

THE QUADRUPOLE FORMALISM APPLIED TO BINARY SYSTEMS

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- Solving Einstein's equations in the weak field, slow motion approximation.
- How to project the solution in the TT -gauge.
- A simple example: radiation emitted by a harmonic oscillator.
- Gravitational radiation emitted by a binary system in circular orbit.
- The Binary pulsar PSR 1931+16 and the double pulsar PSR J0737-3039.
- The gravitational wave energy flux.
- Gravitational wave luminosity of a binary system.
- Orbital evolution of a binary system due to gravitational wave emission: period variation, wave amplitude, the signal phase.

How to estimate the GW-signal emitted by an evolving system: THE QUADRUPOLE FORMALISM

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1$$

in a suitable gauge Einstein's equations become

$$\square_F \bar{h}_{\mu\nu}(t, x^i) = -K T_{\mu\nu}(t, x^i), \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h.$$

where $\square_F = \left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right]$ and $K = \frac{16\pi G}{c^4}$

Fourier-expand $T_{\mu\nu}$ and $\bar{h}_{\mu\nu}$

$$T_{\mu\nu}(t, x^i) = \int_{-\infty}^{+\infty} T_{\mu\nu}(\omega, x^i) e^{-i\omega t} d\omega,$$

$$\bar{h}_{\mu\nu}(t, x^i) = \int_{-\infty}^{+\infty} \bar{h}_{\mu\nu}(\omega, x^i) e^{-i\omega t} d\omega, \quad i = 1, 3$$

\square_F and \int operators commute and the wave equation becomes

$$\int_{-\infty}^{+\infty} \square_F [\bar{h}_{\mu\nu}(\omega, x^i) e^{-i\omega t}] d\omega = -K \int_{-\infty}^{+\infty} T_{\mu\nu}(\omega, x^i) e^{-i\omega t} d\omega,$$

i.e.

$$\int_{-\infty}^{+\infty} \left[\nabla^2 + \frac{\omega^2}{c^2}\right] \bar{h}_{\mu\nu}(\omega, x^i) e^{-i\omega t} d\omega = -K \int_{-\infty}^{+\infty} T_{\mu\nu}(\omega, x^i) e^{-i\omega t} d\omega,$$

this equation can be solved for each assigned value of the frequency:

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right] \bar{h}_{\mu\nu}(\omega, x^i) = -K T_{\mu\nu}(\omega, x^i)$$

SLOW-MOTION APPROXIMATION

We shall solve the wave equation

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right] \bar{h}_{\mu\nu}(\omega, x^i) = -K T_{\mu\nu}(\omega, x^i)$$

assuming that the region where the source is confined

$$|x^i| \leq \epsilon, \quad T_{\mu\nu} \neq 0,$$

is **much smaller** than the wavelength of the emitted radiation $\lambda_{GW} = \frac{2\pi c}{\omega}$.

$$\frac{2\pi c}{\omega} \gg \epsilon \quad \rightarrow \quad \epsilon \omega \ll c \quad \rightarrow \quad v \ll c$$

The wave equation will be solved inside and outside the source, and the two solutions will be matched on the source boundary

Let us first integrate the equations **OUTSIDE** the source

$$\left[\nabla^2 + \frac{\omega^2}{c^2} \right] \bar{h}_{\mu\nu}(\omega, x^i) = 0$$

In polar coordinates, the Laplacian operator is

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial}{\partial \theta} \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

The simplest solution does not depend on ϕ and θ

$$\bar{h}_{\mu\nu}(\omega, r) = \frac{A_{\mu\nu}(\omega)}{r} e^{i\frac{\omega}{c}r} + \frac{Z_{\mu\nu}(\omega)}{r} e^{-i\frac{\omega}{c}r},$$

This solution represents a spherical wave, with an **ingoing** part ($\sim e^{-i\frac{\omega}{c}r}$), and an **outgoing** ($\sim e^{i\frac{\omega}{c}r}$) part.

Since we are interested only in the wave **emitted from** the source, we set $Z_{\mu\nu} = 0$, and the solution becomes

$$\bar{h}_{\mu\nu}(\omega, r) = \frac{A_{\mu\nu}(\omega)}{r} e^{i\frac{\omega}{c}r}$$

This is the solution outside the source, and on its boundary $x = \epsilon$

How do we find $A_{\mu\nu}(\omega)$?

To answer this question we need to integrate the equations inside the source

INSIDE THE SOURCE

$$\left[\nabla^2 + \frac{\omega^2}{c^2} \right] \bar{h}_{\mu\nu}(\omega, x^i) = -K T_{\mu\nu}(\omega, x^i) \quad (1)$$

This equation can be solved for each assigned value of the indices μ, ν .

Let us integrate each term over the source volume V

$$A) \quad \int_V \left[\nabla^2 + \frac{\omega^2}{c^2} \right] \bar{h}_{\mu\nu}(\omega, x^i) d^3x = -K \int_V T_{\mu\nu}(\omega, x^i) d^3x$$

$$\begin{aligned}
1) \quad & \int_V \nabla^2 \bar{h}_{\mu\nu} d^3x = \int_V \text{div}[\vec{\nabla} \bar{h}_{\mu\nu}] d^3x \\
& = \int_S [\vec{\nabla} \bar{h}_{\mu\nu}]^k dS_k \simeq 4\pi \epsilon^2 \left(\frac{d}{dr} \bar{h}_{\mu\nu} \right)_{r=\epsilon} \\
& = 4\pi \epsilon^2 \left(\frac{d}{dr} \frac{A_{\mu\nu}(\omega)}{r} e^{i\frac{\omega}{c}r} \right)_{r=\epsilon} \\
& = 4\pi \epsilon^2 \left[-\frac{A_{\mu\nu}}{r^2} e^{i\frac{\omega}{c}r} + \frac{A_{\mu\nu}}{r} \left(\frac{i\omega}{c} \right) e^{i\frac{\omega}{c}r} \right]_{r=\epsilon} \sim -4\pi A_{\mu\nu}(\omega)
\end{aligned}$$

neglecting all terms of order $\geq \epsilon$

$$\int_V \nabla^2 \bar{h}_{\mu\nu}(\omega, x^i) d^3x \simeq -4\pi A_{\mu\nu}(\omega)$$

Eq. A) now becomes

$$\begin{aligned}
-4\pi A_{\mu\nu}(\omega) + \int_V \frac{\omega^2}{c^2} \bar{h}_{\mu\nu}(\omega, x^i) d^3x & = -K \int_V T_{\mu\nu}(\omega, x^i) d^3x \\
\int_V \frac{\omega^2}{c^2} \bar{h}_{\mu\nu}(\omega, x^i) d^3x & \lesssim \frac{\omega^2}{c^2} |\bar{h}_{\mu\nu}|_{\max} \frac{4}{3}\pi\epsilon^3 \quad \text{negligible}
\end{aligned}$$

The final solution inside the source gives

$$\begin{aligned}
-4\pi A_{\mu\nu}(\omega) & = -K \int_V T_{\mu\nu}(\omega, x^i) d^3x \quad K = \frac{16\pi G}{c^4} \\
A_{\mu\nu}(\omega) & = \frac{K}{4\pi} \int_V T_{\mu\nu}(\omega, x^i) d^3x = \frac{4G}{c^4} \int_V T_{\mu\nu}(\omega, x^i) d^3x
\end{aligned}$$

SUMMARIZING:

By integrating the wave equation outside the source we find

$$\bar{h}_{\mu\nu}(\omega, r) = \frac{A_{\mu\nu}(\omega)}{r} e^{i\frac{\omega}{c}r}$$

by integrating over the source volume we find

$$A_{\mu\nu}(\omega) = \frac{4G}{c^4} \int_V T_{\mu\nu}(\omega, x^i) d^3x,$$

therefore

$$\bar{h}_{\mu\nu}(\omega, r) = \frac{4G}{c^4} \cdot \frac{e^{i\omega\frac{r}{c}}}{r} \int_V T_{\mu\nu}(\omega, x^i) d^3x,$$

or, in terms of the outgoing coordinate $(t - \frac{r}{c}, \mathbf{x}^i)$

$$\boxed{\bar{h}_{\mu\nu}(t, r) = \frac{4G}{c^4 r} \cdot \int_V T_{\mu\nu}(t - \frac{r}{c}, x^i) d^3x,} \quad (2)$$

This integral can be further simplified

NOTE THAT:

1) We still have to project onto the TT-gauge

2) By this approach we obtain an order of magnitude estimate of the emitted radiation

We shall now simplify the integral over $T_{\mu\nu}$ in eq. (2).

We are in flat spacetime, therefore

$$\frac{\partial}{\partial x^\nu} T^{\mu\nu} = 0, \quad \rightarrow \quad \frac{1}{c} \frac{\partial}{\partial t} T^{\mu 0} = -\frac{\partial}{\partial x^k} T^{\mu k}, \quad k = 1, 3$$

Integrate over the source volume:

$$\int_V \frac{1}{c} \frac{\partial}{\partial t} T^{\mu 0} d^3x = - \int_V \frac{\partial}{\partial x^k} T^{\mu k} d^3x;$$

Apply Gauss's theorem to the R.H.S.

$$\int_V \frac{\partial}{\partial x^k} T^{\mu k} d^3x = \int_S T^{\mu k} dS_k$$

where S is the surface which encloses V .

On S , $T^{\mu\nu} = 0$, **therefore** $\int_S T^{\mu k} dS_k = 0$, and

$$\frac{1}{c} \frac{\partial}{\partial t} \int_V T^{\mu 0} d^3x = 0,$$

i.e.

$$\int_V T^{\mu 0} d^3x = \text{const}, \quad \rightarrow \quad \bar{h}^{\mu 0} = \text{const}.$$

Since we are interested in the **time-dependent** part of the field, we put

$$\bar{h}^{\mu 0}(t, r) = \bar{h}_{\mu 0}(t, r) = 0;$$

(this condition is automatically satisfied when transforming to the TT-gauge) To simplify the space components of $\bar{h}_{\mu\nu}$:

$$\bar{h}_{ik}(t, r) = \frac{4G}{c^4 r} \int_V T_{ik}(t - \frac{r}{c}, x^i) d^3x, \quad i, k = 1, 3$$

we shall use the **Tensor-Virial Theorem**

Tensor-Virial Theorem

Let us consider the space components of the conservation law

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0 \quad \rightarrow \quad \mathbf{1)} \quad \frac{1}{c} \frac{\partial T^{n0}}{\partial t} + \frac{\partial T^{ni}}{\partial x^i} = 0, \quad n, i = 1, 3$$

multiply **1)** by x^k and integrate over the volume

$$\begin{aligned} \frac{1}{c} \frac{\partial}{\partial t} \int_V T^{n0} x^k dx^3 &= - \int_V \frac{\partial T^{ni}}{\partial x^i} x^k dx^3 \\ &= - \left[\int_V \frac{\partial (T^{ni} x^k)}{\partial x^i} dx^3 - \int_V T^{ni} \frac{\partial x^k}{\partial x^i} dx^3 \right] \quad \left(\frac{\partial x^k}{\partial x^i} = \delta_i^k \right) \\ &= - \int_S (T^{ni} x^k) dS_i + \int_V T^{nk} dx^3 \end{aligned}$$

as before, $\int_S (T^{ni} x^k) dS_i = 0$, therefore

$$\frac{1}{c} \frac{\partial}{\partial t} \int_V T^{n0} x^k dx^3 = \int_V T^{nk} dx^3$$

Since T^{nk} is symmetric in n and k ,

$$\frac{1}{c} \frac{\partial}{\partial t} \int_V T^{k0} x^n dx^3 = \int_V T^{kn} dx^3$$

and adding the two we get

$$\frac{1}{2c} \frac{\partial}{\partial t} \int_V (T^{n0} x^k + T^{k0} x^n) dx^3 = \int_V T^{nk} dx^3.$$

We shall now use the 0- component of the conservation law:

$$\frac{\partial T^{0\nu}}{\partial x^\nu} = 0 \quad \rightarrow \quad \mathbf{2)} \quad \frac{1}{c} \frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i} = 0$$

multiply **2)** by $x^k x^n$ and integrate

$$\begin{aligned} \frac{1}{c} \frac{\partial}{\partial t} \int_V T^{00} x^k x^n dx^3 &= - \int_V \frac{\partial T^{0i}}{\partial x^i} x^k x^n dx^3 \\ &= - \left[\int_V \frac{\partial (T^{0i} x^k x^n)}{\partial x^i} dx^3 - \int_V \left(T^{0i} \frac{\partial x^k}{\partial x^i} x^n + T^{0i} x^k \frac{\partial x^n}{\partial x^i} \right) dx^3 \right] \\ &= - \int_S (T^{0i} x^k x^n) dS_i + \int_V (T^{0k} x^n + T^{0n} x^k) dx^3 \end{aligned}$$

as before, the first integral vanishes, and

$$\frac{1}{c} \frac{\partial}{\partial t} \int_V T^{00} x^k x^n dx^3 = \int_V (T^{0k} x^n + T^{0n} x^k) dx^3.$$

Let us differentiate with respect to $x^0 = ct$:

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_V T^{00} x^k x^n dx^3 = \frac{1}{c} \frac{\partial}{\partial t} \int_V (T^{0k} x^n + T^{0n} x^k) dx^3$$

and since we just found

$$\frac{1}{2c} \frac{\partial}{\partial t} \int_V (T^{n0} x^k + T^{k0} x^n) dx^3 = \int_V T^{nk} dx^3,$$

finally

$$2 \int_V T^{kn} dx^3 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_V T^{00} x^k x^n dx^3$$

The quantity $\frac{1}{c^2} \int_V T^{00} x^k x^n dx^3$ is the **Quadrupole Moment Tensor** of the system

$$\boxed{q^{kn}(t) = \frac{1}{c^2} \int_V T^{00}(t, x^i) x^k x^n dx^3}, \quad (3)$$

and it is a function of **time only**. **In conclusion**

$$\int_V T^{kn}(t, x^i) dx^3 = \frac{1}{2} \frac{d^2}{dt^2} q^{kn}(t)$$

and since

$$\bar{h}^{kn}(t, r) = \frac{4G}{c^4 r} \cdot \int_V T^{kn}(t - \frac{r}{c}, x^i) d^3x,$$

we finally find

$$\begin{cases} \bar{h}^{\mu 0} = 0, & \mu = 0, 3 \\ \bar{h}^{kn}(t, r) = \frac{2G}{c^4 r} \cdot \left[\frac{d^2}{dt^2} q^{kn}(t - \frac{r}{c}) \right], & k, n = 1, 3 \end{cases} \quad (4)$$

This is the gravitational wave emitted by a mass-energy system evolving in time

NOTE THAT

1) $\frac{G}{c^4} \sim 8 \cdot 10^{-50} \text{ s/g cm}$ **GW are extremely weak!**

2) **We are not yet in the TT-gauge**

3) these equations are derived on very strong assumptions: one is that $\mathbf{T}^{\mu\nu}_{,\nu} = \mathbf{0}$, i.e. the motion of the bodies is dominated by non-gravitational forces. **However**, and remarkably, the result depends only on the sources motion and not on the forces acting on them. **Gravitational radiation has a quadrupolar nature.**

A system of accelerated charged particles has a time-varying dipole moment

$$\vec{d}_{EM} = \sum_i q_i \vec{r}_i$$

and it will emit dipole radiation, the flux of which depends on the second time derivative of \vec{d}_{EM} .

For an isolated system of masses we can define a gravitational dipole moment

$$\vec{d}_G = \sum_i m_i \vec{r}_i,$$

which satisfies the conservation law of the total momentum of an isolated system

$$\frac{d}{dt} \vec{d}_G = 0.$$

For this reason, gravitational waves do not have a dipole contribution.

It should be stressed that for a spherical or axisymmetric distribution of matter (or energy) the quadrupole moment is a constant, even if the body is rotating: thus, a spherical or axisymmetric star does not emit gravitational waves; similarly, a star which collapses in a perfectly spherically symmetric way has a vanishing q^{ik} and does not emit gravitational waves.

To produce waves we need a certain degree of asymmetry, as it occurs for instance in the non-radial pulsations of stars, in a non spherical gravitational collapse, in the coalescence of massive bodies etc.

HOW TO SWITCH TO THE TT-GAUGE

$$\begin{cases} \bar{h}^{\mu 0} = 0, & \mu = 0, 3 \\ \bar{h}^{ik}(t, r) = \frac{2G}{c^4 r} \cdot \left[\frac{d^2}{dt^2} q^{ik} \left(t - \frac{r}{c} \right) \right] \end{cases}$$

$$q^{ik} \left(t - \frac{r}{c} \right) = \frac{1}{c^2} \int_V T^{00} \left(t - \frac{r}{c}, x^n \right) x^i x^k dx^3$$

we shall make an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}$$

which does not spoil the harmonic gauge condition $g^{\mu\nu} \Gamma^{\lambda}_{\mu\nu} = 0$, i.e. choosing ϵ^{μ} such that $\square_F \epsilon^{\mu} = 0$, and imposing that

$$\begin{aligned} \delta^{ik} h^{TT}_{ik} &= 0, & i, k = 1, 3 & \quad \text{vanishing trace} \\ n^i h^{TT}_{ik} &= 0 & & \quad \text{transverse wave condition} \end{aligned}$$

where

$$\vec{n} = \frac{\vec{x}}{r}$$

is the unit vector normal to the wavefront.

How to do it:

As a first thing we define the operator which projects a vector onto the plane orthogonal to the direction of \vec{n} :

$$P_{jk} \equiv \delta_{jk} - n_j n_k. \quad (5)$$

P_{jk} is symmetric, it is a projector, because

$$P_{jk} P_{kl} = P_{jl},$$

and it is transverse:

$$n^j P_{jk} = 0.$$

Next, we define the transverse-traceless projector

$$\mathcal{P}_{jkmn} \equiv P_{jm} P_{kn} - \frac{1}{2} P_{jk} P_{mn}, \quad (6)$$

which “extracts” the transverse-traceless part of a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor. We want to compute

$$h^{TT}_{jk} = (\mathcal{P}h)_{jk} = \mathcal{P}_{jklm} h_{lm}. \quad (7)$$

It is easy to check that \mathcal{P}_{jkmn} satisfies the following properties

- it is a projector, i.e.

$$\mathcal{P}_{jkmn}\mathcal{P}_{mnr s} = \mathcal{P}_{jkr s};$$

- it is transverse, i.e.

$$n^j\mathcal{P}_{jkmn} = n^k\mathcal{P}_{jkmn} = n^m\mathcal{P}_{jkmn} = n^n\mathcal{P}_{jkmn} = 0;$$

- it is traceless, i.e.

$$\delta^{jk}\mathcal{P}_{jkmn} = \delta^{mn}\mathcal{P}_{jkmn} = 0. \quad (8)$$

It is worth mentioning that

$$h_{jk}^{TT} = \mathcal{P}_{jkmn}h_{mn} = \mathcal{P}_{jkmn}\bar{h}_{mn},$$

indeed, since $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ h_{jk} and \bar{h}_{jk} differ only by the trace, which is projected out by \mathcal{P} because of (8). By applying the projector to \bar{h}_{jk} given in eq. (4) we find

$$\begin{cases} \bar{h}^{TT\mu 0} = 0, & \mu = 0, 3 \\ \bar{h}^{TT ik}(t, r) = \frac{2G}{c^4 r} \cdot \left[\frac{d^2}{dt^2} q^{TT ik}\left(t - \frac{r}{c}\right) \right] \end{cases}$$

where $q^{TT ij}$ is the quadrupole moment of the source projected in the TT-gauge, i.e. it is the **Transverse-Traceless quadrupole moment**

$$q_{ij}^{TT} = \mathcal{P}_{ijlm}q_{lm}$$

Sometimes is useful to define the **reduced quadrupole moment**

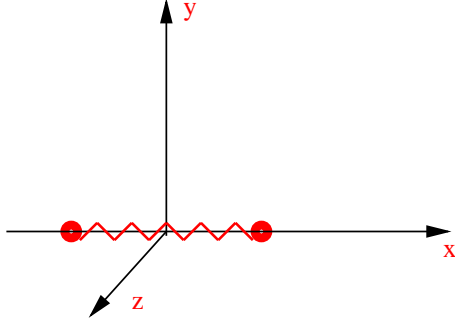
$$Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij}q^k{}_k$$

the trace of which is zero by definition, and consequently

$$Q_{ij}^{TT} = \mathcal{P}_{ijlm}Q_{lm} = \mathcal{P}_{ijlm}q_{lm}$$

Waves emitted by a harmonic oscillator

We shall now compute the GW-radiation emitted by a harmonic oscillator composed of two masses m , oscillating at a frequency ω with amplitude A .



$$\begin{cases} x_1 = -\frac{1}{2}l_0 - A \cos \omega t \\ x_2 = +\frac{1}{2}l_0 + A \cos \omega t \end{cases}$$

Let us compute the quadrupole moment

$$q^{ik}(t) = \frac{1}{c^2} \int_V T^{00}(t, x^n) x^i x^k dx^3$$

$$T^{00} = \sum_{n=1}^2 c p^0 \delta(x - x_n) \delta(y) \delta(z)$$

since $v \ll c$, $\rightarrow \gamma \sim 1 \rightarrow p^0 = mc$, therefore

$$\begin{aligned} q^{xx} = q_{xx} &= \frac{1}{c^2} \int_V m_1 c^2 \delta(x - x_1) x^2 dx \delta(y) dy \delta(z) dz \\ &+ \frac{1}{c^2} \int_V m_2 c^2 \delta(x - x_2) x^2 dx \delta(y) dy \delta(z) dz \\ &= m [x_1^2 + x_2^2] = m \left[\frac{1}{2} l_0^2 + 2A^2 \cos^2 \omega t + 2Al_0 \cos \omega t \right] \\ &= m [\cos t + A^2 \cos 2\omega t + 2Al_0 \cos \omega t] \end{aligned}$$

$$(\cos 2\alpha = 2 \cos^2 \alpha - 1)$$

$$\begin{aligned} q^{yy} &= \frac{1}{c^2} \int_V m_1 c^2 \delta(x - x_1) dx y^2 \delta(y) dy \delta(z) dz \\ &+ \frac{1}{c^2} \int_V m_2 c^2 \delta(x - x_2) dx y^2 \delta(y) dy \delta(z) dz \end{aligned}$$

since $\int_V y^2 \delta(y) dy = 0$, $q_{yy} = 0$;

in a similar way, since the motion is confined on the x -axis, the remaining components of q_{ij} vanish.

We now want to compute the wave emerging in the z -direction

In this case $\vec{n} = \frac{\vec{z}}{r} \rightarrow (0, 0, 1)$ and

$$P_{jk} = \delta_{jk} - n_j n_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} \bar{h}^{TT}_{\mu 0} = 0, & \mu = 0, 3 \\ \bar{h}^{TT}_{jk}(t, z) = \frac{2G}{c^4 z} \cdot \left[\frac{d^2}{dt^2} q_{jk}^{TT} \left(t - \frac{z}{c} \right) \right] \end{cases}$$

where

$$q_{jk}^{TT}(t) = \mathcal{P}_{jklm} q_{lm}(t) = (P_{jl}P_{km} - \frac{1}{2}P_{jk}P_{lm})q_{lm}(t) .$$

Since the only non-vanishing component of q_{ij} is q_{xx} , we find

$$q_{jk}^{TT} = \left(P_{jx}P_{kx} - \frac{1}{2}P_{jk}P_{xx} \right) q_{xx} = \left(P_{jx}P_{kx} - \frac{1}{2}P_{jk} \right) q_{xx}$$

therefore

$$q_{xx}^{TT} = \left(P_{xx}P_{xx} - \frac{1}{2}P_{xx}^2 \right) q_{xx} = \frac{1}{2}q_{xx}$$

$$q_{yy}^{TT} = \left(P_{yx}P_{yx} - \frac{1}{2}P_{yy} \right) q_{xx} = -\frac{1}{2}q_{xx} ;$$

$$q_{xy}^{TT} = \left(P_{xy}P_{yx} - \frac{1}{2}P_{xy} \right) q_{xx} = 0$$

similarly, the remaining components $q_{zz}^{TT}, q_{zx}^{TT}, q_{zy}^{TT}$ can be shown to vanish. The wave which travels along z therefore is

$$\begin{cases} h_{\mu 0}^{TT} = 0, & h_{xy}^{TT} = 0, & h_{zi}^{TT} = 0, \\ h_{xx}^{TT} = -h_{yy}^{TT} = \frac{G}{c^4 z} \frac{d^2}{dt^2} q_{xx}(t - \frac{z}{c}), \end{cases}$$

Since

$$q_{xx}(t) = m \left[\cos t + A^2 \cos 2\omega t + 2Al_0 \cos \omega t \right]$$

the only non vanishing components of the wave traveling along z are

$$\begin{aligned} h_{xx}^{TT} &= -h_{yy}^{TT} = \\ & \frac{Gm}{c^4 z} \cdot \frac{d^2}{dt^2} \left[\cos t + A^2 \cos 2\omega(t - \frac{z}{c}) + 2Al_0 \cos \omega(t - \frac{z}{c}) \right] \\ &= -\frac{Gm}{c^4 z} \omega^2 \left[4A^2 \cos 2\omega(t - \frac{z}{c}) + 2Al_0 \cos \omega(t - \frac{z}{c}) \right] \end{aligned}$$

If, for instance, we consider two masses $m = 10^3$ kg, with $l_0 = 1$ m, $A = 10^{-4}$ m, and $\omega = 10^4$ Hz:

$$h^{TT} \sim -2 \frac{Gm}{c^4 z} \omega^2 Al_0 \cos \omega(t - \frac{z}{c}) \sim \frac{10^{-35}}{r} !!!$$

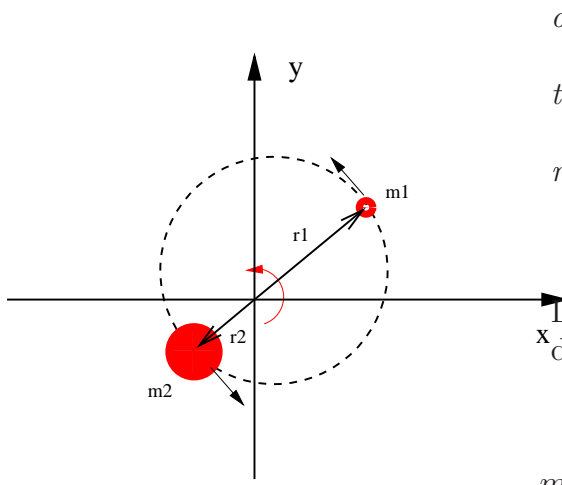
in conclusion

- 1) the radiation emitted along z is linearly polarized
- 2) because of symmetry, the wave emitted along y will be the same
- 3) to find the wave emitted along x , change

$$z \rightarrow x, \quad y \rightarrow z, \quad x \rightarrow y$$

you will find no radiation

GW-emission by a binary system in circular orbit (far from coalescence)



orbital separation l_0

total mass $M \equiv m_1 + m_2$

reduced mass $\mu \equiv \frac{m_1 m_2}{M}$.

Let us consider a coordinate frame with origin coincident with the center of mass

$$l_0 = r_1 + r_2, \quad r_1 m_1 + r_2 m_2 = 0$$

$$r_1 = \frac{m_2 l_0}{M}, \quad r_2 = \frac{m_1 l_0}{M}.$$

The orbital frequency ω_K can be found from Kepler's law: for each mass

$$G \frac{m_1 m_2}{l_0^2} = m_1 \omega_K^2 \frac{m_2 l_0}{M}, \quad G \frac{m_1 m_2}{l_0^2} = m_2 \omega_K^2 \frac{m_1 l_0}{M}$$

i.e.

$$\omega_K = \sqrt{\frac{GM}{l_0^3}}$$

The equations of motion are

$$\begin{cases} x_1 = \frac{m_2}{M} l_0 \cos \omega_K t \\ y_1 = \frac{m_2}{M} l_0 \sin \omega_K t \end{cases} \quad \begin{cases} x_2 = -\frac{m_1}{M} l_0 \cos \omega_K t \\ y_2 = -\frac{m_1}{M} l_0 \sin \omega_K t \end{cases}$$

Let us compute

$$q^{ik}(t) = \frac{1}{c^2} \int_V T^{00}(t, x^n) x^i x^k dx^3$$

where

$$T^{00} = \sum_{n=1}^2 m_n c^2 \delta(x - x_n) \delta(y - y_n) \delta(z)$$

$$\begin{aligned} q_{zz} &= m_1 \int_V \delta(x - x_1) dx \delta(y - y_1) dy z^2 \delta(z) dz \\ &+ m_2 \int_V \delta(x - x_2) dx \delta(y - y_2) dy z^2 \delta(z) dz = 0, \end{aligned}$$

since $\int_V z^2 \delta(z) dz = 0$.

$$\begin{aligned}
q_{xx} &= m_1 \int_V x^2 \delta(x - x_1) dx \delta(y - y_1) dy \delta(z) dz \\
&+ m_2 \int_V x^2 \delta(x - x_2) dx \delta(y - y_2) dy \delta(z) dz \\
&= m_1 x_1^2 + m_2 x_2^2 \\
&= \mu l_0^2 \cos^2 \omega_K t = \frac{\mu}{2} l_0^2 \cos 2\omega_K t + cost
\end{aligned}$$

($\cos 2\alpha = 2 \cos^2 \alpha - 1$). Computing the remaining components in a similar way, we find

$$\begin{aligned}
q_{xx} &= \frac{\mu}{2} l_0^2 \cos 2\omega_K t + cost \\
q_{yy} &= -\frac{\mu}{2} l_0^2 \cos 2\omega_K t + cost \\
q_{xy} &= \frac{\mu}{2} l_0^2 \sin 2\omega_K t .
\end{aligned}$$

therefore, the time-varying part of q_{ij} is:

$$q_{xx} = -q_{yy} = \frac{\mu}{2} l_0^2 \cos 2\omega_K t \quad q_{xy} = \frac{\mu}{2} l_0^2 \sin 2\omega_K t$$

waves are emitted at twice the orbital frequency.

Let us compute the wave emerging in the z-direction: $\vec{n} = \frac{\vec{z}}{r} \rightarrow (0, 0, 1)$ and

$$P_{jk} = \delta_{jk} - n_j n_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} h^{TT\mu 0} = 0, & \mu = 0, 3 \\ h^{TTik}(t, z) = \frac{2G}{c^4 z} \cdot \left[\frac{d^2}{dt^2} q^{TT ik} \left(t - \frac{z}{c} \right) \right] \end{cases}$$

where

$$q^{TT}{}_{jk} = \mathcal{P}_{jkmn} q_{mn} = \left(P_{jm} P_{kn} - \frac{1}{2} P_{jk} P_{mn} \right) q_{mn}$$

The non-vanishing components are q_{xx} , q_{yy} and q_{xy} ,

$$\begin{aligned}
q^{TT}{}_{xx} &= \left(P_{xm} P_{xn} - \frac{1}{2} P_{xx} P_{mn} \right) q_{mn} \\
&= \left(P_{xx} P_{xx} - \frac{1}{2} P_{xx}^2 \right) q_{xx} - \frac{1}{2} P_{xx} P_{yy} q_{yy} = \frac{1}{2} (q_{xx} - q_{yy})
\end{aligned}$$

$$q^{TT}{}_{yy} = \left(P_{ym} P_{yn} - \frac{1}{2} P_{yy} P_{mn} \right) q_{mn} = -\frac{1}{2} (q_{xx} - q_{yy})$$

$$q^{TT}{}_{xy} = \left(P_{xm} P_{yn} - \frac{1}{2} P_{xy} P_{mn} \right) q_{mn} = P_{xx} P_{yy} q_{xy} = q_{xy}$$

and the remaining components vanish.

The final result for the radiation emerging in the z-direction is

$$\begin{cases} h^{TT}_{\mu 0} = 0, & h^{TT}_{zi} = 0, \\ h^{TT}_{xx} = -h^{TT}_{yy} = \frac{G}{c^4 z} \frac{d^2}{dt^2} (q_{xx} - q_{yy}), \\ h^{TT}_{xy} = \frac{2G}{c^4 z} \frac{d^2}{dt^2} q_{xy} \end{cases}$$

and since

$$\begin{aligned} q_{xx} &= -q_{yy} = \frac{\mu}{2} l_0^2 \cos 2\omega_K t \\ q_{xy} &= \frac{\mu}{2} l_0^2 \sin 2\omega_K t \end{aligned}$$

$$\begin{aligned} h^{TT}_{xx} &= -h^{TT}_{yy} = -\frac{G}{c^4 z} \mu l_0^2 (2\omega_K)^2 \cos 2\omega_K \left(t - \frac{z}{c}\right) \\ h^{TT}_{xy} &= -\frac{G}{c^4 z} \mu l_0^2 (2\omega_K)^2 \sin 2\omega_K \left(t - \frac{z}{c}\right). \end{aligned}$$

In conclusion

- 1) radiation is emitted at **twice the orbital frequency**
- 2) the wave along z has **both polarizations**
- 3) since $h^{TT}_{xx} = i h^{TT}_{xy}$ the wave is **circularly polarized**

A more general expression for the TT -wave.

Since

$$\begin{cases} q_{xx} = -q_{yy} = \frac{\mu}{2} l_0^2 \cos 2\omega_K t \\ q_{xy} = \frac{\mu}{2} l_0^2 \sin 2\omega_K t \end{cases}, q_{ij} = \frac{\mu}{2} l_0^2 \begin{pmatrix} \cos 2\omega_K t & \sin 2\omega_K t & 0 \\ \sin 2\omega_K t & -\cos 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If we define

$$A_{ij} = \begin{pmatrix} \cos 2\omega_K t & \sin 2\omega_K t & 0 \\ \sin 2\omega_K t & -\cos 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \boxed{q_{ij} = \frac{\mu}{2} l_0^2 A_{ij}}.$$

In the TT -gauge the wave is

$$h_{ij}^{TT} = \frac{2G}{rc^4} \ddot{q}_{ij}^{TT} \equiv \frac{2G}{rc^4} \mathcal{P}_{ijkl} q_{kl}$$

$$h_{ij}^{TT} = \frac{2G}{rc^4} \frac{\mu}{2} l_0^2 (2\omega_K)^2 [\mathcal{P}_{ijkl} A_{kl}] \equiv \frac{4\mu MG^2}{rl_0 c^4} A_{kl}^{TT}$$

where we have used $\omega_K = \sqrt{GM/l_0^3}$. Thus, the wave amplitude is (order of magnitude)

$$\boxed{h_0 = \frac{4\mu MG^2}{rl_0 c^4}}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \quad M = m_1 + m_2. \quad (9)$$

and the general form for the TT -wave is

$$\boxed{h_{ij}^{TT} = h_0 [\mathcal{P}_{ijkl} A_{kl}]}. \quad (10)$$

if $\hat{n} = \hat{z} \rightarrow P_{ij} = \text{diag}(1, 1, 0)$

$$A_{ij}^{TT} = \begin{pmatrix} \cos 2\omega_K t & \sin 2\omega_K t & 0 \\ \sin 2\omega_K t & -\cos 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

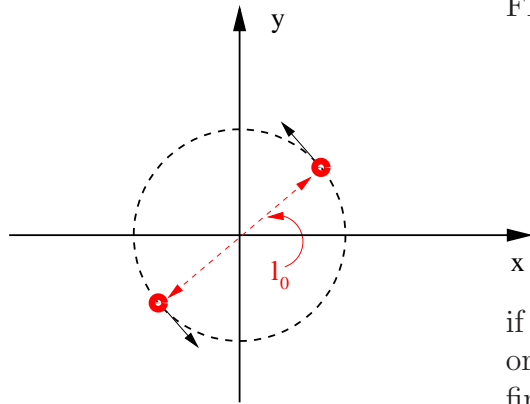
if $\hat{n} = \hat{x} \rightarrow P_{ij} = \text{diag}(0, 1, 1)$ the wave is linearly polarized:

$$A_{ij}^{TT} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}\cos 2\omega_K t & 0 \\ 0 & 0 & \frac{1}{2}\cos 2\omega_K t \end{pmatrix};$$

if $\hat{n} = \hat{y} \rightarrow P_{ij} = \text{diag}(1, 0, 1)$ the wave is linearly polarized:

$$A_{ij}^{TT} = \begin{pmatrix} \frac{1}{2}\cos 2\omega_K t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\cos 2\omega_K t \end{pmatrix}.$$

Binary Pulsar PSR 1931+16 (Taylor - Weisberg 1982)



FROM OBSERVATIONS:

$$\begin{aligned}
 M_1 &\sim M_2 \sim 1.4M_\odot, \\
 T &= 7h\ 45m\ 7s, \\
 \nu_K &\sim 3.58 \cdot 10^{-5} \text{ Hz} \\
 l_0 &= 0.19 \cdot 10^{12} \text{ cm} \simeq 2R_\odot
 \end{aligned}$$

if we assume that the **orbit is circular**, (however, the orbit is eccentric, $\epsilon \simeq 0.62$), from the observed data we find

$$\nu_{GW} = 2\nu_K \sim 7.16 \cdot 10^{-5} \text{ Hz} .$$

The distance of the system from Earth is

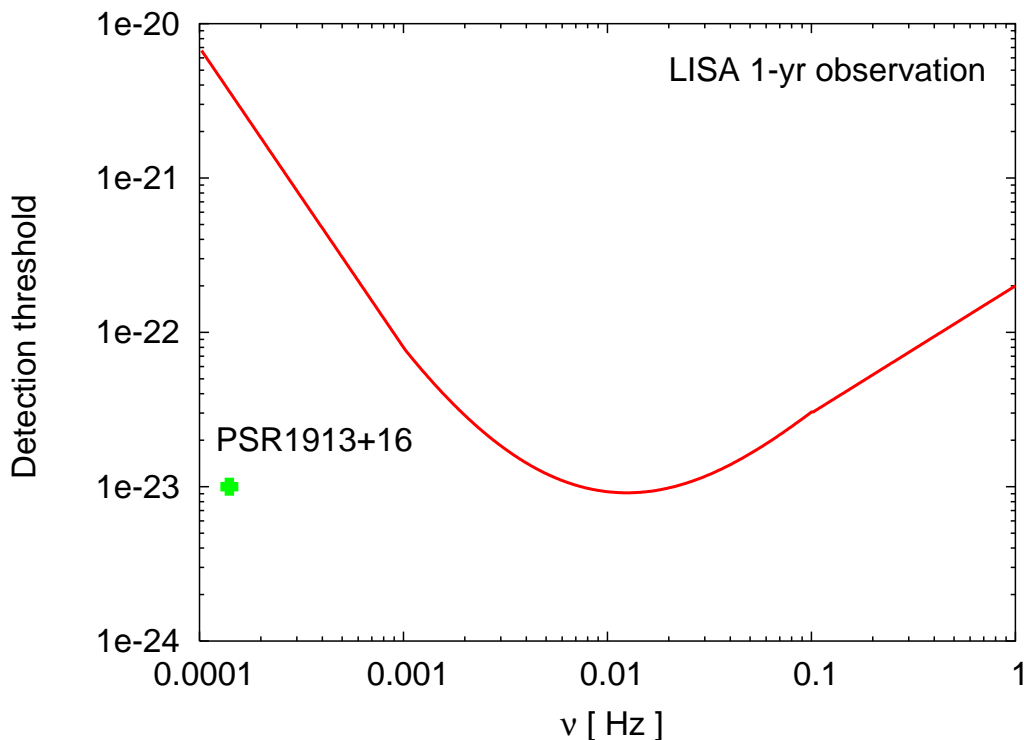
$$r = 5 \text{ kpc}, \quad 1 \text{ pc} = 3.08 \cdot 10^{18} \text{ cm}, \rightarrow r = 1.5 \cdot 10^{22} \text{ cm} ,$$

the wave amplitude is

$$h_0 = \frac{4\mu MG^2}{rl_0c^4} \sim 6 \cdot 10^{-23} .$$

Ground-based and space-based interferometers are sensitive in the frequency regions:

$$\begin{aligned}
 \text{LIGO} &[40 \text{ Hz} - 1 - 2 \text{ kHz}] & \text{LISA} &[10^{-4} - 10^{-1}] \text{ Hz} \\
 \text{VIRGO} &[10 \text{ Hz} - 1 - 2 \text{ kHz}] & &
 \end{aligned}$$



Waves from PSR cannot be detected directly by current detectors.

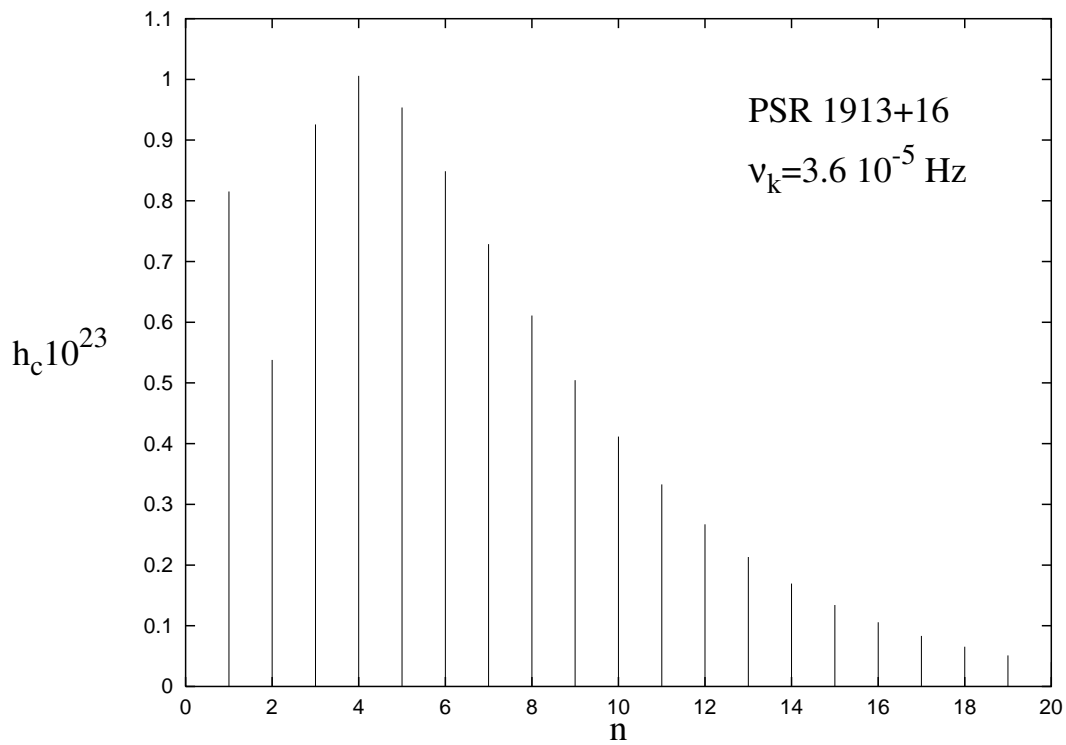
Let us check whether we are in the condition to apply the quadrupole formalism:

$$\lambda_{GW} = \frac{c}{\nu_{GW}} \sim 10^{14} \text{ cm} \quad \lambda_{GW} \gg l_0$$

Yes we are!

Circular orbit \rightarrow GWs are emitted at twice the orbital frequency

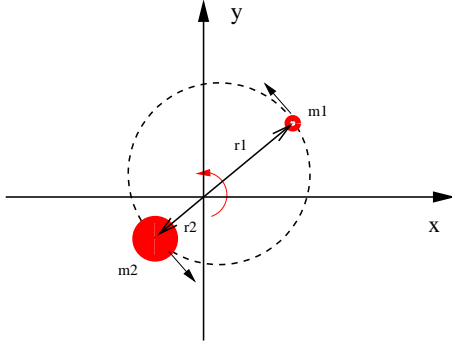
If the orbit is elliptic, waves are emitted at frequencies multiple of the orbital frequency, and the number of equally spaced spectral lines increases with ellipticity.



For a detailed description:

Michele Maggiore, *Gravitational waves*, Vol. I: Theory and experiments. Oxford University Press.

New Binary Pulsar PSR J0737-3039 discovered in 2003.



$$m_1 = 1.337M_{\odot}, \quad m_2 \sim 1.250M_{\odot},$$

$$T = 2.4h, \quad e = 0.08$$

$$r = 500 \text{ pc} \quad l_0 \sim 1.2R_{\odot}$$

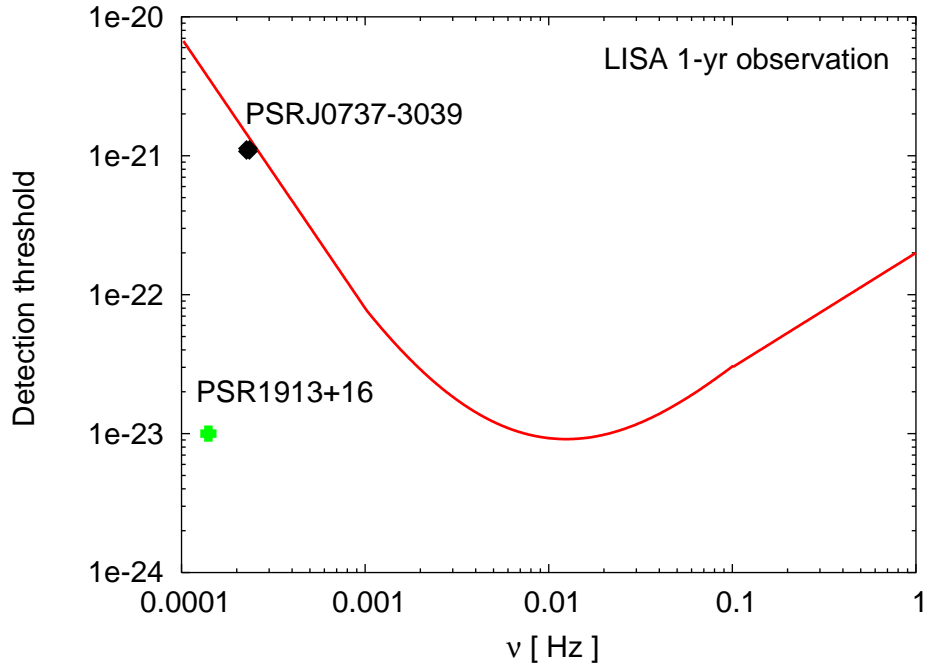
the orbit is nearly circular.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = 0.646M_{\odot}$$

$$h_0 = \frac{4\mu MG^2}{r l_0 c^4} \sim 1.1 \cdot 10^{-21}$$

and waves are emitted at the frequency

$$\nu_{GW} = 2 \nu_K = 2.3 \cdot 10^{-4} \text{ Hz}$$



There are many other binaries LISA could detect:

Cataclismic variables:

semi-detached systems with small orbital period

primary star: white dwarf

secondary star: star filling the Roche lobe and accreting matter on the companion

emission frequency: few digits $\cdot 10^{-4}$ Hz,

$h \sim [10^{-22} - 10^{-21}]$

Double-degenerate binary systems (WD-WD, WD-NS)

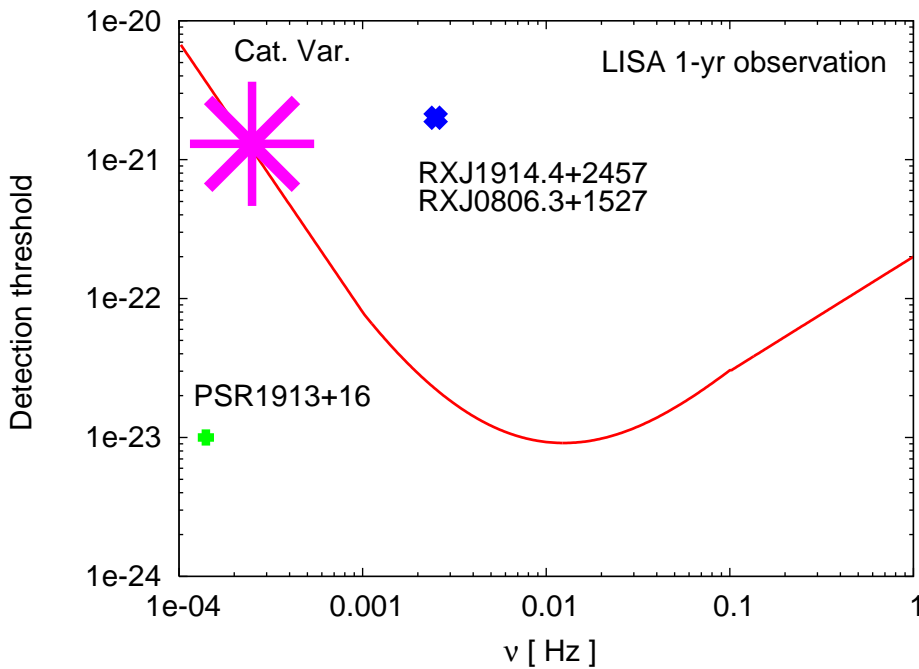
Ultra-short period : < 10 minutes, $\nu_{GW} > 10^{-3}$ s

Strong X-ray emitter

RXJ1914.4+2457

$M_1=0.5 M_{\odot}$ $M_2=0.1 M_{\odot}$

$P=9.5$ min



THE ENERGY-MOMENTUM PSEUDOTENSOR

$T^{\mu\nu}$ of matter satisfies the (covariant) divergenceless equation

$$T^{\mu\nu}{}_{;\nu} = 0 \quad (11)$$

We know it is not a conservation law, because it cannot be written as an ordinary divergence.

In a locally inertial frame (LIF): eq. (11) becomes

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0, \quad (12)$$

this means that $T^{\mu\nu}$ can be written as:

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha} \eta^{\mu\nu\alpha}, \quad (13)$$

where $\eta^{\mu\nu\alpha}$ is **antisymmetric** in ν and α ; **INDEED**

$$\frac{\partial^2 \eta^{\mu\nu\alpha}}{\partial x^\nu \partial x^\alpha} = 0,$$

because the derivative operator is symmetric in ν and α

We want to find the expression of $\eta^{\mu\nu\alpha}$: write Einstein eqs.

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \quad \longrightarrow \quad T^{\mu\nu} = \frac{c^4}{8\pi G} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right). \quad (14)$$

In a LIF $R^{\mu\nu}$ is:

$$R^{\mu\nu} = \frac{1}{2} g^{\mu\alpha} g^{\nu\beta} g^{\gamma\delta} \left(\frac{\partial^2 g_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 g_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} \right).$$

By replacing in eq. (14), $T^{\mu\nu}$ becomes

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha} \left\{ \frac{c^4}{16\pi G} \frac{1}{(-g)} \frac{\partial}{\partial x^\beta} [(-g) (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta})] \right\} \quad (15)$$

The part within $\{ \}$ is antisymmetric in ν and α , symmetric in μ and ν , and it is the quantity $\eta^{\mu\nu\alpha}$ we were looking for.

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha} \eta^{\mu\nu\alpha}, \quad \eta^{\mu\nu\alpha} = \left\{ \frac{c^4}{16\pi G} \frac{1}{(-g)} \frac{\partial}{\partial x^\beta} [(-g) (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta})] \right\}$$

Since in a LIF $g_{\mu\nu,\alpha} = 0$ we can extract $\frac{1}{(-g)}$ and write this equation as

$$(-g)T^{\mu\nu} = \frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}, \quad (16)$$

where

$$\zeta^{\mu\nu\alpha} = (-g)\eta^{\mu\nu\alpha} = \frac{c^4}{16\pi G} \frac{\partial}{\partial x^\beta} [(-g) (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta})]. \quad (17)$$

EQ. (16) has been derived in a locally inertial frame. In any other frame $(-g)T^{\mu\nu}$ will not

equate $\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}$, therefore, **in a generic frame**

$$\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha} - (-g)T^{\mu\nu} \neq 0.$$

We shall call this difference $(-g)t^{\mu\nu}$, i.e.

$$(-g)t^{\mu\nu} = \frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha} - (-g)T^{\mu\nu}.$$

The quantities $t^{\mu\nu}$ are symmetric, because $T^{\mu\nu}$ and $\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}$ are symmetric in μ and ν .
It follows that

$$(-g)(T^{\mu\nu} + t^{\mu\nu}) = \frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha} \quad \longrightarrow \quad \frac{\partial}{\partial x^\nu} [(-g)(T^{\mu\nu} + t^{\mu\nu})] = 0,$$

this is

THE CONSERVATION LAW OF THE TOTAL ENERGY AND MOMENTUM OF MATTER + GRAVITATIONAL FIELD VALID IN ANY REFERENCE FRAME.

$$(-g)t^{\mu\nu} = \frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha} - (-g)T^{\mu\nu}.$$

If we express $T^{\mu\nu}$ in terms of $g_{\mu\nu}$ and by using Einstein's eqs.

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \quad \longrightarrow \quad T^{\mu\nu} = \frac{c^4}{8\pi G} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right).$$

and eq. (17) it is possible to show that $t^{\mu\nu}$ can be written as follows

$$\begin{aligned} t^{\mu\nu} = & \frac{c^4}{16\pi G} \left\{ (2\Gamma^\delta_{\alpha\beta}\Gamma^\sigma_{\delta\sigma} - \Gamma^\delta_{\alpha\sigma}\Gamma^\sigma_{\beta\delta} - \Gamma^\delta_{\alpha\delta}\Gamma^\sigma_{\beta\sigma}) (g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}) \right. \\ & + g^{\mu\alpha}g^{\beta\delta} (\Gamma^\nu_{\alpha\sigma}\Gamma^\sigma_{\beta\delta} + \Gamma^\nu_{\beta\delta}\Gamma^\sigma_{\alpha\sigma} - \Gamma^\nu_{\delta\sigma}\Gamma^\sigma_{\alpha\beta} - \Gamma^\nu_{\alpha\beta}\Gamma^\sigma_{\delta\sigma}) \\ & + g^{\nu\alpha}g^{\beta\delta} (\Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\delta} + \Gamma^\mu_{\beta\delta}\Gamma^\sigma_{\alpha\sigma} - \Gamma^\mu_{\delta\sigma}\Gamma^\sigma_{\alpha\beta} - \Gamma^\mu_{\alpha\beta}\Gamma^\sigma_{\delta\sigma}) \\ & \left. + g^{\alpha\beta}g^{\delta\sigma} (\Gamma^\mu_{\alpha\delta}\Gamma^\nu_{\beta\sigma} - \Gamma^\mu_{\alpha\beta}\Gamma^\nu_{\delta\sigma}) \right\} \end{aligned}$$

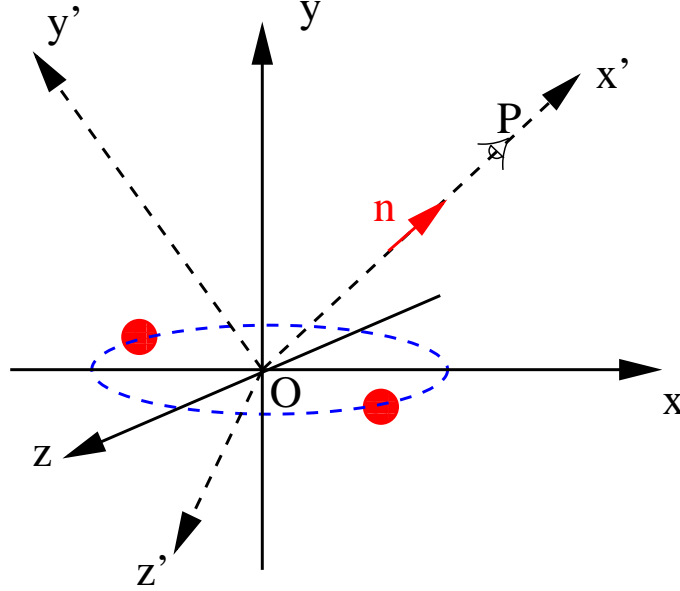
This is the stress-energy pseudotensor of the gravitational field.

$t^{\mu\nu}$ it is not a tensor because :

- 1) it is the **ordinary** derivative, (not the covariant one) of a tensor
- 2) it is a combination of the Γ 's that are not tensors.

However, as the Γ 's, it behaves as a tensor under a linear coordinate transformation.

Let us consider an emitting source and the associated 3-dimensional coordinate frame $O(x, y, z)$. Be an observer located at $P = (x_1, y_1, z_1)$ at a distance $r = \sqrt{x_1^2 + y_1^2 + z_1^2}$ from the origin. The observer wants to detect the wave traveling along the direction identified by the versor $n = \frac{r}{|r|}$.



Consider a second frame $O'(x', y', z')$, with origin coincident with O , and having the x' -axis aligned with n . Assuming that the wave traveling along x' is linearly polarized and has only one polarization, the corresponding metric tensor will be

$$g_{\mu'\nu'} = \begin{pmatrix} (t) & (x) & (y) & (z) \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & [1 + h_+^{TT}(t, x')] & 0 \\ 0 & 0 & 0 & [1 - h_+^{TT}(t, x')] \end{pmatrix},$$

The observer wants to measure the energy which flows per unit time across the unit surface orthogonal to x' , i.e. $t^{0x'}$; therefore he needs to compute the Christoffel symbols i.e. the derivatives of $h_{\mu'\nu'}^{TT}$. The metric perturbation has the form $h^{TT}(t, x') = \frac{const}{x'} \cdot f(t - \frac{x'}{c})$, the only derivatives which come into play are those with respect to time and x'

$$\begin{aligned} \frac{\partial h^{TT}}{\partial t} &\equiv \dot{h}^{TT} = \frac{const}{x'} \dot{f}, \\ \frac{\partial h^{TT}}{\partial x'} &\equiv h^{TT'} = -\frac{const}{x'^2} f + \frac{const}{x'} f' \sim -\frac{1}{c} \frac{const}{x'} \dot{f} = -\frac{1}{c} \dot{h}^{TT}, \end{aligned}$$

where we have retained only the dominant $1/x'$ term. Thus, the non-vanishing Christoffel symbols are:

$$\begin{aligned} \Gamma^0_{y'y'} &= -\Gamma^0_{z'z'} = \frac{1}{2} \dot{h}_+^{TT} & \Gamma^{y'}_{0y'} &= -\Gamma^{z'}_{0z'} = \frac{1}{2} \dot{h}_+^{TT} \\ \Gamma^{x'}_{y'y'} &= -\Gamma^{x'}_{z'z'} = \frac{1}{2c} \dot{h}_+^{TT} & \Gamma^{y'}_{y'x'} &= -\Gamma^{z'}_{z'x'} = -\frac{1}{2c} \dot{h}_+^{TT}. \end{aligned}$$

from which we find

$$t^{0x} \equiv \frac{dE_{GW}}{dx^0 dS} = \frac{c^2}{16\pi G} \left[\left(\frac{d \mathbf{h}^{TT}(\mathbf{t}, \mathbf{x}')}{dt} \right)^2 \right].$$

Thus the energy flux is

$$\boxed{\frac{dE_{GW}}{dt dS} = \frac{c^3}{16\pi G} \left[\left(\frac{d \mathbf{h}^{TT}(\mathbf{t}, \mathbf{x}')}{dt} \right)^2 \right]}.$$

In general, if both polarization are present

$$g_{\mu'\nu'} = \begin{pmatrix} (t) & (x) & (y) & (z) \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & [1 + h_+^{TT}(t, x')] & h_\times^{TT}(t, x') \\ 0 & 0 & h_\times^{TT}(t, x') & [1 - h_+^{TT}(t, x')] \end{pmatrix},$$

$$t^{0x'} = \frac{c^2}{16\pi G} \left[\left(\frac{dh_+^{TT}}{dt} \right)^2 + \left(\frac{dh_\times^{TT}}{dt} \right)^2 \right] = \frac{c^2}{32\pi G} \left[\sum_{jk} \left(\frac{dh_{jk}^{TT}}{dt} \right)^2 \right].$$

$c t^{0x'}$ is the energy flowing across a unit surface orthogonal to the direction x' per unit time.

However, the direction x' is arbitrary; if the observer is located in a different position and computes the energy flux he receives, he will find formally the same but with h_{jk}^{TT} referred to the TT-gauge associated with the new direction. Therefore, if we consider a generic direction $\vec{r} = r\vec{n}$

$$t^{0r} = \frac{c^2}{32\pi G} \left[\sum_{jk} \left(\frac{dh_{jk}^{TT}(t, r)}{dt} \right)^2 \right].$$

Since in GR the energy of the gravitational field cannot be defined locally, to find the GW-flux we need to average over several wavelengths, i.e.

$$\frac{dE_{GW}}{dt dS} = \langle ct^{0r} \rangle = \frac{c^3}{32\pi G} \left\langle \sum_{jk} \left(\frac{dh_{jk}^{TT}}{dt} \right)^2 \right\rangle.$$

We shall now express the energy flux directly in terms of the source quadrupole moment. Since

$$\begin{cases} \bar{h}_{\mu 0}^{TT} = 0, & \mu = 0, 3 \\ \bar{h}_{ik}^{TT}(t, r) = \frac{2G}{c^4 r} \cdot \left[\frac{d^2}{dt^2} q_{ik}^{TT} \left(t - \frac{r}{c} \right) \right] \end{cases}$$

by direct substitution we find

$$\begin{aligned} \frac{dE_{GW}}{dt dS} &= \frac{c^3}{32\pi G} \left\langle \sum_{jk} \left(\frac{dh_{jk}^{TT}}{dt} \right)^2 \right\rangle = \frac{G}{8\pi c^5 r^2} \left\langle \sum_{jk} \left[\overset{\dots TT}{q}_{jk} \left(t - \frac{r}{c} \right) \right]^2 \right\rangle \\ &= \frac{G}{8\pi c^5 r^2} \left\langle \sum_{jk} \left[\mathcal{P}_{jkmn} \ddot{q}_{mn} \left(t - \frac{r}{c} \right) \right]^2 \right\rangle. \end{aligned}$$

From this formula we can compute the gravitational luminosity $L_{GW} = \frac{dE_{GW}}{dt}$:

$$\begin{aligned} L_{GW} &= \int \frac{dE_{GW}}{dt dS} dS = \int \frac{dE_{GW}}{dt dS} r^2 d\Omega \\ &= \frac{G}{2c^5} \frac{1}{4\pi} \int d\Omega \left\langle \sum_{jk} \left(\mathcal{P}_{jkmn} \ddot{q}_{mn} \left(t - \frac{r}{c} \right) \right)^2 \right\rangle. \end{aligned}$$

To evaluate the integral over the solid angle it is convenient to replace the quadrupole moment q_{mn} with the **reduced quadrupole moment**

$$Q_{ij} = q_{ij} - \frac{1}{3} \delta_{ij} q^k{}_k;$$

we remind that, since its trace is zero by definition, we have

$$\mathcal{P}_{ijlm} Q_{lm} = \mathcal{P}_{ijlm} q_{lm}.$$

The GW luminosity becomes

$$L_{GW} = \frac{G}{2c^5} \frac{1}{4\pi} \int d\Omega \left\langle \sum_{jk} \left(\mathcal{P}_{jkmn} \ddot{Q}_{mn} \left(t - \frac{r}{c} \right) \right)^2 \right\rangle.$$

Let us compute the integral over the solid angle. By using the definitions (5) and (6) and the properties of \mathcal{P}_{mnjk} we find

$$\begin{aligned} \sum_{jk} \left(\mathcal{P}_{jkmn} \ddot{Q}_{mn} \right)^2 &= \sum_{jk} \mathcal{P}_{jkmn} \ddot{Q}_{mn} \mathcal{P}_{jkrs} \ddot{Q}_{rs} = \\ \sum_{jk} \mathcal{P}_{mnjk} \mathcal{P}_{jkrs} \ddot{Q}_{mn} \ddot{Q}_{rs} &= \mathcal{P}_{mnr s} \ddot{Q}_{mn} \ddot{Q}_{rs} = \\ \left[(\delta_{mr} - n_m n_r) (\delta_{ns} - n_n n_s) - \frac{1}{2} (\delta_{mn} - n_m n_n) (\delta_{rs} - n_r n_s) \right] &\ddot{Q}_{mn} \ddot{Q}_{rs}. \end{aligned}$$

If we expand this expression, and remember that

- $\delta_{mn} \ddot{Q}_{mn} = \delta_{rs} \ddot{Q}_{rs} = 0$
because the trace of Q_{ij} vanishes by definition, and

- $n_m n_r \delta_{ns} \ddot{Q}_{mn} \ddot{Q}_{rs} = n_n n_s \delta_{mr} \ddot{Q}_{mn} \ddot{Q}_{rs}$
because Q_{ij} is symmetric, we find

$$\sum_{jk} \left(\mathcal{P}_{jkmn} \ddot{Q}_{mn} \right)^2 = \ddot{Q}_{rn} \ddot{Q}_{rn} - 2n_m n_r \ddot{Q}_{ms} \ddot{Q}_{sr} + \frac{1}{2} n_m n_n n_r n_s \ddot{Q}_{mn} \ddot{Q}_{rs}.$$

and by replacing in the equation for L_{GW}

$$L_{GW} = \frac{G}{2c^5} \frac{1}{4\pi} \int d\Omega \left[\ddot{Q}_{rn} \ddot{Q}_{rn} - 2n_m n_r \ddot{Q}_{ms} \ddot{Q}_{sr} + \frac{1}{2} n_m n_n n_r n_s \ddot{Q}_{mn} \ddot{Q}_{rs} \right].$$

The integrals to be calculated over the solid angle are:

$$\frac{1}{4\pi} \int d\Omega n_i n_j, \quad \text{and} \quad \frac{1}{4\pi} \int d\Omega n_i n_j n_r n_s.$$

In polar coordinates, the versor \mathbf{n} is

$$n_i = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta). \quad (18)$$

Thus, for parity reasons

$$\frac{1}{4\pi} \int d\Omega n_i n_j = \frac{1}{4\pi} \int_0^\pi d\vartheta \sin \vartheta \int_0^{2\pi} d\varphi n_i n_j = 0 \quad \text{when } i \neq j.$$

Furthermore, it is easy to show that

$$\frac{1}{4\pi} \int d\Omega n_1^2 = \int d\Omega n_2^2 = \int d\Omega n_3^2 = \frac{1}{3} \quad \rightarrow \quad \frac{1}{4\pi} \int d\Omega n_i n_j = \frac{1}{3} \cdot \delta_{ij}.$$

The second integral can be computed in a similar way and gives

$$\frac{1}{4\pi} \int d\Omega n_i n_j n_r n_s = \frac{1}{15} (\delta_{ij} \delta_{rs} + \delta_{ir} \delta_{js} + \delta_{is} \delta_{jr}).$$

Consequently

$$\begin{aligned} \frac{1}{4\pi} \int d\Omega \left(\ddot{Q}_{rn} \ddot{Q}_{rn} - 2n_m \ddot{Q}_{ms} \ddot{Q}_{sr} n_r + \frac{1}{2} n_m n_n n_r n_s \ddot{Q}_{mn} \ddot{Q}_{rs} \right) \\ = \frac{2}{5} \ddot{Q}_{rn} \ddot{Q}_{rn} \end{aligned} \quad (19)$$

and, finally, the emitted power is

$$\boxed{L_{GW} = \frac{dE_{GW}}{dt} = \frac{G}{5c^5} \left\langle \sum_{jk=1,3} \ddot{Q}_{jk} \left(t - \frac{r}{c} \right) \ddot{Q}_{jk} \left(t - \frac{r}{c} \right) \right\rangle}. \quad (20)$$

Equation (20) was first derived by Einstein. **We shall now compute the GW-luminosity of a binary system**

We need to compute the reduced quadrupole moment

$$Q_{ij} = q_{ij} - \frac{1}{3} \delta_{ij} q^k_k.$$

For a circular orbit the time-varying part of q_{ij} is:

$$\boxed{q_{ij}(t) = \frac{\mu}{2} l_0^2 A_{ij}(t)} \quad \text{where} \quad A_{ij}(t) = \begin{pmatrix} \cos 2\omega_K t & \sin 2\omega_K t & 0 \\ \sin 2\omega_K t & -\cos 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

The trace of q_{ij} is

$$q^k{}_k = \eta^{kl} q_{kl} = q_{xx} + q_{yy} = 0 ,$$

therefore, the time-varying part of $Q_{ij}(t - r/c)$ is

$$Q_{ij} = \frac{\mu}{2} l_0^2 \begin{pmatrix} \cos 2\omega_K t_{ret} & \sin 2\omega_K t_{ret} & 0 \\ \sin 2\omega_K t_{ret} & -\cos 2\omega_K t_{ret} & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

its third time-derivative is

$$\ddot{Q}_{ij} = \frac{\mu}{2} l_0^2 8 \omega_K^3 \begin{pmatrix} \sin 2\omega_K t_{ret} & -\cos 2\omega_K t_{ret} & 0 \\ -\cos 2\omega_K t_{ret} & -\sin 2\omega_K t_{ret} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \omega_K = \sqrt{\frac{GM}{l_0^3}}$$

and

$$\sum_{jk} \ddot{Q}_{jk} \ddot{Q}_{jk} = 32 \mu^2 l_0^4 \omega_K^6 = 32 \mu^2 G^3 \frac{M^3}{l_0^5} .$$

By substituting this expression in eq. (20) we find

$$\boxed{L_{GW} \equiv \frac{dE_{GW}}{dt} = \frac{32}{5} \frac{G^4}{c^5} \frac{\mu^2 M^3}{l_0^5}} . \quad (21)$$

For the binary pulsar **PSR 1913+16**

$$\boxed{L_{GW} = 0.7 \cdot 10^{31} \text{ erg/s}} .$$

Power radiated in gravitational waves by a binary system in circular orbit:

$$L_{GW} \equiv \frac{dE_{GW}}{dt} = \frac{32}{5} \frac{G^4}{c^5} \frac{\mu^2 M^3}{l_0^5} .$$

This expression has to be considered as an average over several wavelengths (or equivalently, over a sufficiently large number of periods), as stated in eq. (20); therefore, in order L_{GW} to be defined, we must be in a regime where the orbital parameters do not change significantly over the time interval taken to perform the average. This assumption is called **adiabatic approximation**, and it is certainly applicable to systems like PSR 1913+16 or PSR J0737-3039 that are very far from coalescence.

In the adiabatic regime, the system has the time to adjust the orbit to compensate the energy lost in gravitational waves with a change in the orbital energy, in such a way that

$$\frac{dE_{orb}}{dt} + L_{GW} = 0. \quad (22)$$

Let us see what are the consequences of this equation.

The orbital energy of the binary is

$$E_{orb} = E_K + U ,$$

where the kinetic and the gravitational energy are, respectively,

$$\begin{aligned} E_K &= \frac{1}{2} m_1 \omega_K^2 r_1^2 + \frac{1}{2} m_2 \omega_K^2 r_2^2 = \frac{1}{2} \omega_K^2 \left[\frac{m_1 m_2^2 l_0^2}{M^2} + \frac{m_2 m_1^2 l_0^2}{M^2} \right] \\ &= \frac{1}{2} \omega_K^2 \mu l_0^2 = \frac{1}{2} \frac{G \mu M}{l_0} \end{aligned}$$

and

$$U = -\frac{G m_1 m_2}{l_0} = -\frac{G \mu M}{l_0} .$$

Therefore

$$E_{orb} = -\frac{1}{2} \frac{G \mu M}{l_0}$$

and its time derivative is

$$\frac{dE_{orb}}{dt} = \frac{1}{2} \frac{G \mu M}{l_0} \left(\frac{1}{l_0} \frac{dl_0}{dt} \right) = -E_{orb} \left(\frac{1}{l_0} \frac{dl_0}{dt} \right). \quad (23)$$

The term $\frac{dl_0}{dt}$ can be expressed in terms of the time derivative of ω_K as follows

$$\omega_K^2 = GM l_0^{-3} \rightarrow 2 \ln \omega_K = \ln GM - 3 \ln l_0 \rightarrow \frac{1}{\omega_K} \frac{d\omega_K}{dt} = -\frac{3}{2} \frac{1}{l_0} \frac{dl_0}{dt},$$

and eq. (23) becomes

$$\frac{dE_{orb}}{dt} = \frac{2}{3} \frac{E_{orb}}{\omega_K} \frac{d\omega_K}{dt}. \quad (24)$$

Since $\omega_K = 2\pi P^{-1}$

$$\frac{1}{\omega_K} \frac{d\omega_K}{dt} = -\frac{1}{P} \frac{dP}{dt}$$

and eq. (24) gives

$$\frac{dE_{orb}}{dt} = -\frac{2}{3} \frac{E_{orb}}{P} \frac{dP}{dt} \quad \rightarrow \quad \frac{dP}{dt} = -\frac{3}{2} \frac{P}{E_{orb}} \frac{dE_{orb}}{dt}. \quad (25)$$

Since by eq. (22) $\frac{dE_{orb}}{dt} = -L_{GW}$ we finally find how the orbital period changes due to the emission of gravitational waves

$$\boxed{\frac{dP}{dt} = \frac{3}{2} \frac{P}{E_{orb}} L_{GW}}. \quad (26)$$

For example if we consider PSR 1913+16, assuming the orbit is circular we find

$$P = 27907 \text{ s}, \quad E_{orb} \sim -1.4 \cdot 10^{48} \text{ erg}, \quad L_{GW} \sim 0.7 \cdot 10^{31} \text{ erg/s}$$

and

$$\frac{dP}{dt} \sim -2.2 \cdot 10^{-13}.$$

The orbit of the real system has a quite strong eccentricity $\epsilon \simeq 0.617$. If we would do the calculations using the equations of motion appropriate for an eccentric orbit we would find

$$\frac{dP}{dt} = -2.4 \cdot 10^{-12}.$$

PSR 1913+16 has now been monitored for more than three decades and the rate of variation of the period, measured with very high accuracy, is

$$\frac{dP}{dt} = -(2.4184 \pm 0.0009) \cdot 10^{-12}.$$

(J. M. Wisberg, J.H. Taylor *Relativistic Binary Pulsar B1913+16: Thirty Years of Observations and Analysis*, in Binary Radio Pulsars, ASP Conference series, 2004, eds. F.AA.Rasio, I.H.Stairs).

Residual differences due to Doppler corrections, due to the relative velocity between us and the pulsar induced by the differential rotation of the Galaxy.

$$\boxed{\frac{\dot{P}_{corrected}}{\dot{P}_{GR}} = 1.0013(21)}$$

For the recently discovered double pulsar PSR J0737-3039

$$P = 8640 \text{ s}, \quad E_{orb} \sim -2.55 \cdot 10^{48} \text{ erg}, \quad L_{GW} \sim 2.24 \cdot 10^{32} \text{ erg/s}$$

and

$$\frac{dP}{dt} \sim -1.2 \cdot 10^{-12},$$

which is also in agreement with observations. **Thus, this prediction of General Relativity is confirmed by observations. This result provided the first indirect evidence of the existence of gravitational waves and for this discovery R.A. Hulse and J.H. Taylor have been awarded of the Nobel prize in 1993.**



ORBITAL EVOLUTION

Knowing the energy lost by the system, we can also evaluate how the orbital separation l_0 changes in time. From eq. (23)

$$\frac{1}{l_0} \frac{dl_0}{dt} = -\frac{1}{E_{orb}} \frac{dE_{orb}}{dt};$$

remembering that $L_{GW} = \frac{32}{5} \frac{G^4}{c^5} \frac{\mu^2 M^3}{l_0^5}$, and $E_{orb} = -\frac{1}{2} \frac{G\mu M}{l_0}$, and that in the adiabatic approximation $\frac{dE_{orb}}{dt} = -L_{GW}$, we find

$$\frac{1}{l_0} \frac{dl_0}{dt} = \frac{L_{GW}}{E_{orb}} \rightarrow \frac{1}{l_0} \frac{dl_0}{dt} = -\left[\frac{64}{5} \frac{G^3}{c^5} \mu M^2 \right] \cdot \frac{1}{l_0^4}$$

Assuming that at some initial time $t = 0$ the orbital separation is $l_0(t = 0) = l_0^{in}$ by integrating this equation we easily find

$$l_0^4(t) = (l_0^{in})^4 - \frac{256}{5} \frac{G^3}{c^5} \mu M^2 t = (l_0^{in})^4 \left[1 - \frac{256}{5} \frac{G^3}{c^5 (l_0^{in})^4} \mu M^2 t \right]$$

If we define

$$t_{coal} = \frac{5}{256} \frac{c^5 (l_0^{in})^4}{G^3 \mu M^2},$$

the previous equation becomes

$$\boxed{l_0(t) = l_0^{in} \left[1 - \frac{t}{t_{coal}} \right]^{1/4}}.$$

which shows that the orbital separation decreases in time.

When $t = t_{coal}$ the orbital separation becomes zero, and this is possible because we have assumed that the bodies composing the binary system are pointlike. Of course, stars and black holes have finite sizes, therefore they start merging and coalesce before $t = t_{coal}$. In addition, when the two stars are close enough, both the slow motion approximation and the weak field assumption on which the quadrupole formalism relies fails to hold and strong field effects have to be considered; however, t_{coal} gives an order of magnitude of the time the system needs to merge starting from a given initial distance l_0^{in} .

WAVEFORM: AMPLITUDE AND PHASE

Since the orbital separation between the two bodies decreases with time as

$$l_0(t) = l_0^{in} \left[1 - \frac{t}{t_{coal}} \right]^{1/4},$$

the Keplerian angular velocity $\omega_K = \sqrt{GM/l_0^3}$ changes in time as

$$\omega_K(t) = \sqrt{\frac{GM}{l_0^3}} = \frac{\omega_K^{in}}{\left[1 - \frac{t}{t_{coal}} \right]^{3/8}}, \quad \omega_K^{in} = \sqrt{\frac{GM}{(l_0^{in})^3}}.$$

Since in the adiabatic regime the orbit evolves through a sequence of **stationary circular orbits**, the frequency of the emitted wave at some time t is twice the orbital frequency at that time, i.e.

$$\nu_{GW}(t) = \frac{\omega_K}{\pi} = \frac{\nu_{GW}^{in}}{\left[1 - \frac{t}{t_{coal}} \right]^{3/8}}, \quad \nu_{GW}^{in} = \frac{1}{\pi} \sqrt{\frac{GM}{(l_0^{in})^3}},$$

Similarly, the instantaneous amplitude of the emitted signal can be found from eq. (9)

$$h_0(t) = \frac{4\mu MG^2}{rl_0(t)c^4};$$

since $\omega_K^2 = GM/l_0^3$,

$$h_0(t) = \frac{4\mu MG^2}{rc^4} \cdot \frac{\omega_K^{2/3}(t)}{G^{1/3}M^{1/3}} = \frac{4G^{5/3}\mu M^{2/3}}{rc^4} \cdot \omega_K^{2/3}(t);$$

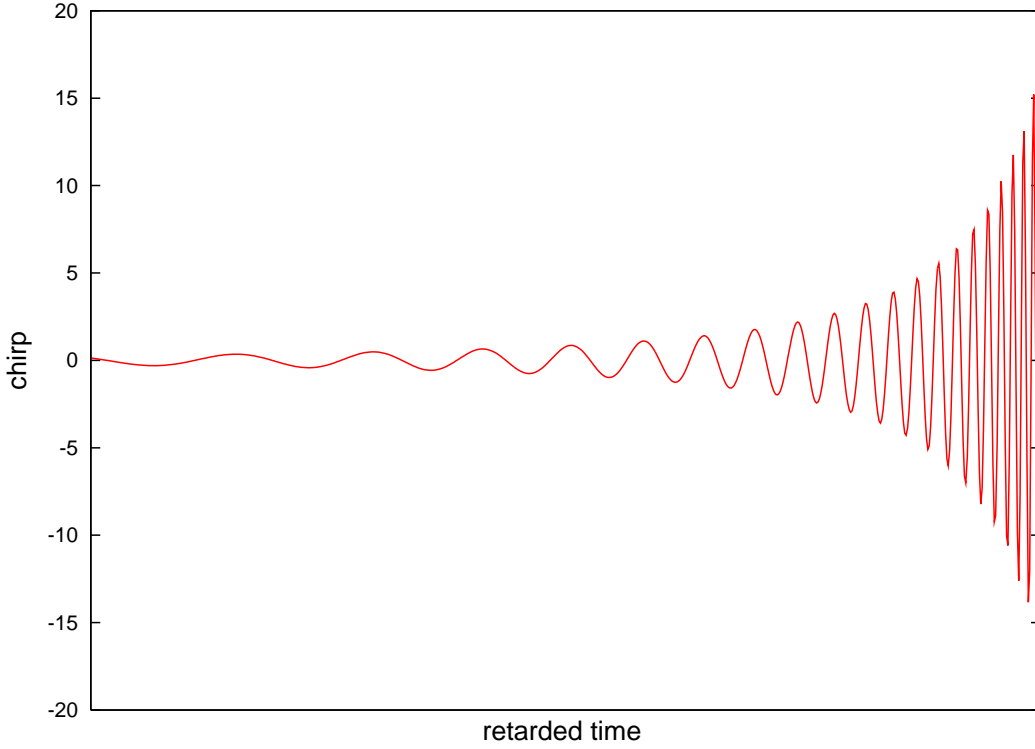
if we now define the quantity \mathcal{M} , called **chirp mass**,

$$\mathcal{M}^{5/3} = \mu M^{2/3} \quad \rightarrow \quad \mathcal{M} = \mu^{3/5} M^{2/5}$$

and use the relation among ω_K and the wave frequency ν_{GW} we find

$$h_0(t) = \frac{4\pi^{2/3} G^{5/3} \mathcal{M}^{5/3}}{c^4 r} \nu_{GW}^{2/3}(t).$$

The amplitude and the frequency of the gravitational signal emitted by a coalescing system increases with time. For this reason this peculiar waveform is called **chirp**, like the chirp of a singing bird



THE PHASE

According to eq. (10) the wave in the TT -gauge is

$$h_{ij}^{TT}(t, r) = h_0 \left[\mathcal{P}_{ijkl} A_{kl} \left(t - \frac{r}{c} \right) \right] \quad (27)$$

where A_{kl} is

$$A_{ij} = \begin{pmatrix} \cos 2\omega_K t & \sin 2\omega_K t & 0 \\ \sin 2\omega_K t & -\cos 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since ω_K is a function of time, the phase appearing in A_{kl} has to be substituted by an integrated phase

$$\Phi(t) = \int^t 2\omega_K(t) dt = \int^t 2\pi\nu_{GW}(t) dt + \Phi_{in}, \quad \text{where } \Phi_{in} = \Phi(t=0)$$

Since

$$\nu_{GW}(t) = \frac{\omega_K}{\pi} = \frac{\nu_{GW}^{in}}{\left[1 - \frac{t}{t_{coal}} \right]^{3/8}}, \quad \nu_{GW}^{in} = \frac{1}{\pi} \sqrt{\frac{GM}{(l_0^{in})^3}},$$

and

$$t_{coal} = \frac{5}{256} \frac{c^5 (l_0^{in})^4}{G^3 \mu M^2},$$

$$\nu_{GW}^{in} t_{coal}^{3/8} = (5^{3/8}) \frac{1}{8\pi} \left(\frac{c^3}{GM} \right)^{5/8}$$

and $\nu_{GW}(t)$ can be written as

$$\nu_{GW}(t) = \frac{1}{8\pi} \left(\frac{c^3}{G\mathcal{M}} \right)^{5/8} \left[\frac{5}{t_{coal} - t} \right]^{3/8} ;$$

consequently, the integrated phase becomes

$$\Phi(t) = -2 \left[\frac{c^3 (t_{coal} - t)}{5G\mathcal{M}} \right]^{5/8} + \Phi_{in}$$

which shows that **if we know the signal phase we can measure the chirp mass.**
In conclusion, the signal emitted during the inspiralling will be

$$h_{ij}^{TT} = -\frac{4\pi^{2/3} G^{5/3} \mathcal{M}^{5/3}}{c^4 r} \nu_{GW}^{2/3}(t) \left[\mathcal{P}_{ijkl} A_{kl} \left(t - \frac{r}{c} \right) \right]$$

where

$$A_{ij}(t) = \begin{pmatrix} \cos \Phi(t) & \sin \Phi(t) & 0 \\ \sin \Phi(t) & -\cos \Phi(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ll} \text{LIGO} [40 \text{ Hz} - 1 - 2 \text{ kHz}] & \text{LISA} [10^{-4} - 10^{-1}] \text{ Hz} \\ \text{VIRGO} [10 \text{ Hz} - 1 - 2 \text{ kHz}] & \end{array}$$

Let us consider 3 binary system

- a)** $m_1 = m_2 = 1.4 M_\odot$ **c)** $m_1 = m_2 = 10^6 M_\odot$
b) $m_1 = m_2 = 10 M_\odot$

Let us first calculate what is the orbital distance between the two bodies on the innermost stable circular orbit (ISCO) and the corresponding emission frequency

$$l_0^{ISCO} \sim \frac{6GM}{c^2}, \quad \omega_K = \sqrt{\frac{GM}{l_0^3}} = \pi \nu_{GW} \rightarrow \nu_{GW}^{ISCO} = \frac{1}{\pi} \sqrt{\frac{GM}{(l_0^{ISCO})^3}}$$

(M is the total mass)

- a)** $l_0^{ISCO} = 24,8 \text{ km}$ $\nu_{GW} = 1570.4 \text{ Hz}$
b) $l_0^{ISCO} = 177,2 \text{ km}$ $\nu_{GW} = 219.8 \text{ Hz}$
c) $l_0^{ISCO} = 17.720.415,3 \text{ km}$ $\nu_{GW} = 2.2 \cdot 10^{-3} \text{ Hz}$
- a) and b)** are interesting for LIGO and VIRGO,
c) will be detected by LISA

Let us consider **LIGO** and **VIRGO**; we want to compute the time a given signal spends in the detector bandwidth before coalescence.

From

$$\nu_{GW}(t) = \frac{\nu_{GW}^{in} t_{coal}^{3/8}}{[t_{coal} - t]^{3/8}}$$

we get

$$t = t_{coal} \left[1 - \left(\frac{\nu_{GW}^{in}}{\nu_{GW}(t)} \right)^{8/3} \right].$$

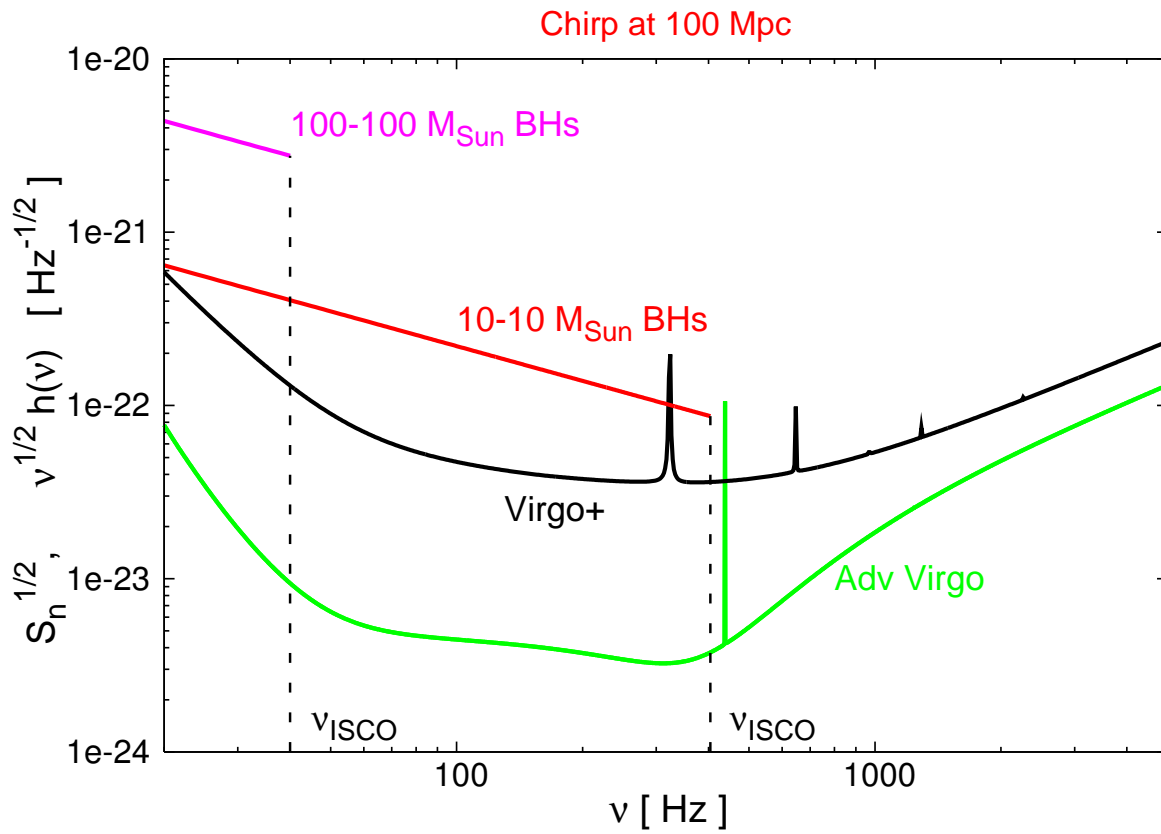
Putting :

ν_{GW}^{in} = lowest frequency detectable by the antenna, and

$\nu_{GW}^{max} = \nu_{GW}^{ISCO}$ we find

	LIGO	VIRGO
a) ($m_1 = m_2 = 1.4 M_\odot$)	[40 – 1570.4 Hz]	[10 – 1570.4 kHz]
	$\Delta t = 24.86 \text{ s}$	$\Delta t = 16.7 \text{ m}$
b) ($m_1 = m_2 = 10 M_\odot$)	[40 – 219.8 Hz]	[10 – 219.8 kHz]
	$\Delta t = 0.93 \text{ s}$	$\Delta t = 37.82 \text{ s}$

VIRGO catches the signal for a longer time.



Planned sensitivity curve for VIRGO+ and ADVANCED VIRGO

The plotted signal is the

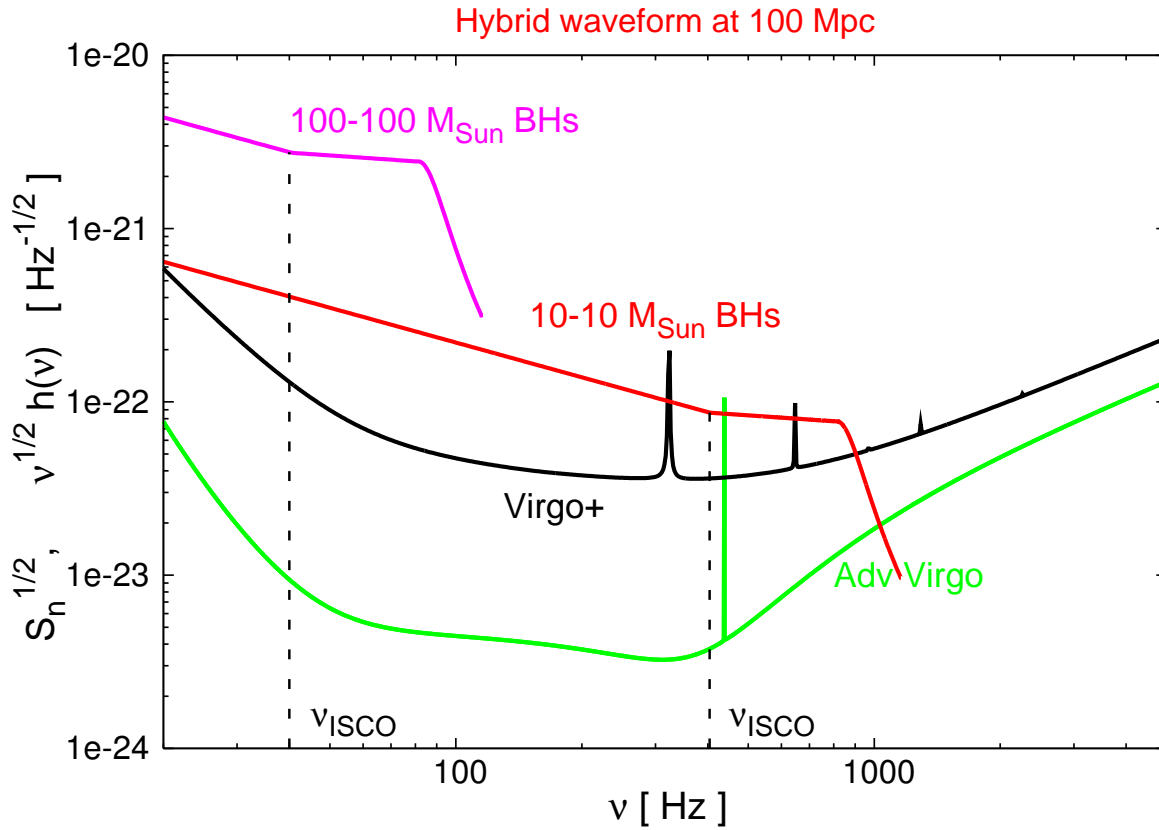
$$\text{strain amplitude} = \nu^{1/2} h(\nu) ,$$

evaluated for the chirp.

Source located at a distance of 100 Mpc.

The chirp is only a part of the signal emitted during the binary coalescence.

Black hole-black hole coalescence



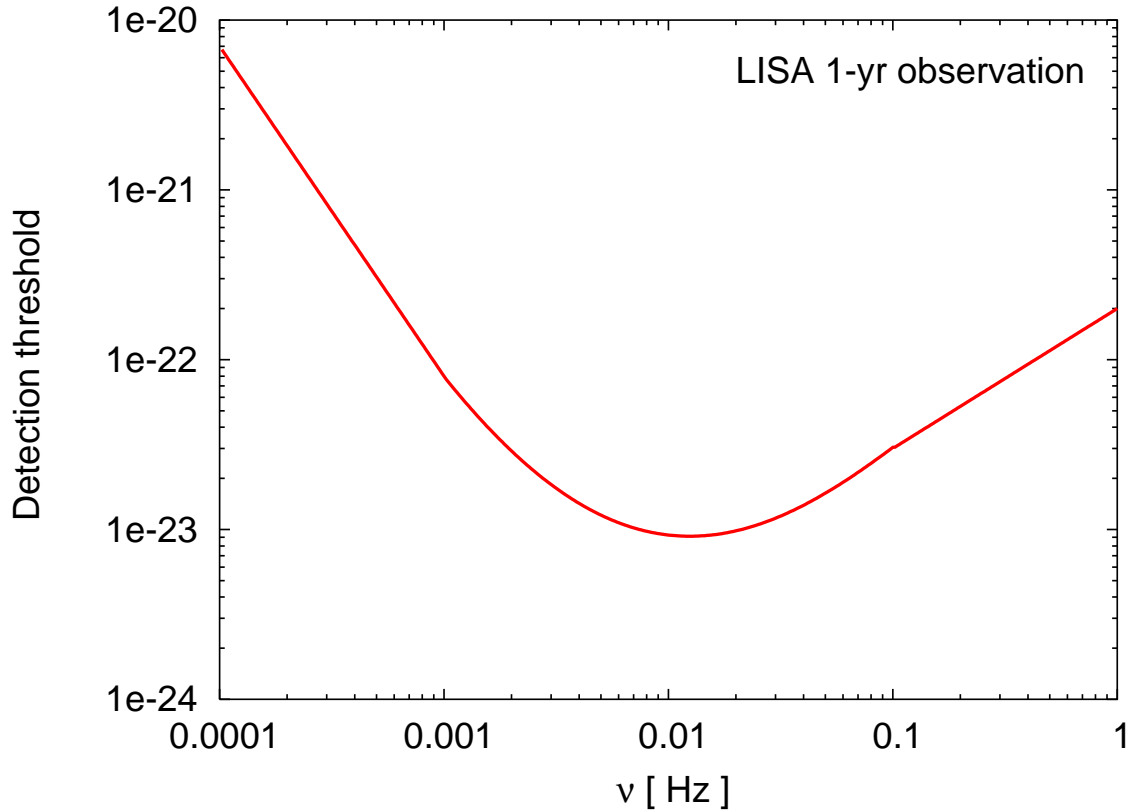
This hybrid waveform is obtained by matching three signals:

- 1) chirp waveform for $\nu < \nu_{ISCO}$
- 2) waveform emitted during the merger phase of two black holes, obtained by numerical integration of Einstein's equations.
- 3) waveform emitted after the final black hole is formed, due to ringdown oscillations.

P. Ajith et al., Phys. Rev. D 77, 104017 2008

WHAT ABOUT LISA?

LISA $[10^{-4} - 10^{-1}] Hz$



Let us consider 2 BH-BH binary systems

- a) $m_1 = m_2 = 10^2 M_\odot$
- b) $m_1 = m_2 = 10^6 M_\odot$

Orbital distance between the two bodies on the innermost stable circular orbit (ISCO) and the corresponding emission frequency

$$l_0^{ISCO} \sim \frac{6GM}{c^2}, \quad \omega_K = \sqrt{\frac{GM}{l_0^3}} = \pi \nu_{GW} \rightarrow \nu_{GW}^{ISCO} = \frac{1}{\pi} \sqrt{\frac{GM}{(l_0^{ISCO})^3}}$$

- a) $l_0^{ISCO} = 1772 km$ $\nu_{GW} = 21.98 Hz$
- b) $l_0^{ISCO} = 17.720.415, 3 km$ $\nu_{GW} = 2.2 \cdot 10^{-3} Hz$

Time a given signal spends in the detector bandwidth before coalescence.

- a) $m_1 = m_2 = 10^2 M_\odot$
- b) $m_1 = m_2 = 10^6 M_\odot$

$$t = t_{coal} \left[1 - \left(\frac{\nu_{in}}{\nu_{GW}(t)} \right)^{8/3} \right]$$

LISA

a) $[10^{-4} - 10^{-1} \text{ Hz}]$

$$\Delta t = 556.885 \text{ years}$$

b) $[10^{-4} - 2.2 \cdot 10^{-3} \text{ Hz}]$

$$\Delta t = 0,12 \text{ years} = 43 \text{ d } 18 \text{ h } 43 \text{ m } 24 \text{ s}$$