1. Apparent intensity and angular resolution [10 pts].

Photons are produced in a spherical cloud of radius $R$ at a uniform rate $\Gamma$ photons cm$^{-3}$ s$^{-1}$. The cloud is a distance $d$ away. Neglect absorption of the photons, i.e., assume the medium is optically thin. A detector on Earth has an effective area $\Delta A$, and an angular acceptance of half-angle $\Delta \theta$ (i.e. the detector is sensitive to incoming photons that are arriving within a cone of half-angle $\Delta \theta$).

(a) Suppose the angular size of the source is much larger than $\Delta \theta$ (the source is completely resolved). What is the observed photon intensity toward the center of the cloud, in photons s$^{-1}$ cm$^{-2}$ sr$^{-1}$? You should find that the answer is independent of $d$ as well as the properties of the detector ($\Delta \theta$ and $\Delta A$).

Solution: First consider the general scenario (that we will eventually make more specific to apply to the unresolved and resolved case). The flux, $F$, normally incident on a detector observing an effective volume $V$ of an isotropically emitting cloud a distance $d$ away is,

$$F = \frac{VT}{4\pi d^2}.\,$$

The flux is related to the observed photon intensity, $I_\nu$, through the first angular moment.

$$F = \int I_\nu \cos \theta d\Omega \approx I_\nu \Delta \Omega$$

Above, $\cos \theta \approx 1$ since we are looking through the center of the cloud, and $\Delta \Omega \approx \pi \Delta \theta^2$ represents the range of solid angle to which the detector is sensitive. Combining these relations we find

$$I_\nu = \frac{VT}{4\pi d^2 \pi (\Delta \theta)^2}$$

Now, we narrow our focus to the resolved case. See Figure 1 for the relevant geometry. In the resolved case, we can approximate the effective volume of the cloud observed as a cylinder of length $2R$ and radius $\Delta \theta d$.

$$V \approx 2R \pi (\Delta \theta d)^2$$

This results in an observed intensity,

$$I_\nu = \frac{\Gamma}{2\pi}$$

Notice that the observed intensity is NOT a property of the detector (no dependence on $\Delta \theta$, $\Delta A$ or $d$).

(b) Now suppose the angular size of the source is much smaller than $\Delta \theta$ (the source is unresolved). What is the observed average intensity when the source is in the beam of the detector? Does it depend on the properties of the detector?

Solution: In the unresolved case, we are observing the entire cloud, $V = \frac{4}{3} \pi R^3$ (Figure 2). Using the same equation as before we have,
Notice that the observed intensity in this case *does* depend on the detector properties. If the detector has a fatter beam (higher $\Delta \theta$), or is further away from the source (higher $d$), the intensity will decrease.

**Additional Note:** In the limiting case (exactly resolved, $d\Delta \theta = R$), this result reduces to,

$$I_\nu \text{crit} = \frac{R^3 \Gamma}{3d^2 \pi (\Delta \theta)^2} \neq \frac{R \Gamma}{2\pi} \text{(from a)}$$

In this limit, the observed intensity is again not a property of the detector, but our answer differs from that in (a), since our approximation of a cylindrical volume is no longer valid.

2. **Angular diameters and effective temperatures [10 pts].**

(a) Show that if you can measure the bolometric flux $F$ and the angular diameter $\phi$ of a star, then you can determine the effective temperature $T_{\text{eff}}$ even if you do not know the distance to the star. Note, “bolometric” means “integrated over all frequencies.”

**Solution:** Let $d$ denote the distance to a star. The angular diameter of the star is $\phi = 2R_*/d$. The bolometric flux measured here on Earth is

$$F = \left( \frac{R_*}{d} \right)^2 \sigma T_{\text{eff}}^4 = \frac{1}{4} \phi^2 \sigma T_{\text{eff}}^4$$

Solving for $T_{\text{eff}}$, we obtain an expression in terms of observable quantities,

$$T_{\text{eff}} = \left( \frac{4F}{\sigma \phi^2} \right)^{1/4}$$

(b) In one recent application of this technique, astronomers used optical interferometry to measure the angular diameters of both stars in the binary system $\beta$ CrB. The results were $0.699 \pm 0.017$ mas for star A, and
0.415 ± 0.017 mas for star B, where “mas” means milli-arcseconds. The bolometric apparent magnitudes of stars A and B are 3.87 ± 0.05 and 5.83 ± 0.10, respectively. The bolometric absolute magnitude of the Sun is 4.75, and the effective temperature of the Sun is 5777 K. Use this information to calculate the effective temperatures of stars A and B. You need not calculate the uncertainties.

(In case you are curious to learn more, the reference is Bruntt et al. 2010, Astron. & Astrophys., 512, 55.)

**Solution:** The measured bolometric flux from a star is related to the star’s apparent magnitude $m$ and the sun’s absolute magnitude $M_\odot$, by

$$F = 10^{0.415(m - m_\odot)} \left( \frac{R_\odot}{10 \text{ pc}} \right)^2 \sigma T_{\text{eff}}^4 \odot$$

Plugging in values we find $T_{\text{effA}} \approx 8159$ K and $T_{\text{effB}} \approx 6743$ K, for stars A and B respectively.

3. **Saha equation and pure hydrogen [20 pts].** Consider a gas of pure hydrogen at fixed density and temperature. The ionization energy of hydrogen is $\chi_0 = 13.6$ eV. You may assume that all the hydrogen atoms (whether neutral or ionized) are in their ground energy state.

(a) Write down the Saha equation relating the number densities of neutral and ionized hydrogen ($n_0$ and $n_1$, respectively). Make reasonable approximations to use numerical values for the partition functions.

**Solution:** It’s easy enough to write down the Saha equation:

$$\frac{n_1}{n_0} = \frac{2Z_1}{n_e Z_0} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_0/kT}.$$  

The partition function for neutral hydrogen is

$$Z_0 = 2(1 + 2^2 e^{-\chi_0(1-1/2^2)/kT} + ...) \approx 2$$ for $kT << \chi_0$.

The partition function for ionized hydrogen is 1 since there are two possible orientations of the free electron’s spin relative to the spin of the proton, and we’ve already written the factor of 2 in the Saha equation. Thus we have

$$\frac{n_1}{n_0} = \frac{1}{n_e} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_0/kT}.$$
(b) To find the individual densities, further constraints are required. Reasonable constraints are charge neutrality \((n_e = n_1)\) and conservation of nucleon number \((n_1 + n_0 = n)\), where the total hydrogen number density \(n\) is a constant if the density \(\rho\) is fixed. Rewrite the Saha equation in terms of the hydrogen ionization fraction \(x = n_1/n\), eliminating \(n_1, n_0,\) and \(n_e\). Does this equation have the expected limiting behavior for \(T \to 0\) and \(T \to \infty\)?

**Solution:** The two constraints are the conservation of charge and nucleon number, which can be written: \(n_e = n_1\) and \(n = n_1 + n_0\). Writing \(x = n_1/n\), the Saha equation becomes

\[
\frac{n x}{n - n x} = \frac{1}{n x} \left( \frac{2 \pi m_e k T}{\hbar^2} \right)^{3/2} e^{-\chi_0/k T}
\]

\[
x^2 = \frac{1}{n} \left( \frac{2 \pi m_2 k T}{\hbar^2} \right)^{3/2} e^{-\chi_0/k T}
\]

From this equation we see that as \(T \to 0\), \(x \to 0\); i.e., no ionization occurs. And as \(T \to \infty\), \(x \to 1\), indicating full ionization. These are the proper limiting behaviors.

(c) Use the relation \(\rho = m_H n\) (where \(m_H = 1\ gm/N_A\), where \(N_A = 6.023 \times 10^{23}\) is Avogadro’s number) to replace \(n\) with \(\rho\). Find an expression for the half-ionized \((x = 0.5)\) path in the \(\rho-T\) plane. Plot this path on a log-log plot for densities in the interesting range from \(10^{-10} - 10^{-2}\) g cm\(^{-3}\).

**Solution:** The mass density is given by \(\rho = m_H n\), where \(m_H\) is the mass of hydrogen, 1 gm/N\(_A\). To get the half-ionization curve, set \(x = 0.5\) in the Saha equation to obtain

\[
\rho(T) = \frac{2}{N_A} \left( \frac{2 \pi m_e k T}{\hbar^2} \right)^{3/2} e^{-\chi_0/k T}.
\]

This is the half-ionization curve shown in Figure 3.
Figure 3: Half ionization curve for pure hydrogen.
4. Saha equation and pure helium [30 pts].

Consider a gas of pure helium at fixed density and temperature. The ionization energies for helium are \( \chi_0 = 24.6 \text{ eV} \) (from neutral to singly ionized) and \( \chi_1 = 54.4 \text{ eV} \) (from singly to doubly ionized). You may assume that all the helium atoms (whether neutral, singly ionized, or doubly ionized) are in their ground energy state. Let \( n_e, n_0, n_1, \) and \( n_2 \) be the number densities of, respectively, free electrons, neutral atoms, singly-ionized atoms, and doubly-ionized atoms. The total number density of neutral atoms and ions is denoted by \( n \). Define \( x_e \) as the ratio \( n_e / n \), and let \( x_i \) be \( n_i / n \) where \( i = 0, 1, 2 \). You should assume that the gas is electrically neutral.

The degeneracy factors you need for the atoms and ions are 2 for He, 4 for \( \text{He}^+ \), and 2 for \( \text{He}^{2+} \).

(a) Construct the ratios \( n_1/n_0 \) and \( n_2/n_1 \) using the Saha equation. In doing so, take care in establishing the zero points of energy for the various constituents.

**Solution:** The Saha equations are

\[
\frac{n_1}{n_0} = \frac{4}{n_e} \left( \frac{2 \pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_0/kT} \\
\frac{n_2}{n_1} = \frac{1}{n_e} \left( \frac{2 \pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_1/kT}
\]

(b) Apply charge neutrality and nucleon number conservation \((n = n_0 + n_1 + n_2)\) and recast the above Saha equations so that only \( x_1 \) and \( x_2 \) appear as unknowns. The resulting two equations have \( T \) and \( n \) [or, equivalently, \( \rho = n m_{\text{He}} = n (4 \text{gm}/N_A) \)] as parameters.

**Solution:** Charge conservation and nucleon number conservation can be written: \( n = n_0 + n_1 + n_2 \) and \( n_e = n_1 + 2n_2 \), so that the Saha equations become

\[
\frac{x_1(x_1 + 2x_2)}{1 - x_1 - x_2} = \frac{4}{n} \left( \frac{2 \pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_0/kT} \\
\frac{x_2(x_1 + 2x_2)}{x_1} = \frac{1}{n} \left( \frac{2 \pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_1/kT}
\]

(c) Simultaneously solve the two Saha equations for \( x_1 \) and \( x_2 \) for temperatures in the range \( 4 \times 10^4 \leq T \leq 2 \times 10^5 \) K. Do this for a fixed density with the three values \( \rho = 10^{-4}, \ 10^{-6}, \) or \( 10^{-8} \) g cm\(^{-3}\). You may find it more convenient to use the logarithm of your equations. Choose a dense grid in temperature because you will soon plot the results. Once you have found \( x_1 \) and \( x_2 \), also find \( x_e \) and \( x_0 \) for the same range of temperature. Note that this is a numerical exercise; you will want to use a tool like Mathematica or Matlab for this.

**Solution:** The mass density is given by \( \rho = m_{\text{He}} n = 4 \text{ gm} n/N_A \), where \( m_{\text{He}} = 4m_H = 4 \text{ gm} / N_A \) is the mass of the helium. The two Saha equations can then be written

\[
f(x_1, x_2) = \frac{x_1^2 + 2x_1x_2}{x_2} + \frac{16}{\rho N_A} \left( \frac{2 \pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_0/kT} (x_1 + x_2 - 1) = 0 \\
g(x_1, x_2) = \frac{x_1x_2 + 2x_2^2}{x_1} - \frac{4}{\rho N_A} \left( \frac{2 \pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_1/kT} x_1 = 0
\]

This set of coupled, nonlinear equations can be solved using nearly any multidimensional root-finding technique. I used a simple Newton-Raphson method, which works similarly to the Newton-Raphson method for solving a single equation. In this method, a guess is made for \( x = (x_1, x_2) \) and then the guess is refined using

\[
x_{\text{new}} = x_{\text{old}} + \delta x,
\]
where
\[ \delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \begin{bmatrix} \partial f/\partial x_1 & \partial f/\partial x_2 \\ \partial g/\partial x_1 & \partial g/\partial x_2 \end{bmatrix} \begin{bmatrix} -f \\ -g \end{bmatrix} \]

Here is the MATLAB code that implements this procedure:

```matlab
function heionization (rho, Tstart, Tspace, Tfinal)
% Plots the abundance of neutral, slightly ionized, and doubly ionized
% helium, as well as the electrons.
% % x0 = n0/n, x1 = n1/n, x2 = n2/n, xe = ne/n, where n = rho * Na/4
% % The temperature range is Tstart:Tspace:Tfinal % Constants
Na = 6.02214e23;
me = 9.1094e-28;
kB = 1.3807e-16;
h = 6.6261e-27;
kBeV = 8.617e-5;
chi0 = 24.6 ;
chi1 = 54.4 ;
x1 = [];
x2 = [];

%Loop through temperature for T = Tstart:Tspace:Tfinal

% Define A and B:
A = 16/Na/rho*(2*pi*me*kB*T/hˆ 2). ˆ (3/2). *exp(-chi0/kBeV./T);  % (3/2).
B = 4/Na/rho*(2*pi*me*kB*T/hˆ 2). ˆ (3/2). *exp(-chi1/kBeV./T);

% Dumb initial guesses:
x1guess = 0.5;
x2guess = 0.5;
% Calculate f and g; correct until within tolerance 0.0001;

f = x1guess .^ 2 + 2 * x1guess * x2guess + A * (x1guess + x2guess - 1);
g = x1guess * x2guess + 2 * x2guess .^ 2 - B * x1guess;
err = max( abs(f), abs(g));
while (err > 0.0001)
    M = [2*x1guess + 2*x2guess + A 2*x1guess + A; x2guess - B x1guess + 4 * x2guess];
    dx = inv(M) * [-f -g];
    x1guess = x1guess + dx(1);
    x2guess = x2guess + dx(2);
end
```

7
f = x1guess^2 + 2*x1guess*x2guess + A*(x1guess+x2guess-1);
g = x1guess*x2guess + 2*x2guess^2 - B*x1guess;
err = max(abs(f), abs(g));
end

% Add solutions to list;
x1 = [x1 x1guess];
x2 = [x2 x2guess];
end

% Calculate xe and x0;
xe = x1 + 2*x2;
x0 = 1 - x1 - x2;
Tvals = Tstart:Tspace:Tfinal;
plot(Tvals, xe);
hold on;
plot(Tvals, x0, ':');
plot(Tvals, x1, '--');
plot(Tvals, x2, '-.');
return;

(d) Plot all your xs as a function of temperature for your chosen value of ρ. (Plot x0, x1, and x2 on the same graph.) Identify the transition temperatures (half-ionization) for the two ionization stages.

**Solution:** Figures 4-6 show the ionization fraction for three densities. The half-ionization temperatures (defined as the lowest temperature at which the ionization fraction of a species is 0.5) are:

- ρ = 10^{-4} g/cm^3: T(x1 = 0.5) = 3.2 \times 10^4 K, T(x2 = 0.5) = 8.1 \times 10^4 K
- ρ = 10^{-6} g/cm^3: T(x1 = 0.5) = 2.2 \times 10^4 K, T(x2 = 0.5) = 5.4 \times 10^4 K
- ρ = 10^{-8} g/cm^3: T(x1 = 0.5) = 1.7 \times 10^4 K, T(x2 = 0.5) = 4.0 \times 10^4 K

5. Limb darkening [20 pts].

In this problem you will derive a relation between the measured limb darkening of a star, and the source function of its photosphere. Let the intensity of the stellar disk be I_ν(r), where r is the distance from the center of the stellar disk in units of the stellar radius (i.e. r = 0 at the center, and r = 1 at the limb).

(a) Instead of r it is traditional to express I_ν as a function of μ = √1 − r^2. Show that μ = cos θ, where θ is the angle between the line of sight and the normal to the stellar surface.

**Solution:** Refer to Fig. 7 for the geometry of the problem. The two rays toward Earth are parallel. Take the normal to the star’s surface at some reduced radius r and continue it through to the center of the circle. This line intersects the two parallel rays, which is why the two angles labeled θ in the figure are the same angle.

Construct the right triangle shown in the figure. The hypotenuse is 1 and the height is r. Therefore the base is √1 − r^2, from the Pythagorean theorem. From the definition of the cosine, cos θ = √1 − r^2 = μ.

(b) We want an expression for the intensity at the stellar surface in terms of the source function. Start from the the radiative transfer equation for a plane-parallel atmosphere. Show that for an upward-propagating ray coming from far below to the top surface, the formal solution is

\[ I_ν(μ) = \int_0^∞ dτ_ν \frac{S_ν(τ_ν)}{μ} e^{-τ_ν/μ}, \]  

(1)
Figure 4: Ionization fractions for pure helium, $\rho = 10^{-4}$ g/cm$^{-3}$.
Figure 5: Ionization fractions for pure helium, $\rho = 10^{-6} \text{ g/cm}^3$.
Figure 6: Ionization fractions for pure helium, $\rho = 10^{-8} \text{ g/cm}^3$. 

$$p = 10^{-8} \text{ g/cm}^3$$
where $\tau_\nu$ is the vertical optical depth.

**Solution:** Let us take this opportunity to remind ourselves of the distinction between optical depth and vertical optical depth. Start with the form of the radiative transfer equation in terms of optical depth,

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu,$$

where $S_\nu = j_\nu/\alpha_\nu$ is the source function, and $\tau_\nu$ is the actual optical depth. Define the new variable

$$d\xi_\nu \equiv -dz = -d\tau_\nu \cos \theta$$

which is the vertical component along the ray, and the sign is chosen so that the “vertical optical depth” starts at 0 at the top (larger $z$) and increases as one goes down (smaller $z$). In terms of $\xi_\nu$, the RTE becomes

$$I_\nu - S_\nu = \cos \theta \frac{dI_\nu}{d\xi_\nu} = \mu \frac{dI_\nu}{d\xi_\nu}.$$

Multiply by the integrating factor $e^{-\xi_\nu/\mu}$. Collect all the terms which have $I_\nu$,

$$e^{-\xi_\nu/\mu}S_\nu = e^{-\xi_\nu/\mu} \left( I_\nu - \mu \frac{dI_\nu}{d\xi_\nu} \right) - \mu \frac{dI_\nu}{d\xi_\nu} \left( e^{-\xi_\nu/\mu} I_\nu \right).$$

Divide through by $\mu$ and now the right hand side is a total derivative. Integrate over $d\xi_\nu$ from 0 to $\infty$,

$$\int_0^\infty d\xi_\nu e^{-\xi_\nu/\mu} \frac{S_\nu}{\mu} = - \int_0^\infty d\xi_\nu \frac{d}{d\xi_\nu} \left( e^{-\xi_\nu/\mu} I_\nu \right) = -e^{-\xi_\nu/\mu} I_\nu \bigg|_{\xi_\nu=0}^{\xi_\nu=\infty} = I_\nu(\xi_\nu = 0, \mu).$$

By an unfortunate convention, the symbol $\tau_\nu$ is used instead of $\xi_\nu$, but please be aware that the meaning is the vertical optical depth.

(c) Suppose the (unknown) source function can be represented by a polynomial,

$$S_\nu(\tau_\nu) = a_0 + a_1 \tau_\nu + a_2 \tau_\nu^2 + \cdots + a_n \tau_\nu^n. \quad (2)$$

Show that under this assumption the emergent intensity is given by

$$I_\nu(\mu) = a_0 + a_1 \mu + 2a_2 \mu^2 + \cdots + (n!)a_n \mu^n, \quad (3)$$

using the definite integral $\int_0^\infty x^n \exp(-x)dx = n!$. In this way the measured limb-darkening law can be used to determine the source function, and therefore the temperature stratification for an LTE atmosphere.

**Solution:** Substitute Eq. (2) into Eq. (1) and change variables to $x = \tau_\nu/\mu$. This gives

$$I_\nu(\mu) = \int_0^\infty dx \sum_{i=0}^n a_i(x\mu)^i e^{-x}.$$  

Each term in the sum may be integrated using the given definite integral, giving Eq. (3).
(d) Show that for a gray LTE atmosphere, the predicted limb darkening law for the wavelength-integrated intensity at the stellar surface is
\[ \frac{I(\theta)}{I(0)} = \frac{2}{5} + \frac{3}{5} \cos \theta. \]

**Solution:** For a gray atmosphere (\( \tau_\nu = \tau \)), and in a plane parallel atmosphere with no energy generation (\( dF/dz = 0 \), flux is conserved from layer to layer), we found in class that
\[ S = \langle I \rangle, \]
where \( \langle f \rangle = \frac{1}{2} \int_{-1}^{1} f d\mu \) is an angular average. This yields the integro-differential equation
\[ \frac{1}{2} \int_{-1}^{1} I d\mu = I - \mu \frac{dI}{d\tau}. \]
We simply state the solution,
\[ S = \frac{3F}{4\pi} [\tau + q(\tau)] \approx \frac{3F}{4\pi} \left( \tau + \frac{2}{3} \right). \]
This satisfies the assumption of Prob. 5c with \( a_1 = \frac{3F}{4\pi}, a_0 = \frac{2}{3} a_1, \) and all other \( a_n \)'s vanishing. Putting this into the result from Prob. 5c find \( I(\mu) = a_0 + a_1 \mu, \) where \( \mu = \cos \theta. \) Evaluating the ratio gives
\[ \frac{I(\theta)}{I(0)} = \frac{a_0 + a_1 \cos \theta}{a_0 + a_1} = \frac{a_1(2/3 + \cos \theta)}{a_1(2/3 + 1)} = \frac{2}{5} + \frac{3}{5} \cos \theta. \]
6. Radiative transfer in spherical coordinates [20 pts].

After this week’s classes you should be familiar with the radiative diffusion equation for a plane-parallel atmosphere, an appropriate model for a thin photosphere. In this problem you will repeat those steps for a spherical atmosphere, as appropriate for the bulk of a star. We will assume the star is spherically symmetric and that consequently \( I_\nu = I_\nu(r, \theta) \), where \( r \) is the radial coordinate and \( \theta \) is the angle of a ray relative to the local radius vector (and \textit{not} the polar angle referring to the position with respect to the stellar center). See Fig. [8]

(a) Use the chain rule,

\[
\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r} \frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta} \frac{d\theta}{ds}, \tag{4}
\]

to show that the radiative transfer equation (RTE) can be written

\[
\cos \theta \frac{\partial I_\nu}{\partial r} - \sin \theta \frac{\partial I_\nu}{\partial \theta} + \rho \kappa_\nu I_\nu - j_\nu = 0, \tag{5}
\]

In this expression, \( \kappa_\nu \) is the opacity, measured in units of \( \text{cm}^2 \, \text{g}^{-1} \); and \( j_\nu \) is the emission coefficient, measured in units of \( \text{erg cm}^{-3} \, \text{s}^{-1} \, \text{sr}^{-1} \, \text{Hz}^{-1} \) [both as defined by Rybicki & Lightman (p. 9-10)].

\textbf{Solution:} Consider a photon traveling a distance \( ds \) along a ray at an angle \( \theta \) from the local radius vector (See Figure [8]). Then, the radial distance the photon has traveled is \( dr = ds \cos \theta \), while the incremental difference in angle between the ray and the local radial vector is \( d\theta = -\frac{ds}{r} \sin \theta \). Thus, we find

\[
\frac{dr}{ds} = \cos \theta, \quad \frac{d\theta}{ds} = -\frac{\sin \theta}{r}
\]

Substitution into the equation of radiative transfer,

\[
\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r} \frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta} \frac{d\theta}{ds} = -\rho \kappa_\nu I_\nu + j_\nu,
\]

with the chain rule, yields the desired result.

(b) Integrate the RTE over all solid angles to show

\[
\frac{dF_\nu}{dr} + 2 \frac{F_\nu}{r} + c \rho \kappa_\nu u_\nu - \rho \epsilon_\nu = 0, \tag{6}
\]

where \( \epsilon_\nu \) is the (angle-averaged) \textit{emissivity} as defined on p. 9 of Rybicki & Lightman.

\textbf{Solution:}

\[
0 = \int \left[ \cos \theta \frac{\partial I_\nu}{\partial r} - \frac{\sin \theta}{r} \frac{\partial I_\nu}{\partial \theta} + \rho \kappa_\nu I_\nu - j_\nu \right] d\Omega
= \frac{\partial}{\partial r} \int \cos \theta I_\nu d\Omega - \frac{1}{r} \int \sin \theta \frac{\partial I_\nu}{\partial \theta} d\Omega + \rho \kappa_\nu \int I_\nu d\Omega - \int j_\nu d\Omega
= \frac{\partial F_\nu}{\partial r} - 2 \pi \int \sin^2 \theta \frac{\partial I_\nu}{\partial \theta} d\theta + \rho \kappa_\nu c u_\nu - 4 \pi j_\nu
= \frac{\partial F_\nu}{\partial r} - 2 \pi \left[ \sin^2 \theta I_\nu \big|_{\theta=0} - \int 2 \cos \theta \sin \theta I_\nu d\theta \right] + \rho \kappa_\nu c u_\nu - \rho \epsilon_\nu
= \frac{\partial F_\nu}{\partial r} + \int 2 \cos \theta I_\nu d\Omega + \rho \kappa_\nu c u_\nu - \rho \epsilon_\nu
= \frac{\partial F_\nu}{\partial r} + \frac{2}{r} F_\nu + \rho \kappa_\nu c u_\nu - \rho \epsilon_\nu
(c) Multiply the RTE by \( \cos \theta \) and integrate over all solid angles to show

\[
e^d\frac{dp_\nu}{d\nu} + \rho \kappa_\nu F_\nu = 0, \tag{7}
\]

where you have assumed \( j_\nu \) to be isotropic, and \( I_\nu \) to be nearly isotropic. Here, \( p_\nu \) is the specific radiation pressure given by

\[
p_\nu = \frac{1}{c} \int I_\nu \cos^2 \theta \, d\Omega. \tag{8}
\]

**Solution:**

\[
0 = \int \cos \theta \left[ \cos \theta \frac{\partial I_\nu}{\partial r} - \frac{\sin \theta}{r} \frac{\partial I_\nu}{\partial \theta} + \rho \kappa_\nu I_\nu - j_\nu \right] \, d\Omega = \frac{d}{dr} \int \cos^2 \theta I_\nu \, d\Omega - \frac{1}{r} \int \cos \theta \sin \theta \frac{\partial I_\nu}{\partial \theta} \, d\Omega + \rho \kappa_\nu F_\nu - \int \cos \theta j_\nu \, d\Omega
\]

\[
e^d \frac{dp_\nu}{d\nu} + \rho \kappa_\nu F_\nu = \int \cos \theta \left[ \cos \theta \frac{\partial I_\nu}{\partial r} - \frac{\sin \theta}{r} \frac{\partial I_\nu}{\partial \theta} + \rho \kappa_\nu I_\nu \right] \, d\Omega - \int \cos \theta \sin \theta \frac{\partial I_\nu}{\partial \theta} \, d\Omega + \rho \kappa_\nu F_\nu - \int \cos \theta j_\nu \, d\Omega
\]

\[
= \int \cos \theta \left[ \cos \theta \frac{\partial I_\nu}{\partial r} - \frac{\sin \theta}{r} \frac{\partial I_\nu}{\partial \theta} + \rho \kappa_\nu I_\nu \right] \, d\Omega - \int \cos \theta \sin \theta \frac{\partial I_\nu}{\partial \theta} \, d\Omega + \rho \kappa_\nu F_\nu - \int \cos \theta j_\nu \, d\Omega
\]

\[
\approx \int \cos \theta \left[ \cos \theta \frac{\partial I_\nu}{\partial r} - \frac{\sin \theta}{r} \frac{\partial I_\nu}{\partial \theta} + \rho \kappa_\nu I_\nu \right] \, d\Omega - \int \cos \theta \sin \theta \frac{\partial I_\nu}{\partial \theta} \, d\Omega + \rho \kappa_\nu F_\nu - \int \cos \theta j_\nu \, d\Omega
\]

Where in the last line we have noted that \( p_\nu = \frac{1}{3} u_\nu \) since \( I_\nu \) is nearly isotropic.

(d) Use the preceding equation, as well as the blackbody formula for radiation pressure, the relation \( F = L/4\pi r^2 \) and the definition of the Rosseland mean opacity \( \kappa_R \) to show

\[
\frac{dT}{dr} = -\frac{3\rho \kappa_R L}{64\pi \sigma r^2 T^3}. \tag{9}
\]

**Solution:** The radiation pressure of a blackbody is given by

\[
p_\nu = \frac{u_\nu}{3} = \frac{4\pi B_\nu}{3c}. \tag{10}
\]

Substituting in our result from part b) we find:

\[
0 = \int \cos \theta \left[ \cos \theta \frac{\partial I_\nu}{\partial r} - \frac{\sin \theta}{r} \frac{\partial I_\nu}{\partial \theta} + \rho \kappa_\nu I_\nu \right] \, d\Omega - \int \cos \theta \sin \theta \frac{\partial I_\nu}{\partial \theta} \, d\Omega + \rho \kappa_\nu F_\nu - \int \cos \theta j_\nu \, d\Omega
\]

\[
0 = \frac{4\pi}{3\kappa_R} \frac{dB_\nu}{dT} + \rho F_\nu \]

\[
= \frac{4\pi}{3} \frac{1}{\kappa_\nu} \left[ \frac{dB_\nu}{dT} + \rho F_\nu \right] \, d\nu
\]

\[
= \frac{4\pi}{3} \frac{dT}{dr} \left[ \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} + \rho F_\nu \right] \, d\nu
\]

\[
= \frac{4\pi}{3} \frac{dT}{dr} \left[ \frac{d}{dT} \left( \frac{\sigma T^4}{\pi} \right) \right] + \rho L \frac{4\pi r^2}{4\pi r^2}
\]

\[
0 = \frac{16\pi T^3}{\kappa_R} \frac{dT}{dr} + \rho L \frac{4\pi r^2}{4\pi r^2}
\]

\[
\frac{dT}{dr} = -\frac{3\rho \kappa_R L}{64\pi \sigma T^3 r^2}.
\]
Figure 8: Geometry relevant to Prob. 6. A photon propagates a distance $ds$ along a direction $\theta$ from the local radius vector. As a result its radial coordinate increases by $dr$ and the angle to the local radius vector decreases by $d\theta$. 

[Diagram of photon propagation with labels $ds$, $dr$, $\theta$, $r$, and $\theta - d\theta$.]