MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

Problem Set 8

Due: Wednesday, April 24, 2018, in class
This problem set is worth **90 points**

1. **Nuclear statistical equilibrium [25 pts]** (Adapted from Hansen, Kawaler, & Trimble, Problem 6.7).

   In the normal course of evolution of a massive star, the end products of nuclear burning are elements in the iron region of nucleon number. If the temperatures get high enough, the radiation field is capable of initiating photodisintegration, and all the iron-peak elements end up as individual nucleons. This reduces the effective adiabatic index below $4/3$, and is responsible for the final collapse of the star once it enters the photodisintegration regime. This can happen on such rapid time scales that the abundances of nuclei (as functions of temperature and density) can be calculated approximately as if the gas were in chemical equilibrium using a version of the Saha equation.

   To look at this in a very simplified way, consider a gas composed only of ^{56}Ni and ^{4}He where the "chemical reaction" between them is

$$14\,^4\mathrm{He} \rightleftharpoons \,^{56}\mathrm{Ni} + Q,$$

   where you may compute the Q-value from the mass excesses,

$$\begin{aligned}(M - Am_n)c^2 &= 2.42 \text{ MeV for } ^4\text{He } (A = 4) \\ (M - Am_n)c^2 &= -53.9 \text{ MeV for } ^{56}\text{Ni } (A = 56).\end{aligned}$$

   The numerical value of the nucleon mass m_n is not needed for this problem, though you can look it up.

   In equilibrium, the chemical potentials obey the relation

$$14\mu_{\mathrm{He}} = \mu_{\mathrm{Ni}}.$$

   Since the nuclei are non-degenerate, the number density of species i (either He or Ni) obey the Maxwell-Boltzmann distribution,

$$n_i = g_i \left(\frac{2\pi m_i kT}{h^2}\right)^{3/2} \exp\left(\frac{\mu_i - m_i c^2}{kT}\right).$$

   (a) Set up an equivalent of the Saha equation for the reaction, pretending that you are dealing with atoms and ions and assuming that both nuclei are in their ground states. To do this, substitute the Maxwell-Boltzmann relation for the number densities into the equilibrium relation for the chemical potentials, and derive an equation whose left hand side is an appropriate ratio of number densities. Let the statistical weights be unity (which is appropriate since the ground state spins are zero).

   Note that there will be two differences from the Saha equation you encountered earlier in the term: First, there are only two species so there will only be two different n's. Second, the multiple He nuclei in the reaction will result in a nonlinear dependence on n_{He}. The Q-value plays the role of the ionization energy.

   (b) Re-cast your Saha equation so that the unknowns are the mass fractions X_4 and X_{56} where $X_4 + X_{56} = 1$.

   (c) Fix the density to be $\rho = 10^7$ g cm^{-3} and solve for X_4 and X_{56} for temperatures in the range $4.5 < T_9 < 6.5$, where T_9 is the temperature in units of 10^9 K.

   (d) Plot your results for the mass fractions versus temperature. At what temperature is $X_4 = X_{56}$?

2. **White dwarf cooling [50 pts]**

Because a white dwarf has no internal power source, it cools and fades. In this problem you will calculate the cooling rate by modeling the white dwarf as a degenerate core containing most of the mass, and a thin non-degenerate atmosphere from which the core's internal energy is radiated into space. This was first done by Leon Mestel in 1952.

The degenerate core is nearly isothermal because the degenerate electrons have a very large thermal conductivity. The total thermal energy is

$$U = \frac{M}{Am_p}\frac{3}{2}kT_c, \tag{1}$$

where M is the core mass, A is the atomic weight, m_p is the mass of a proton, and T_c is the core temperature. Radiation from the atmosphere causes the energy to be lost at a rate $L = 4\pi R^2 \sigma T_{\rm eff}^4$, but $T_{\rm eff}$ is not the same as T_c. We need to derive the connection between L and T_c, which depends on the opacity of the atmosphere.

(a) Suppose the atmosphere is radiative (not convective), and the opacity obeys Kramer's law, $\kappa \propto \rho T^{-3.5}$. Specifically

$$\kappa = \kappa_0 \left(\frac{\rho}{1\ \text{g cm}^{-3}}\right)\left(\frac{T}{1\ K}\right)^{-3.5}$$

where $\kappa_0 = 4.34 \times 10^{24}\ Z(1+X)\ \text{cm}^2\ \text{g}^{-1}$. ($X$ is hydrogen mass fraction, and Z is the "metal" mass fraction; numerical values for these parameters will be provided later in the problem when they are needed.) Combine the equation of hydrostatic equilibrium and the equation of radiative diffusion to obtain

$$\frac{dP}{dT} = \frac{4ac}{3}\ \frac{4\pi GM(r)}{\kappa_0 L}\ \frac{T^{6.5}}{\rho}, \tag{2}$$

where $a \equiv 4\sigma/c$ is the "radiation constant."

(b) Because the atmosphere is thin, M and L are nearly constant throughout the atmosphere. Use this fact, and the ideal gas law, to integrate the preceding equation from top of the atmosphere ($P \approx 0, T \approx 0$) to a point within the atmosphere (P, T). Then show that within the atmosphere,

$$\rho = \left(\frac{4}{17}\ \frac{16\pi ac}{3}\ \frac{GM}{L}\ \frac{\mu m_p}{\kappa_0 k}\right)^{1/2} T^{13/4}. \tag{3}$$

(c) The preceding equation is valid all the way down to the degenerate core (i.e., the bottom of the atmosphere). By setting the non-relativistic electron degeneracy pressure equal to the ideal gas pressure, show that this boundary occurs when the temperature T and pressure P obey

$$\frac{T}{(\rho/\mu_e)^{2/3}} \approx 1.20 \times 10^5\ \text{K cm}^2\ \text{g}^{-2/3}, \tag{4}$$

where μ_e is the mean molecular weight per electron. Recall that the non-relativistic degenerate electron equation of state is

$$P_{\rm NR} = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e m_p^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3} \tag{5}$$

Problem continues on the next page

(d) Use this condition and your result from part (b) to show that L and T_c obey

$$L \propto \frac{\mu}{\mu_e^2} \frac{M}{\kappa_0} T_c^{7/2}. \tag{6}$$

You should also specify the proportionality constant (you will need it below).

(e) Evaluate $L/L_\odot$ numerically for a carbon-oxygen white dwarf with $\mu_e = 2$, $M = M_\odot$ and $T_c = 10^7$ K. Assume the atmosphere is fully ionized with hydrogen mass fraction $X = 0$, helium mass fraction $Y = 0.9$, and metal mass fraction $Z = 0.1$.

(f) Set $L = -dU/dt$ to derive the Mestel cooling law,

$$T_c(t) = T_0 \left(1 + \frac{5}{2}\frac{t}{\tau_0}\right)^{-2/5}, \tag{7}$$

where τ_0 is a characteristic timescale you should calculate in terms of the properties of the white dwarf and fundamental constants.

(g) Evaluate the numerical value of τ_0 for a carbon-oxygen white dwarf with $\mu_e = 2$, $M = M_\odot$ and $L = 10^4\ L_\odot$, typical of the planetary nebula phase.

Note: This calculation neglects the electrostatic energy of the ions. In real white dwarfs, *crystallization* occurs when the electrostatic potential energy between neighboring ions dominates their thermal energy. An associated latent heat of crystallization is released, providing a new source of thermal energy that delays the further cooling of the star. See, e.g., chapter 4 of Shapiro & Teukolsky.

3. **Maximum rotation speed [15 pts]**

In this problem you will estimate the rotation speed for a neutron star at which it will break up.

(a) Find an expression for the minimum rotation period, $P_{\rm min}$, of a neutron star of a function of its mass M and radius R. To do this, consider a parcel of mass on the surface of the neutron star, near the equator. When does the centrifugal force begin to exceed the gravitational force? (You may neglect the effects of general relativity; GR changes things at the level of about 20%, which is more precise than we need here.)

(b) Evaluate $P_{\rm min}$ for a neutron star with $M = 1.4\ M_\odot$ and $R = 10$ km. For comparison, the fastest known millisecond pulsar is PSR J1748-2446ad, which has a spin period of 1.3959 ms (Hessels et al. 2006, *Science*, 311, 1901).

(c) Newton studied the equatorial bulge of a homogeneous fluid body of mass M that is rotating with angular velocity Ω. He showed that the equatorial radius R_e exceeds the polar radius R_p by an amount given by

$$\frac{R_e - R_p}{R_m} = \frac{5\Omega^2 R_m^3}{4GM}, \tag{8}$$

where $R_m \equiv (R_e + R_p)/2$. Use this formula to estimate $R_e - R_p$ for a neutron star with $M = 1.4\ M_\odot$, $R_m = 10$ km, and a rotation period equal to twice the minimum period you found in Prob. 3b.