).

Putting this back into the WKB formula gives

\[
\text{Trans. Prob.} \approx \exp \left( -bE^{-1/2} \right), \quad b = \frac{\pi \sqrt{2 \mu} Z_1 Z_2 e^2}{\hbar}.
\]
6. Nuclear binding energies [10 pts]

The \( Q \) value of a nuclear reaction is the amount of energy released (or absorbed) in the reaction; \( Q > 0 \) means energy is released. Compute the \( Q \) value in MeV for each of the following nuclear reactions.

(a) \( ^{12}\text{C} + ^{12}\text{C} \rightarrow ^{24}\text{Mg} \)
(b) \( ^{12}\text{C} + ^{12}\text{C} \rightarrow ^{16}\text{O} + ^{2}\text{He} \)
(c) \( ^{19}\text{F} + ^{1}\text{H} \rightarrow ^{16}\text{O} + ^{4}\text{He} \)
(d) \( ^{1}\text{H} + ^{1}\text{H} \rightarrow ^{2}\text{H} + e^+ + \nu \)
(e) \( ^{15}\text{N} + ^{1}\text{H} \rightarrow ^{12}\text{C} + ^{4}\text{He} \)

A useful source for this problem is the NIST table of atomic weights and compositions:
http://physics.nist.gov/cgi-bin/Compositions/stand_alone.pl
You’ll need to convert from atomic mass units to MeV.

**Solution:**

(a) \( ^{12}\text{C} + ^{12}\text{C} \rightarrow ^{24}\text{Mg} \) \( Q = 13.9559 \text{ MeV} \)
(b) \( ^{12}\text{C} + ^{12}\text{C} \rightarrow ^{16}\text{O} + ^{2}\text{He} \) \( Q = -0.1121 \text{ MeV} \)
(c) \( ^{19}\text{F} + ^{1}\text{H} \rightarrow ^{16}\text{O} + ^{4}\text{He} \) \( Q = 8.1290 \text{ MeV} \)
(d) \( ^{1}\text{H} + ^{1}\text{H} \rightarrow ^{2}\text{H} + e^+ + \nu \) \( Q = 1.4453 \text{ MeV} \)
(e) \( ^{15}\text{N} + ^{1}\text{H} \rightarrow ^{12}\text{C} + ^{4}\text{He} \) \( Q = 4.9728 \text{ MeV} \)

All the reactions are exothermic except for (b).

7. Temperature dependence of thermonuclear reaction rates [20 pts]

Next you will derive the leading-order dependence of a thermonuclear reaction rate on temperature. For the reaction \( \text{A} + \text{B} \rightarrow \text{C} \) which liberates an energy \( Q \), the rate of energy production per unit volume can be written

\[
\epsilon_V \ [\text{erg s}^{-1} \text{ cm}^{-3}] = Q n_A n_B \langle \sigma v \rangle,
\]

where \( \langle \sigma v \rangle \) is the product of the cross-section and relative velocity, averaged over the Maxwell-Boltzmann distribution of relative energies. The cross-section may be written

\[
\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}},
\]

where the exponential factor arises from the tunneling probability, the \( E \) in the denominator arises from the inverse square of the de Broglie wavelength, and \( S(E) \) represents the purely nuclear energy dependence.

(a) Show that \( \epsilon_V \) is proportional to

\[
\epsilon_V \propto Q n_A n_B T^{-3/2} \int_0^\infty S(E) \exp \left[-(bE^{-1/2} + E/kT)\right] dE,
\]

with the same constant \( b \) that appeared in the previous problem.

**Solution:** We need to show what exactly is meant by \( \langle \sigma v \rangle \). This is supposed to be integrating over all of phase space in a thermal state. We need the probability density function in phase space. The density of some microstate will be proportional to the Boltzmann factor of that state, which depends on the energy. Here, the energy is \( E = \frac{1}{2} \mu \vec{v} \cdot \vec{v} \), with \( \mu \) the reduced mass for the collision \( \text{A} + \text{B} \), and \( \vec{v} \) the relative velocity. The density is therefore proportional to

\[
p(\vec{v})d\vec{v} \propto \exp \left(-\mu \vec{v} \cdot \vec{v} / 2kT \right) d\vec{v} \quad \Rightarrow \quad p(v)dv \propto v^2 \exp \left(-\mu v^2 / 2kT \right) dv,
\]

where \( v = |\vec{v}| \) and where the second equality comes from the assumption of isotropy and the volume element in 3-dimensional velocity space.
We would like to do the integral in energy space, so we use \( p(v)dv = p(E)dE \) and the Jacobian between velocity and energy, \( dE = \mu vdv \) (from the energy above). Putting it together, the density is proportional to

\[
p(E)dE = p(v)dv \propto E^{1/2} e^{-E/kT} dE.
\]

One can integrate over \( E \) from 0 to \( \infty \) to find the normalization constant, which is important since it contains the temperature. This gives

\[
p(E)dE = 2 \sqrt{\frac{T}{\pi}} (kT)^{-3/2} E^{1/2} e^{-E/kT} dE.
\]

Finally, we can write what is meant by \( \langle \sigma v \rangle \).

\[
\langle \sigma v \rangle = \int_0^\infty \sigma(E)v(E)p(E)dE
\]

\[
= \int_0^\infty \frac{S(E)}{E} e^{-bE^{-1/2}} \cdot \sqrt{\frac{2E}{\mu}} \cdot 2 \sqrt{\frac{T}{\pi}} (kT)^{-3/2} E^{1/2} e^{-E/kT} dE
\]

\[
\propto T^{-3/2} \int_0^\infty S(E) \exp \left[ -(bE^{-1/2} + E/kT) \right] dE.
\]

Since the integral cannot be done analytically, we will need to make an approximation. The quantity \( (bE^{-1/2} + E/kT) \) in the exponent is a falling function of \( E \) plus a rising function of \( E \). The minimum in this quantity corresponds to a maximum in the value of the exponential. For most situations in stellar interiors, the only significant contributions to the integral occur when \( (bE^{-1/2} + E/kT) \) is near its minimum \( E_0 \). Therefore, we will expand the exponent in a Taylor series about \( E_0 \), and we will assume that \( S(E) \) is nearly constant over the narrow range surrounding \( E_0 \).

(b) Show that the Taylor series for the exponent is of the form:

\[
-(bE^{-1/2} + E/kT) = -\frac{3b^2/3}{(4kT)^{1/3}} - f(T)(E - E_0)^2 + \ldots
\]

where \( f(T) \) is a function of the temperature.

**Solution:** Define the negative of the exponent as \( \phi(E) = (bE^{-1/2} + E/kT) \). We want to find the minimum, where \( d\phi/dE = 0 \). This is straightforward,

\[
\frac{d\phi}{dE} \bigg|_{E_0} = -\frac{b}{2E_0^{3/2}} + \frac{1}{kT} = 0 \quad \implies \quad E_0 = (bkT/2)^{2/3}.
\]

The Taylor expansion of \( \phi \) around its minimum is

\[
\phi(E) = \phi(E_0) + \frac{d\phi}{dE} \bigg|_{E_0} (E-E_0) + \frac{1}{2} \frac{d^2\phi}{dE^2} \bigg|_{E_0} (E-E_0)^2 + O((E-E_0)^3)
\]

\[
= \phi(E_0) + \frac{1}{2} \frac{d^2\phi}{dE^2} \bigg|_{E_0} (E-E_0)^2 + \ldots
\]

since, at a minimum, the first derivative vanishes. Now evaluating the functions which appear in the expansion

\[
\phi(E) = \frac{3b^2/3}{(4kT)^{1/3}} + f(T)(E - E_0)^2 + \ldots, \quad f(T) = \frac{3}{(4b)^{2/3}(kT)^{5/3}}
\]

which is the negative of the exponent appearing in the integral \( \langle \sigma v \rangle \).
(c) Complete the integration of over the Gaussian to get an expression for the temperature dependence of \( \rho \epsilon \). You may drop any numerical prefactors, but be careful not to drop any factors that depend on temperature. You should find that the result is proportional to \( e^{-B/T_{6}^{1/3}} \), where \( T_6 \) is the temperature expressed in millions of degrees K, and \( B \) is a constant. Show in particular that \( B = 42.6 (Z_1 Z_2)^{2/3} \left( \frac{A_1 A_2}{A_1 + A_2} \right)^{1/3} \), where \( Z_1 \) and \( Z_2 \) are the atomic numbers (nuclear charges) or the reactants, and \( A_1 \) and \( A_2 \) are their atomic masses (sum of neutron and proton numbers).

\[ S = \frac{\pi^2 m_p e^4}{2 \hbar^2 k (10^6 \text{ K})} \left( \frac{A_1 A_2}{A_1 + A_2} \right)^{1/3} T_{6}^{-1/3}. \]

(If in SI units, replace \( e^2 \) with \( e^2/4\pi\epsilon_0 \)). Plugging in values for the proton mass \( m_p \), the elementary charge \( e \), the reduced Planck constant \( \hbar \), and the Boltzmann constant \( k \), find

\[ \phi(E_0) = B/T_{6}^{1/3}, \quad B = 42.6 (Z_1 Z_2)^{2/3} \left( \frac{A_1 A_2}{A_1 + A_2} \right)^{1/3}. \]