Exercise 3.2: Assuming a uniform value of β throughout the star and defining $U = \int (u_{\text{gas}} + u_{\text{rad}})dm$, show that the virial theorem (2.23) leads to

$$E = \frac{\beta}{2}\Omega = -\frac{\beta}{2-\beta}U$$

for a classical (nonrelativistic) gas. Note in particular the limits $\beta \to 1$ and $\beta \to 0$. If the star contracts, maintaining the same uniform β, which fraction of the gravitational potential energy released is radiated away and which fraction is turned into heat?

3.6 The adiabatic exponent

Thermodynamic processes of a special kind, which will be of interest in later discussions, are those occurring in a system without exchange of heat with the environment. Such processes are called *adiabatic*. From the first law of thermodynamics (mentioned in Section 2.2) it follows that adiabatic processes satisfy the condition

$$du + Pd\left(\frac{1}{\rho}\right) = 0. \tag{3.48}$$

In the previous section we have seen – at least for simple systems – that the specific energy u is always proportional to P/ρ. We may therefore write

$$u = \phi\frac{P}{\rho}, \tag{3.49}$$

which, by differentiating and substituting into Equation (3.48), leads to

$$\phi Pd\left(\frac{1}{\rho}\right) + \phi\frac{1}{\rho}dP + Pd\left(\frac{1}{\rho}\right) = (\phi+1)Pd\left(\frac{1}{\rho}\right) + \phi\frac{1}{\rho}dP = 0. \tag{3.50}$$

Accordingly, the dependence of the pressure on density is described by a power law

$$P \propto \rho^{\frac{\phi+1}{\phi}}. \tag{3.51}$$

The power $(d\ln P/d\ln\rho)$ is called the *adiabatic exponent*, denoted γ_a; the proportionality factor (to be denoted K_a) is determined by the properties of the system (it is a direct function of the *entropy*). In conclusion, adiabatic processes are characterized by the law

$$P = K_a\rho^{\gamma_a}. \tag{3.52}$$

It is easily seen that for the systems we have considered, the value of γ_a is $5/3$ in the case of a nonrelativistic ideal gas or a completely degenerate electron gas, and $4/3$ in the case of a relativistic degenerate electron gas or of pure radiation.

Intermediate values will obtain for mixtures, such as gas and radiation, and nonextreme cases, such as a moderately relativistic degenerate electron gas.

So far we have considered gases of a fixed number of particles: either (almost) fully ionized, as in the deep stellar interior, or (almost) fully recombined, as in the outer layers of a cool stellar atmosphere. When ionization takes place and the number of particles changes with the other physical properties, the adiabatic exponent changes too. Since this will prove to be of particular importance to the stability of stars, it deserves some discussion. We shall only consider the very simple case of a singly ionized pure gas (rather than a mixture of gases), say, hydrogen. Hence we have to deal with three different types of particles: neutral atoms, whose number density we denote by n_0, ions of number density n_+, and free electrons of number density n_e (obviously, $n_e = n_+$). The pressure exerted by the gas is proportional to $n_0 + n_+ + n_e$, while the mass density is proportional to $n_0 + n_+$. The *degree of ionization* is defined by

$$x = \frac{n_+}{n_0 + n_+}. \tag{3.53}$$

The densities of ions and neutrals are related by Saha's equation (after Meghnad Saha, who derived it in 1920)

$$\frac{n_+ n_e}{n_0} = \frac{g}{h^3}(2\pi m_e kT)^{3/2}e^{-\chi/kT}, \tag{3.54}$$

where g is a constant and χ is the ionization potential (the energy required to create an ion by removing an electron from an atom). In terms of the degree of ionization, we have

$$P = (1+x)(n_0 + n_+)kT = (1+x)\mathcal{R}\rho T, \tag{3.55}$$

and Saha's equation becomes

$$\frac{x^2}{1-x^2} = \frac{g}{h^3}\frac{(2\pi m_e)^{3/2}(kT)^{5/2}}{P}e^{-\chi/kT}. \tag{3.56}$$

In the case of a partially ionized gas, the specific energy has an additional term, $\chi n_+/\rho = \chi n_+/[(n_0 + n_+)m_H] = \chi x/m_H$, which is due to the available potential energy of ionization. Thus

$$u = \frac{3}{2}\frac{P}{\rho} + \frac{\chi}{m_H}x \tag{3.57}$$

replaces Equation (3.49). Using Equations (3.55) and (3.56) to express the degree of ionization as a function of pressure and density $x = x(P, \rho)$, differentiating Equation (3.57), and substituting into Equation (3.48) yields

$$\frac{3}{2}\left(\frac{1}{\rho}\right)dP + \frac{3}{2}Pd\left(\frac{1}{\rho}\right) + \frac{\chi}{m_H}\frac{\partial x}{\partial P}dP + \frac{\chi}{m_H}\frac{\partial x}{\partial \rho}d\rho + Pd\left(\frac{1}{\rho}\right) = 0. \tag{3.58}$$

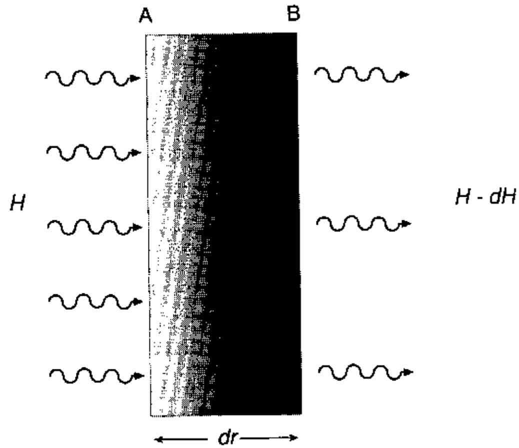

Figure 3.2 Radiation flux passing through a slab.

Multiplying by ρ/P and assembling terms, we have

$$\left[\frac{3}{2}+\frac{\chi}{kT}\left(\frac{P}{1+x}\right)\left(\frac{\partial x}{\partial P}\right)_{\rho}\right]\frac{dP}{P}-\left[\frac{5}{2}-\frac{\chi}{kT}\left(\frac{\rho}{1+x}\right)\left(\frac{\partial x}{\partial \rho}\right)_{P}\right]\frac{d\rho}{\rho}=0, \tag{3.59}$$

from which, after not inconsiderable manipulation, $\gamma_{\mathrm{a}}(x)$ may be calculated:

$$\gamma_{\mathrm{a}}(x)=\frac{5+\left(\frac{5}{2}+\frac{\chi}{kT}\right)^{2}x(1-x)}{3+\left[\frac{3}{2}+\left(\frac{3}{2}+\frac{\chi}{kT}\right)^{2}\right]x(1-x)}. \tag{3.60}$$

In the limit $x=0$ or $x=1$, we obtain $\gamma_{\mathrm{a}}=5/3$, as before; the minimum value is obtained for $x=0.5$; it is 1.63 for $\chi/kT=1$, for example, and 1.21 for $\chi/kT=10$.