1. The radial velocity equation \([20 \text{ pts}]\). In this problem you will derive the equation for the radial (line-of-sight) velocity variations of a star in Keplerian motion. This equation is used to model spectroscopic binaries.

(a) Suppose two stars with masses \(M_1\) and \(M_2\) are in a Keplerian orbit with period \(P\) and eccentricity \(e\). Show that the radial velocity of star 1 can be written

\[
v_{\text{rad},1}(t) = \left(\frac{2\pi}{P}\right) \frac{a_1 \sin I}{\sqrt{1-e^2}} \{\cos [\phi(t) + \omega] + e \cos \omega\}.
\]

Here, \(a_1\) is the semimajor axis of star 1’s orbit, \(I\) is the inclination, \(\phi(t)\) is the “true anomaly” (the polar angle in the orbital plane, measured from pericenter), and \(\omega\) is the argument of pericenter; see Figure 1 for a description of the angles \(I\) and \(\omega\). One way to proceed is as follows:

- Show that the radial coordinate of the relative orbit can be written

\[
Z = r(\phi) \sin(\phi + \omega) \sin I,
\]

with reference to the coordinate system depicted in Figure 1.
- Take the time derivative of \(Z\) and use the conservation of angular momentum,

\[
L = \mu r^2 \dot{\phi} = \sqrt{GM\mu^2 a(1-e^2)},
\]

as well as the polar equation for the ellipse,

\[
r = \frac{a(1-e^2)}{1 + e \cos \phi},
\]

to obtain the desired form for \(v_{\text{rad}}\) of the effective one-body orbit.
- Scale the result for \(v_{\text{rad}}\) appropriately to obtain the radial velocity of star 1.

Solution: Figure 1 illustrates two coordinate systems: an orbit coordinate system \((x, y, \text{ and } z)\) and the sky coordinate system \((X, Y, \text{ and } Z)\). The orbit coordinates are defined so that \(i)\) its origin lies at the center of mass of the star-planet system, \(ii)\) the orbital plane coincides with the \(x-y\) plane and \(iii)\) the \(\hat{x}\) axis points to the pericenter of the star’s orbit. The sky coordinate system so that \(i)\) its origin again lies at the center of mass of the star-planet system, \(ii)\) the \(\hat{Z}\) axis points along the observer’s line of sight, and \(iii)\) the \(\hat{X}\) axis is the reference direction (eg: pointing to the vernal equinox). The radial velocity of the star corresponds to its motion along the \(\hat{Z}\) direction.

To begin, we project \(\hat{r}\), the radial component of the relative orbit, onto the line of sight, \(\hat{Z}\).

\[
Z = \hat{r} \cdot \hat{Z} = r(\phi) \hat{r} \cdot \hat{Z} = r(\phi) \sin(\phi + \omega) \sin I.
\]

The \(\sin(\phi + \omega)\) projects \(\hat{r}\) perpendicular to the line of nodes within the orbital plane. The \(\sin I\) factor then projects from there to the \(\hat{Z}\) direction.

More rigorously, the orbit \((x, y, \text{ and } z)\) and sky \((X, Y, \text{ and } Z)\) co-ordinate systems are related to each other through the orbital elements \((I\ the \ inclination, \ \Omega \ the \ longitude \ of \ the \ ascending \ node, \ and \ \omega \ the \ argument \ of \ pericenter)\), as

\[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}
= P_3 P_2 P_1
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]
Figure 1: Definition of the orbital elements $\Omega$, $I$, and $\omega$ (problem 1). For observational astronomy, the $z$-axis is chosen to be the line of sight.

where,

$$
P_1 = \begin{pmatrix}
\cos \omega & -\sin \omega & 0 \\
\sin \omega & \cos \omega & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

$$
P_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos I & -\sin I \\
0 & \sin I & \cos I
\end{pmatrix}
$$

$$
P_3 = \begin{pmatrix}
\cos \Omega & -\sin \Omega & 0 \\
\sin \Omega & \cos \Omega & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

are all 3D rotations. Performing all the matrix multiplications in equation 1a we obtain

$$
Z = x \sin(I) \sin(\omega) + y \sin(I) \cos(\omega) + z \cos(I).
$$

Substituting the orbit of the relative coordinate,

$$
x = r(\phi) \cos \phi \\
y = r(\phi) \sin \phi \\
z = 0,
$$

we find

$$
Z = r(\phi) \cos \phi \sin(I) \sin(\omega) + r(\phi) \sin \phi \sin(I) \cos(\omega) = r(\phi) \sin(\phi + \omega) \sin I.
$$

The orbit of the relative coordinate is
\[ r = \frac{a(1 - e^2)}{1 + e \cos \phi}, \]
\[ \frac{dr(\phi)}{d\phi} = \frac{a(1 - e^2)e \sin \phi}{(1 + e \cos \phi)^2} = \frac{r^2 e \sin \phi}{a(1 - e^2)} \]

We differentiate \( Z \) to find the velocity of the relative coordinate along the line of sight.

\[
\dot{Z} = \left( \frac{dr(\phi)}{d\phi} \sin(\phi + \omega) + r(\phi) \cos(\phi + \omega) \right) \dot{\phi} \sin I
\]
\[
= \left( \frac{r^2 e \sin \phi}{(1 - e^2)} \sin(\phi + \omega) + \frac{1 + e \cos \phi}{a(1 - e^2)} \cos(\phi + \omega) \right) \frac{L}{\mu a(1 - e^2)} \sin I
\]
\[
= \left( e \cos \omega + \cos(\phi + \omega) \right) \frac{\sqrt{GM}}{a(1 - e^2)} \sin I
\]
\[
= \left( e \cos \omega + \cos(\phi + \omega) \right) \left( \frac{2\pi}{P} \right) \frac{a}{\sqrt{1 - e^2}} \sin I.
\]

In the second to last line we used the conservation of angular momentum \( L = \sqrt{GM\mu^2 a(1 - e^2)} \), while the last line follows from Kepler’s third law.

Finally, we find the motion of the star of mass \( M_1 \) relative to the center of mass by scaling \( Z_1 = \frac{M_2}{M} Z \), and \( a_1 = \frac{M_2}{M} a \).

\[ v_{\text{rad},1}(t) = \dot{Z}_1 = \frac{M_2}{M} \dot{Z} = \left( \frac{2\pi}{P} \right) \frac{a_1 \sin I}{\sqrt{1 - e^2}} \left\{ \cos(\phi(t) + \omega) + e \cos \omega \right\}. \]

(b) Show that the radial velocity may also be written
\[ v_{\text{rad},1}(t) = \left( \frac{2\pi G}{P} \right)^{1/3} \frac{M_2 \sin I}{(M_1 + M_2)^{2/3} \sqrt{1 - e^2}} \left\{ \cos(\phi(t) + \omega) + e \cos \omega \right\}. \]

**Solution:** The desired result follows by substituting
\[ a_1 = \frac{M_2}{M_1 + M_2} a, \]
and Kepler’s third law
\[ a = \left( G (M_1 + M_2) \left( \frac{P}{2\pi} \right)^2 \right)^{1/3}, \]
into the answer from a).

(c) To calculate \( v_{\text{rad},1}(t) \), you will need to compute \( \phi(t) \). Last week you derived parametric equations for \( r(t) \) but not for \( \phi(t) \). Show that \( \phi(t) \) can be obtained by
\[ \tan \frac{\phi}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{u}{2}, \]
where \( u \) is the the same parameter as in problem 5(b) of problem set 1. (This parameter is known as the “eccentric anomaly.”)

One way to proceed is
• Use the polar equation for an ellipse to solve for \( \cos \phi \) in terms of \( r, a, \) and \( e. \n\)
• Use the previous result \( r = a(1 - e \cos u) \) to eliminate \( r \) in favor of \( u. \)
• Write expressions for \( (1 + \cos \phi) \) and \( (1 - \cos \phi) \) in terms of \( u \) and \( e. \)
• Use trigonometric identities to derive the desired result.

**Solution:** Solving for \( \cos \phi \) in the polar equation for an ellipse, we find

\[
\cos \phi = a(1 - e^2) - r.
\]

Substituting \( r = a(1 - e \cos u) \) to eliminate \( r \) in favor of \( u, \)

\[
\cos \phi = \frac{\cos u - e}{1 - e \cos u}.
\]

Finally, we use the trigonometric identity \( \tan \frac{\phi}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}. \)

\[
\tan \frac{\phi}{2} = \sqrt{\frac{1 - \cos \phi}{1 + \cos \phi}}
= \sqrt{\frac{1 - \cos u - e}{1 - e \cos u}}
= \sqrt{\frac{1 + e}{1 - e} \tan u}
= \sqrt{\frac{1 + e}{1 - e}} \tan u
\]

(d) HD 80606b is a giant planet with an orbital eccentricity exceeding 0.9, presenting an extreme example of the “eccentric exoplanet” problem: the observation that exoplanets often have eccentric orbits, despite the 20th century expectation that more circular orbits would be common. The planet and star have masses \( 4.2 \ M_{\text{Jup}} \) and \( 1.05 \ M_\odot \) respectively. The orbital parameters are \( P = 111.44 \) days, \( e = 0.933, I = 89.32 \) degrees, and \( \omega = 300.83 \) degrees.

Write a computer program to plot the radial velocity of the star as a function of time. The radial velocity should be in units of m s\(^{-1}\), and the time should be in units of days relative to pericenter. Plot at least two full orbits.

**Solution:** Evaluate Equation[5] with the parameters of the HD80606 system. To relate \( \phi \) to \( t \), we first use Equation[6] to evaluate \( u(\phi) \) (being careful to choose the appropriate root in the \( \arctan \) function), and then recall

\[
t - t_0 = \left( \frac{P}{2\pi} \right) (u - e \sin u)
\]
from PSet 1.
Figure 2: HD80606 radial velocity.
2. Visual-spectroscopic binary [30 pts]. Sirius, the brightest star in the night sky, is a binary system consisting of a normal A-type star (referred to as Sirius A) and a white dwarf (Sirius B). It is both a visual binary and a spectroscopic binary. Moreover, it is close enough to Earth for its parallax to be measured accurately.

A plot of the Sirius orbit is shown in Figure 2. In this plot, the more massive A star is taken to be a fixed reference point, and the position of the white dwarf relative to the A star is plotted as a function of time.

(a) Estimate the orbital period $P$.

Solution: In 2000, Sirius B returns to its 1950 position relative to Sirius A.

$P \approx 50$ years

(b) The plotted orbital shape is elliptical. However, even a circular orbit will appear elliptical if the orbital plane is tipped at an angle with respect to our line of sight. Present an argument, based on the plot, that the orbit is truly eccentric.

Solution: If the orbit were circular, the time for Sirius B to travel between $\phi_0$ to $\phi_0 + \pi$ would be $P/2$ for all $\phi_0$. This is not the case for the Sirius system (e.g. Sirius B takes $\sim 17$ years to go from the West to East (relative to Sirius A) and $\sim 33$ years to go from East to West) so the orbit must be eccentric.

(c) Measure the apparent semimajor axis of the binary in arcsec.

A careful analysis of the trajectory (which you need not perform) would reveal $e$, $\omega$, $\Omega$, and $i$, from which you could deduce that the actual semimajor axis is obtained by dividing the apparent semimajor axis by 0.95.

Solution: To measure the apparent semimajor axis, we first have to determine time of pericenter and apocenter. We know that the Sirius A, the apocenter, and the pericenter must all lie along a line in the sky plane. Further, the travel time between apocenter and pericenter is $P/2$. Based on these two criteria, we see that apocenter passage occurred circa 1994, while the last pericenter passage was in 1969.

On this figure, we measure 10.3 cm separating the 1994 and 1969 positions of Sirius B, with a scale of $\sim 7''$ per 5 cm.

Thus, the apparent angular semi-major axis is $\sim \frac{10.3}{2} \frac{\text{cm}}{5 \text{ cm}} \approx 7.6''.$ The actual semimajor axis is $7.2/0.95 = 7.6$ arcsec.

(d) The parallax of the center of mass of the binary is $\pi = 0.379''$. Use the implied distance to find the physical size of the semimajor axis in AU, and the sum of the masses, $M_A + M_B$.

Solution: The distance to the binary is $d = \frac{1}{\pi} = 2.64$ pc. From, c) the actual semimajor axis subtends 7.6'', corresponding to a physical size of

\[ a = (7.6'')(2.64 \text{ pc}) = 20.0 \text{ AU} \]

We use Kepler’s third law to find the total mass of the system,

\[ M_A + M_B = \left( \frac{a}{AU} \right)^3 \left( \frac{\text{year}}{P} \right)^2 M_\odot = 3.2 M_\odot. \]

(e) If the orbits of both stars had been plotted around the center of mass of the binary, the orbit of the white dwarf would be 2.4 times larger than that of Sirius A. What is the mass ratio $M_A/M_B$? What are the values of $M_A$ and $M_B$?

Solution: We calculate the mass ratio, from the ratio of semi-major axes in the center of mass frame,

\[ \frac{M_A}{M_B} = \frac{a_B}{a_A} = 2.4 \]

The masses of the individual stars then follow: $M_A = 2.3 M_\odot, M_B = 0.9 M_\odot$.
(f) The apparent bolometric magnitudes of Sirius A and B are $-2.1$ and $+8.3$, respectively. Compute the luminosity of each star in terms of the solar luminosity, $L_\odot$. The absolute bolometric magnitude of the Sun is $+4.75$.

**Solution:** The absolute magnitudes, $\mathcal{M}$, of Sirius A and B are related to their apparent magnitudes $m$ through the distance modulus.

\[
\mathcal{M} = m - 5 \log_{10} \left( \frac{d}{10 \text{pc}} \right) = m + 2.9
\]

\[
\mathcal{M}_A = 0.8 \\
\mathcal{M}_B = 11.2
\]

Comparing to the absolute magnitude of the sun yields the bolometric luminosity in solar units,

\[
L = 10^{2/5(\mathcal{M}_\odot-\mathcal{M})} L_\odot
\]

Evaluating, yields $L_A = 38 L_\odot$, $L_B = 0.0026 L_\odot$.

(g) Use the Stefan-Boltzmann law and the known surface temperatures of the Sun and Sirius A & B (5777 K, 11200 K, and 28500 K, respectively) to compute the radii of Sirius A & B in units of the Sun’s radius, $R_\odot$.

**Solution:** Treating the stars as blackbodies (a pretty good approximation), the Stefan-Boltzmann law relates the stellar luminosity, $L$, to the stellar radius, $R$, and effective temperature $T_{\text{eff}}$.

\[
L = 4 \pi \sigma R^2 T_{\text{eff}}^4
\]

Solving for $R$,

\[
R = \left( \frac{L}{L_\odot} \right)^{1/2} \left( \frac{T_{\text{eff}}^{\odot}}{T_{\text{eff}}} \right)^2 R_\odot.
\]

Plugging in numbers, we find $R_A = 1.6 R_\odot$, $R_B = 0.0021 R_\odot$.

(h) Compute the density (in g cm$^{-3}$) of Sirius A and B. Compare to the Earth’s mean density of 5.5 g cm$^{-3}$.

**Solution:** Evaluating $\rho = M / (4/3 \pi R^3)$, we find $\rho_A = 0.72$ g cm$^{-3}$ and $\rho_B = 1.4 \times 10^8$ g cm$^{-3}$. The mean density of Sirius A is less dense than the mean density of Earth, and lower even than the density of water at STP. In contrast Sirius B is more than $2 \times 10^7$ times more dense than the Earth. It’s an amazing object: a white dwarf!
Figure 3: The relative orbit of Sirius A and B (problem 2).
3. Transiting planet [12 pts]. A planet is observed to transit a star. The transits recur with period \( P \). The duration of each transit is \( T \), and during each transit the apparent brightness of the star drops by a fraction \( \delta \). By modeling the transit light curve, it is found that the orbital inclination is \( i = 90^\circ \) (i.e. the orbital axis is perpendicular to the line of sight). Doppler measurements of the star reveal a sinusoidal radial-velocity variation with an amplitude \( K \).

In what follows you may assume \( M_p \ll M_\star \) and \( R_p \ll R_\star \ll a \), where \( M_p \) is the planetary mass, \( M_\star \) is the stellar mass, \( R_p \) is the planetary radius, \( R_\star \) is the stellar radius, and \( a \) is the orbital semimajor axis.

(a) Derive an approximate formula for the star’s mean density, \( \langle \rho_\star \rangle \), in terms of observable quantities.

\[ \text{Solution:} \quad \langle \rho_\star \rangle = \frac{M_\star}{\frac{4}{3} \pi R_\star^3} = \left( \frac{4\pi^2 a^3}{GP^2} \right) \frac{3}{4\pi} \left( \frac{P}{\pi aT} \right)^3 = \frac{3P}{\pi^2 GT^3}. \]

Above, we have used Kepler’s 3rd law to substitute for \( M_\star \) and our \( T \) expression to substitute for \( R_\star \).

(b) Derive an approximate expression for the planet’s surface gravity \( g_p \), in terms of observable quantities. (The surface gravity is defined as \( GM_p/R_p^2 \), where \( M_p \) and \( R_p \) are the mass and radius of the planet, respectively.)

\[ \text{Solution:} \quad g_p = \frac{GM_p}{R_p^2} = \frac{G K P M_\star}{2\pi a} \frac{1}{\delta R_\star^2} = \left( \frac{G K P M_\star}{2\pi a \delta} \right) \left( \frac{P}{\pi aT} \right)^2 = \frac{2KP}{\pi \delta T^2}. \]

In the second line, we used our expressions for \( K \) and \( \delta \) to replace \( M_p \) and \( R_p \), respectively. In the third line we used our expression for \( T \) to replace \( R_\star \). The final simplifications follow from Kepler’s third law.
4. **Gravitational-wave chirp [30 pts]** Consider a binary consisting of two masses \( m_1 \) and \( m_2 \) in a circular orbit of radius \( R \). Consider the orbit to be adequately described using Newtonian gravity.

(a) Compute the rate at which energy is carried away from the system by gravitational waves. Express your answer in terms of the reduced mass \( \mu \), the total mass \( M \), the orbital frequency \( \Omega \), and the orbital separation \( R \).

**Solution:**

As discussed in lecture, this binary will have a quadrupole moment tensor

\[
I_{ij} = \mu R^2 \begin{pmatrix}
\cos^2(\Omega t) & \cos(\Omega t) \sin(\Omega t) & 0 \\
\cos(\Omega t) \sin(\Omega t) & \sin^2(\Omega t) & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\equiv \frac{1}{2} \mu R^2 \begin{pmatrix}
[1 + \cos(2\Omega t)] & \sin(2\Omega t) & 0 \\
\sin(2\Omega t) & [1 - \cos(2\Omega t)] & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

On the second line, we’ve used some trig identities to write things in a particularly convenient way. To compute the rate at which gravitational waves carry away energy, we need the third derivative of this quantity:

\[
\frac{d^3 I_{ij}}{dt^3} = 4\mu \Omega^3 R^2 \begin{pmatrix}
\sin(2\Omega t) & -\cos(2\Omega t) & 0 \\
-\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

We can now compute the rate at which energy is carried off:

\[
\frac{dE_{GW}}{dt} = \frac{1}{5c^5} \left( \frac{d^3 I_{ij}}{dt^3} \cdot \frac{d^3 I_{ij}}{dt^3} \right) = \frac{16G\mu^2 \Omega^6 R^4 [\sin^2(2\Omega t) + \cos^2(2\Omega t) + \cos^2(2\Omega t) + \sin^2(2\Omega t)]}{5c^5} = \frac{32G\mu^2 \Omega^6 R^4}{5c^5}
\]

Since we assume that the orbit is described using Newtonian gravity, we can further reduce by plugging in the Keplerian relation \( \Omega = \sqrt{G M / R^3} \); we will save that for the next step.
Due to this loss of energy, the radius of the orbit will gradually shrink, and the frequency of the binary will “chirp” to higher frequencies as time passes.

(b) By asserting global conservation of energy in the following form,

\[
\frac{d}{dt} (E_{\text{kinetic}} + E_{\text{potential}}) + \frac{dE_{\text{GW}}}{dt} = 0 \, ,
\]

derive an equation for \( \frac{dR}{dt} \), the rate at which the orbital radius shrinks. \( \text{Hint:} \) You may find it useful to express all the terms in this equation as functions of \( R \) before evaluating terms and solving for \( \frac{dR}{dt} \).

**Solution:**

To begin, let us write the kinetic and potential energies of the orbit in a convenient form. Bearing in mind that we use Newtonian gravity and take the orbit to be circular, we have

\[
E_{\text{potential}} = -\frac{GM\mu}{R} \, ,
\]

and

\[
E_{\text{kinetic}} = \frac{1}{2} \mu R^2 \Omega^2 = \frac{1}{2} \frac{GM}{R} \, .
\]

Add these terms together and take their time derivative:

\[
\frac{d}{dt} (E_{\text{potential}} + E_{\text{kinetic}}) = \frac{d}{dt} \left( -\frac{GM\mu}{2R} \right) = \frac{1}{2} \frac{GM\mu}{R^2} \frac{dR}{dt} \, .
\]

Next, let us use the Kepler relation to rewrite the rate of change of energy as a function only of orbital radius:

\[
\frac{dE_{\text{GW}}}{dt} = \frac{32G^4 \mu^2 M^3}{5c^5 R^5} \, .
\]

Finally, put everything together and solve for \( \frac{dR}{dt} \):

\[
\frac{d}{dt} (E_{\text{kinetic}} + E_{\text{potential}}) + \frac{dE_{\text{GW}}}{dt} = 0
\]

becomes

\[
\frac{1}{2} \frac{GM\mu}{R^2} \frac{dR}{dt} + \frac{32G^4 \mu^2 M^3}{5c^5 R^5} = 0
\]

so

\[
\frac{dR}{dt} = -\frac{64G^3 \mu M^2}{5c^5 R^5} \, .
\]
(c) Find the rate of change of the orbital frequency $\Omega$ caused by gravitational-wave emission. You should find that the masses only appear in the combination $\mathcal{M} = \mu^{3/5} M^{2/5}$, perhaps raised to some power. This combination of masses is known as the “chirp mass,” since it sets the rate at which the frequency “chirps.”

**Solution:**

Begin by writing $d\Omega/dt$ in terms of $dR/dt$:

$$
\frac{d\Omega}{dt} = \frac{d}{dt} \sqrt{\frac{GM}{R^3}} = -\frac{3}{2} \sqrt{\frac{GM}{R^5}} \frac{dR}{dt} = \frac{96}{5} \frac{G^{7/2} \mu M^{5/2}}{c^5 R^{11/2}}.
$$

To clean this up, let us replace $R$ with $\Omega$: inverting the Kepler relation, we find

$$
R = \left(\frac{GM}{\Omega^2}\right)^{1/3}.
$$

Plugging this in yields

$$
\frac{d\Omega}{dt} = \frac{96}{5} \frac{G^{5/3}}{c^5} \mu M^{2/3} \Omega^{11/3} = \frac{96}{5} \left(\frac{GM}{c^3}\right)^{5/3} \Omega^{11/3}.
$$

The final equality uses the chirp mass defined in the problem. Note that the factor $G/c^3$ converts mass into time: $GM/c^3 \approx 4.92 \times 10^{-6}$ seconds (a useful factor to have memorized if you work in gravitational-wave astrophysics).

(d) Integrate the $d\Omega/dt$ you obtained in part (c) to obtain $\Omega(t)$, the time evolution of the binary’s orbital frequency. Let $T_c$ be the “coalescence time,” the time at which the frequency formally goes to infinity. Your answer should be a power law in $T_c - t$.

**Solution:**

Since our equation is a power law for $\Omega$, integrating is simple. Rearrange

$$
\Omega^{-11/3} d\Omega = \frac{96}{5} \left(\frac{GM}{c^3}\right)^{5/3} dt
$$

and then integrate both sides from $t$ to $T_c$:

$$
-\frac{3}{8} \left[ \Omega(T_c)^{-8/3} - \Omega(t)^{-8/3} \right] = \frac{96}{5} \left(\frac{GM}{c^3}\right)^{5/3} (T_c - t).
$$

Using the formal definition of $T_c$, this simplifies to

$$
\Omega(t) = \left[ \frac{256}{5} \left(\frac{GM}{c^5}\right)^{5/3} (T_c - t) \right]^{-3/8}.
$$

---

1 In reality, various approximations we have introduced break down before we reach this time. $T_c$ is nonetheless not a bad proxy for the time at which the members of the binary merge due to gravitational-wave emission.
(e) Suppose that a signal from a neutron star binary with \( m_1 = 1.3 \, M_\odot \), \( m_2 = 1.4 \, M_\odot \) enters the band of the LIGO detector when the gravitational-wave frequency \( f = 30 \, \text{Hz} \). Bearing in mind that the orbital frequency \( \Omega \) is related to the gravitational-wave frequency \( f \) by

\[
f = 2 \frac{\Omega}{2\pi},
\]

estimate how long the system will be in band before the neutron stars collide (i.e., before our formula predicts that \( \Omega \) will diverge).

**Solution:**

For this combination of masses, the chirp mass takes the value

\[
\mathcal{M} = \mu^{3/5} M^{2/5} = \frac{m_1^{3/5} m_2^{3/5}}{(m_1 + m_2)^{1/5}} = 1.17 \, M_\odot,
\]

and so the combination

\[
\frac{G\mathcal{M}}{c^3} = 1.17 \times 4.92 \times 10^{-6} \, \text{seconds} = 5.76 \times 10^{-6} \, \text{seconds}.
\]

When the gravitational-wave frequency \( f = 30 \, \text{Hz} \), then \( \Omega = 30\pi \, \text{sec}^{-1} \). Using this value and the formula we found earlier, we solve for \( T_c - t \):

\[
30\pi \, \text{sec}^{-1} = \left[ \frac{256}{5} \left( 5.76 \times 10^{-6} \, \text{sec} \right)^{5/3} (T_c - t) \right]^{-3/8}
\]

\[
\rightarrow \quad T_c - t = 57.4 \, \text{seconds}.
\]

The neutron stars will collide roughly 1 minute after entering the band of the LIGO detector.

---

\(^2\)The \(1/2\pi\) is the usual relation between angular frequency and frequency; the 2 is due to the radiation between quadrupolar in nature.
5. **Habitable zone [12 pts]** The “habitable zone” of a star is defined as the range of orbital distances where a planet would have a surface temperature appropriate for water to exist as a liquid.

A planet’s surface temperature can be estimated as follows. Suppose it is located a distance \( d \) from a star of radius \( R_\star \) and effective temperature \( T_{\text{eff}} \). Assume the planet has a Bond albedo \( A \), meaning that it reflects a fraction \( A \) of the incident power, and absorbs the complementary fraction \( 1 - A \). Further assume the planet is in thermal equilibrium, meaning that the energy absorbed by the planet in a given time interval is equal to the energy that it re-radiates during the same time interval. Finally, assume that the planet radiates as a blackbody, and that its atmosphere efficiently redistributes the heat over the planet’s surface, causing the energy to be re-radiated isotropically.

(a) Find \( T_{\text{eq}} \), the planet’s equilibrium temperature, in terms of \( T_{\text{eff}}, d, R_\star, \) and \( A \).

**Solution:** The stellar luminosity, \( L_\star \) is given by the Stefan-Boltzmann law,

\[
L_\star = 4\pi\sigma R_\star^2 T_{\text{eff}}^4.
\]

At the planet’s distance, the flux from the star \( F_\star \) is,

\[
F_\star = \frac{L_\star}{4\pi d^2} = \sigma \left( \frac{R_\star}{d} \right)^2 T_{\text{eff}}^4.
\]

The energy absorbed by the planet per unit time is,

\[
dE_{\text{in}} \frac{dt}{dt} = (1 - A) \pi R_p^2 F_\star = (1 - A) \pi \sigma R_p^2 \left( \frac{R_\star}{d} \right)^2 T_{\text{eff}}^4.
\]

The energy reradiated by the planet per unit time may also be expressed using the Stefan-Boltzmann law,

\[
dE_{\text{out}} \frac{dt}{dt} = 4\pi\sigma R_p^2 T_{\text{eq}}^4.
\]

In equilibrium, \( \frac{dE_{\text{in}}}{dt} = \frac{dE_{\text{out}}}{dt} \), from which we find

\[
T_{\text{eq}} = (1 - A)^{1/4} \left( \frac{R_\star}{2d} \right)^{1/2} T_{\text{eff}}.
\]

A naive calculation of the habitable zone would require \( T_{\text{eq}} \) to be in the range 273-373 K. However, the greenhouse effect of the planet’s atmosphere will likely raise the surface temperature. To account for this and other atmospheric effects, it has been suggested that the habitable zone should be defined by the criterion \( 175 \text{ K} < T_{\text{eq}} < 270 \text{ K} \), where \( T_{\text{eq}} \) is calculated as above (Kaltenegger & Sasselov 2011, ApJ Letters, 736, 25).

(b) Using this definition of the habitable zone, and assuming \( A = 0.3 \) (the Earth’s approximate Bond albedo), determine the range of orbital distances and orbital periods corresponding to the habitable zone of

- the Sun (\( R = R_\odot, M = M_\odot, T_{\text{eff}} = 5777 \text{ K} \)).
- Vega (\( R = 2.26 R_\odot, M = 2.14 M_\odot, T_{\text{eff}} = 9602 \text{ K} \)).
- Proxima Centauri (\( R = 0.141 R_\odot, M = 0.123 M_\odot, T_{\text{eff}} = 3042 \text{ K} \)).

**Solution:** Solving for \( d \) in our answer to a), we find

\[
d = \frac{R_\star}{2} (1 - A)^{1/2} \left( \frac{T_{\text{eff}}}{T_{\text{eq}}} \right)^2.
\]

Using this expression for \( d \) and Kepler’s third law, we estimate the boundaries to the habitable zone listed in the table below.
Table 1: Boundaries of the habitable zone.

<table>
<thead>
<tr>
<th>Star</th>
<th>Inner Boundary</th>
<th>Outer Boundary</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$d$</td>
<td>$P$</td>
</tr>
<tr>
<td>Sun</td>
<td>0.89 AU</td>
<td>307 days</td>
</tr>
<tr>
<td>($M = M_\odot$, $R = R_\odot$, $T_{eff} = 5777$ K)</td>
<td>2.1 AU</td>
<td>3.1 years</td>
</tr>
<tr>
<td>Vega</td>
<td>5.5 AU</td>
<td>9.0 years</td>
</tr>
<tr>
<td>($M = 2.14 M_\odot$, $R = 2.26 R_\odot$, $T_{eff} = 9602$ K)</td>
<td>13 AU</td>
<td>33 years</td>
</tr>
<tr>
<td>Proxima Centauri</td>
<td>0.035 AU</td>
<td>6.8 days</td>
</tr>
<tr>
<td>($M = 0.123 M_\odot$, $R = 0.141 R_\odot$, $T_{eff} = 3042$ K)</td>
<td>0.083 AU</td>
<td>25 days</td>
</tr>
</tbody>
</table>

6. **Blackbody radiation [16 pts]**. The Planck radiation spectrum is given by

\[ B_\nu = \frac{2h\nu^3}{c^2} \exp\left(\frac{h\nu}{kT}\right) - 1 \] (erg cm$^{-2}$ s$^{-1}$ Hz$^{-1}$ steradian$^{-1}$),

per unit frequency.

(a) **Wavelength spectrum.** Show by explicit calculation that the equivalent Planck radiation spectrum per unit wavelength is given by

\[ B_\lambda = \frac{2hc^2}{\lambda^5} \exp\left(\frac{hc}{\lambda kT}\right) - 1 \] (erg cm$^{-2}$ s$^{-1}$ cm$^{-1}$ steradian$^{-1}$),

starting from the expression for $B_\nu$.

**Solution:** To translate the Plank radiation spectrum per unit frequency to a spectrum per unit wavelength, we use the fact that the flux per steradian should be equal if we look at a certain frequency or wavelength interval:

\[ B_\nu |d\nu| = B_\lambda |d\lambda|, \]

where the absolute value signs are needed because $\nu$ and $\lambda$ are inversely related: $\nu = c/\lambda$. So we have

\[ B_\lambda = B_\nu \left| \frac{d\nu}{d\lambda} \right| = \left(\frac{2hc}{\lambda^3} \exp\left(\frac{hc}{\lambda kT}\right) - 1 \right) \left(\frac{c}{\lambda^2}\right) = \frac{2hc^2}{\lambda^5} \exp\left(\frac{hc}{\lambda kT}\right) - 1 \text{ erg cm}^{-2} \text{s}^{-1} \text{cm}^{-1} \text{steradian}^{-1}. \]

(b) **Stefan-Boltzmann law.** Derive the Stefan-Boltzmann law ($F = \sigma T^4$) by integrating the Planck blackbody spectrum over all wavelengths or frequencies. (Note that there is an extra factor of $\pi$ to convert from brightness per unit solid angle to total brightness, so that $F = \pi \int \int B_\nu |d\nu| = \pi \int B_\lambda |d\lambda|$.) You may use the fact that

\[ \int_0^\infty \frac{u^3}{e^u - 1} du = \frac{\pi^4}{15}. \]

Give an expression for the Stefan-Boltzmann constant $\sigma$ in terms of fundamental physical constants, and check its numerical value and units, $\sigma = 5.67 \times 10^{-5}$ erg cm$^{-2}$ s$^{-1}$ K$^{-4}$.

**Solution:** The flux, $F$, is the energy output per unit time per unit area. We’ll have to integrate over frequency (or wavelength) and solid angle to get the flux. To integrate over solid angle, we need to first multiply $B_\nu$ by a factor of $\cos \theta$ to account for the different effective area seen at different angles. So
\[
F = \int B_\nu d\nu = \int_0^{\pi/2} \int_0^\infty B_\nu d\cos\theta d\Omega = \int_0^\pi \int_0^\infty B_\nu d\nu 2\pi \sin\theta d\theta = 2\pi \int_0^{15} \int_0^\infty B_\nu d\nu d\cos\theta = \pi \int_0^\infty B_\nu d\nu.
\]

This gives the factor of \( \pi \) discussed in the problem statement. Plugging in for \( B_\nu \) from above, we have

\[
F = \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}.
\]

Making the change of variables \( u = h\nu/kT \), we obtain the Stefan-Boltzmann law:

\[
F = \frac{2\pi h}{c^2} \left( \frac{kT}{h} \right)^4 \int_0^\infty \frac{u^3 du}{e^u - 1} = \frac{2\pi h}{c^2} \left( \frac{kT}{h} \right)^4 \frac{\pi^4}{15} = \frac{2\pi^5}{15} \frac{k^4}{c^2h^3} T^4 = \sigma T^4,
\]

where

\[
\sigma = \frac{2\pi^5}{15} \frac{k^4}{c^2h^3} = 5.67 \times 10^{-5} \text{erg cm}^{-2} \text{s}^{-1} \text{K}^{-4}.
\]

(c) **Wavelength of radiation peak.** Derive the Wien displacement law, which relates the wavelength of the radiation at the peak of the Planck function \( B_\lambda \) to the temperature: \( T\lambda_{\text{max}} = 0.29 \text{ cm K} \). [When you differentiate to find the maximum of \( B_\lambda \), you will obtain a nonlinear equation of the form \( 5(1 - e^{-y}) - y = 0 \) which you can solve numerically.]

**Solution:** The wavelength of the radiation peak can be found by setting \( \partial B_\lambda / \partial \nu = 0 \):

\[
\frac{\partial B_\lambda}{\partial \nu} = - \frac{2hc^2}{\lambda^6} \frac{5(e^y - 1) - ye^y}{(e^y - 1)^2} = 0 \quad \Rightarrow \quad 5(1 - e^{-y}) - y = 0,
\]

where \( y = hc/\lambda kT \). The solution is \( y = 4.965 \), so

\[
T\lambda_{\text{max}} = \frac{hc}{yk} = 0.29 \text{ cm K}.
\]

(d) **Frequency of radiation peak.** Repeat the previous part, but this time find the relation between the frequency at the peak of the Planck function \( B_\nu \) and the temperature: \( \nu_{\text{max}}/T = 5.9 \times 10^{10} \text{ Hz K}^{-1} \). For a given temperature \( T \), does the photon energy corresponding to \( \nu_{\text{max}} \) agree with the photon energy corresponding to \( \lambda_{\text{max}} \) you found in the previous part? Should they agree?

**Solution:** The frequency of the radiation peak can be found by the same method:

\[
\frac{\partial B_\nu}{\partial \nu} = \frac{2h}{c^2} \left[ \frac{3\nu^2}{e^{h\nu/kT} - 1} - \frac{\nu^3(h/kT)e^{h\nu/kT}}{(e^{h\nu/kT} - 1)^2} \right] = 0 \quad \Rightarrow \quad 3(1 - e^{-x}) - x = 0,
\]

where \( x = h\nu/kT \). The solution is \( x = 2.821 \), so

\[
\frac{\nu_{\text{max}}}{T} = \frac{kx}{h} = 5.9 \times 10^{10} \text{ Hz K}^{-1}.
\]

These two expression multiplied together should equal \( c \) if \( \lambda_{\text{max}} \) is to correspond to \( \nu_{\text{max}} \). However, \( (T\lambda_{\text{max}})(\nu_{\text{max}}/T) = \lambda_{\text{max}} \nu_{\text{max}} = 0.57c \neq c ! \) The reason the peaks of \( B_\nu \) and \( B_\lambda \) don’t correspond to the same frequency is that the two functions represent two physically different quantities: \( B_\nu \) is the flux per steradian per unit frequency, and \( B_\lambda \) is the flux per steradian per unit wavelength. As we saw in part (a), this means that

\[
B_\nu |d\nu| = B_\lambda |d\lambda| \quad \Rightarrow \quad B_\nu = B_\lambda \frac{d\lambda}{d\nu} \quad \Rightarrow \quad B_\nu = B_\lambda \frac{c}{\nu^2} = B_\lambda \frac{\lambda^2}{c}.
\]

Notice the factor of \( \lambda^2 \) in the correspondence between \( B_\nu \) and \( B_\lambda \): \( \lambda^2 B_\lambda \) will not have the same peak as \( B_\lambda \), explaining why the two notions of “radiation peak” give somewhat different photon energies.