MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

Problem Set 7

Due: Friday, April 12, 2018, in class
This problem set is worth **105 points**

1. **Evolution on the red giant branch [20 pts]**

   Once a low-mass star leaves the main sequence and ascends the red giant branch, its luminosity is provided by hydrogen burning in a thin shell surrounding a degenerate helium core. As hydrogen is burned in the shell, the mass of the helium core grows and the star continues to ascend the giant branch. It turns out that both the luminosity and the radius of red giants can be written as steep functions of their helium core mass alone, nearly independent of any other quantity. These expressions are

   $$L \simeq 2 \times 10^5 L_\odot \left(\frac{m_c}{M_\odot} \right)^6, \quad R \simeq 3700 R_\odot \left(\frac{m_c}{M_\odot} \right)^4,$$

   where m_c is the helium core mass, and L and R are the luminosity and radius of the (entire) red giant, respectively. Note that these expressions are "empirical" in that they are based on the results of stellar evolution calculations and are *not* derived analytically.

   (a) [10 pts] Use these relations to compute the evolution of a star (L and R as functions of time) as it ascends the red giant branch. Take the initial core mass value to be $m_c = 0.1 M_\odot$ and the final value (at the tip of the giant branch) to be $m_c = 0.45 M_\odot$. Note that the luminosity L tells us the rate at which hydrogen is burned in the shell around the core, and thereby tells us the rate of growth of the degenerate helium core: $L = 0.007 \dot{m}_c c^2$.

   (b) [10 pts] Plot the radius, luminosity, and effective temperature of the star as a function of time on the giant branch. Also, plot the track of the evolving giant on an H-R diagram (i.e., $\log L$ versus $\log T_{\text{eff}}$, with $\log T_{\text{eff}}$ increasing to the left).

2. **Core helium flash in red giants [15 pts]**
(Adapted from Hansen, Kawaler, & Trimble, Problem 6.8)

Suppose you have a gram of pure ^{4}He in the center of a pre–helium-flash red giant. The initial density and temperature of the gram are, respectively, $\rho = 2 \times 10^5$ g cm^{-3} and $T = 1.5 \times 10^8$ K. This is hot enough to burn helium by the triple-α reaction, which is the only reaction you will use. The energy generation rate per unit mass for this reaction is given by

$$\epsilon_{3\alpha} = \frac{5.1 \times 10^8 \, \rho^2 \, Y^3}{T_9^3} \, e^{-4.4027/T_9} \text{ erg g}^{-1} \text{ s}^{-1},$$

where Y is the helium mass fraction (which we will take to be 1) and T_9 is the temperature in units of 10^9 K.

In this problem, you will follow the time evolution of the gram as helium burning proceeds by computing the temperature T as a function of time. $\epsilon_{3\alpha}$ tells you how much energy is produced per unit time in this gram. Compute the rate of change of temperature by coupling this to specific heat capacity:

$$\delta T = \frac{(\delta E/\Delta m)}{c_V} \, .$$

The specific heat c_V has contributions from the electrons and from the helium ions: $c_V = c_{V,e} + c_{V,\text{He}}$, with

$$c_{V,e} = \frac{1.35 \times 10^5}{(\rho/\text{g cm}^{-3})} (T/\text{K}) \, x \, (1 + x^2)^{1/2} \text{ erg g}^{-1} \text{ K}^{-1},$$

and

$$c_{V,\text{He}} = \frac{3k}{4m_p}.$$

In the equation for $c_{V,e}$, x is the dimensionless Fermi momentum; it can be computed using the fact that $\rho \approx 2 \times 10^6 x^3$ g cm^{-3}.

Start the clock runnning at $t = 0$ with the stated initial conditions. For simplicity, assume that the density and the helium concentration remain constant for all time and that no heat is allowed to leave the gram. Compute $T(t)$ numerically until that time when the material begins to become nondegenerate. (Use the nonrelativistic demarcation line $\rho \approx 10^{-8}$g cm^{-3} $(T/K)^{3/2}$; since we assume ρ is constant, this effectively means to compute the temperature T at which the given ρ becomes non-degenerate, and then to integrate until you reach that temperature.)

(a) [8 pts] Plot temperature $T(t)$ versus time in days. You will be able to recognize the flash when it happens because the temperature will suddenly skyrocket after not too many days of burning.

(b) [7 pts] After how many days does the helium flash occur? As a test of the quality of your numerical integration scheme, do your best to determine this onset time precisely. You may need to adjust the parameters of your numerical integrator to pin this down accurately.

3. **White dwarf structure: analytic results [30 pts]**

In this problem you will use a polytrope model to derive the mass-radius relation for white dwarfs. Before beginning, you may want to review the previous polytrope problems as well as the discussion of polytropes in section 5.3 of Chaudhuri.

(a) [10 pts] Show that for a polytrope with index n, the total mass M and outer radius R are related as

$$M = 4\pi R^{(3-n)/(1-n)} \left[\frac{(n+1)K}{4\pi G} \right]^{n/(n-1)} \xi_1^{2-(n-3)/(n-1)} \left| \frac{d\phi_n}{d\xi} \right|_{\xi_1}, \qquad (1)$$

where ξ_1 is defined by $\phi_n(\xi_1) = 0$, and K is the constant in the polytropic relation $p = K\rho^{1+1/n}$.

(b) [10 pts] For a white dwarf of sufficiently low mass, the electrons are non-relativistic and an $n = 1.5$ polytrope is a good model. Show that in this case

$$R/R_\oplus = 1.76 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}} \right)^{-1/6} \left(\frac{\mu_e}{2} \right)^{-5/6}, \qquad (2)$$

$$M/M_\odot = 0.496 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}} \right)^{1/2} \left(\frac{\mu_e}{2} \right)^{-5/2}, \qquad (3)$$

$$M/M_\odot = 2.7 \left(\frac{R}{R_\oplus} \right)^{-3} \left(\frac{\mu_e}{2} \right)^{-5}, \qquad (4)$$

using your results from part (a) as well as from your previous work on polytropes. In these expressions, ρ_c is the central density, and μ_e is the mean molecular weight per electron. (Low mass white dwarfs are composed mostly of degenerate helium, hence $\mu_e = 2$ for full ionization.) For completely ionized neutral matter, $\mu_e = \rho/n_e m_p$. Also, $R_\oplus = 6378.1$ km is the radius of the Earth, and $R_\oplus/R_\odot = 0.00917$.

(c) [10 pts] For larger masses and higher central densities, the electrons become relativistic, and an $n = 3$ polytrope is a better description. Show that in this case,

$$R/R_\oplus = 5.25 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}} \right)^{-1/3} \left(\frac{\mu_e}{2} \right)^{-2/3}, \qquad (5)$$

$$M/M_\odot = 1.46 \left(\frac{\mu_e}{2} \right)^{-2}. \qquad (6)$$

Thus there is a uniquely defined mass, the *Chandrasekhar mass*, which is the maximum mass of an object that is supported by electron degeneracy pressure.

4. **White dwarf structure: numerical models [40 pts]**

In the previous problem you considered the cases of non-relativistic and relativistic electrons separately. In this problem you will build numerical models of white dwarfs, using an equation of state that makes a smooth transition from the non-relativistic to the relativistic regime.

The Fermi pressure of a non-relativistic (NR) degenerate electron gas is

$$P_{\mathrm{NR}} = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e m_p^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3} = K_{\mathrm{NR}} \left(\frac{\rho}{\mu_e}\right)^{5/3} \rho^{5/3}, \tag{7}$$

where ρ is the total mass density, μ_e is the mean molecular mass per electron, and the constant K_{NR} is approximately 1.00×10^{13} in cgs units.

The Fermi pressure of an ultrarelativistic (UR) degenerate electron gas is

$$P_{\mathrm{UR}} = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8 m_p^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{4/3} = K_{\mathrm{UR}} \left(\frac{\rho}{\mu_e}\right)^{4/3}, \tag{8}$$

where $K_{\mathrm{UR}} \approx 1.24 \times 10^{15}$ in cgs units.

On a log-log plot, the functions $P_{\mathrm{NR}}(\rho)$ and $P_{\mathrm{UR}}(\rho)$ are straight lines with slopes $5/3$ and $4/3$ respectively. They intersect at a density

$$\rho_0 \equiv \mu_e \left(\frac{K_{\mathrm{UR}}}{K_{\mathrm{NR}}}\right)^3 \approx 3.79 \times 10^6 \text{ g cm}^{-3}, \tag{9}$$

where here and for the rest of this problem we have assumed $\mu_e = 2$. We will patch together the two expressions in an approximate way:

$$P = \frac{P_{\mathrm{NR}} P_{\mathrm{UR}}}{\sqrt{P_{\mathrm{NR}}^2 + P_{\mathrm{UR}}^2}} = \frac{K_{\mathrm{NR}}(\rho/\mu_e)^{5/3}}{\sqrt{1 + (\rho/\rho_0)^{2/3}}}. \tag{10}$$

This expression has the correct asymptotic behavior for both low ρ and high ρ. It is therefore a simple way to interpolate between the two regimes. (Note, you could use instead the exact formula for the Fermi pressure of an arbitrarily relativistic degenerate gas that you derived in a previous problem set. This approximation turns out to give nearly identical answers, and is easier to work with.)

Now that we have a relation $P(\rho)$, we can proceed as we did for the polytropic model, by solving the equation of hydrostatic equilibrium,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{1}{\rho} \frac{dP}{dr}\right) = -4\pi G \rho. \tag{11}$$

(a) [7 pts] Nondimensionalize the preceding equation by defining $\theta \equiv \rho/\rho_0$, and $s \equiv r/a$, where a is a constant with the dimensions of length that is formed from the constants K_{NR}, ρ_0, and G. You should find that $a \approx 1.557 \times 10^8$ cm.

(b) [5 pts] Turn the second-order differential equation into two first-order coupled differential equations, by defining $V \equiv d\theta/ds$.

Next, write a code to integrate the first-order coupled equations for θ and V. Start from the center, using the boundary conditions $\theta(0) = \theta_c$ and $V(0) = 0$, where θ_c is a constant that specifies the central density of the white dwarf. Stop when θ goes to zero, thereby defining the outer radius. (To avoid numerical difficulties you may want to stop when $\theta/\theta_c = 10^{-3}$.) Perform the integration for 17 different choices for the central density: $\rho_c = 10^4, 10^{4.5}, 10^5, 10^{5.5}, ..., 10^{11.5}, 10^{12}$ g cm^{-3}.

While integrating, keep track of the dimensionless mass of the star,

$$\mathcal{M} = \int_0^{s_{\max}} \theta(s) s^2 ds, \tag{12}$$

which can be converted into the actual mass,

$$M = 4\pi \rho_0 a^3 \mathcal{M} = (0.090 M_\odot) \mathcal{M}. \tag{13}$$

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(c) [7 pts] Plot the masses of your white dwarf models (in $M_\odot$) as a function of $\log_{10} \rho_c$. Calculate the maximum stable mass of a white dwarf.

(d) [7 pts] Plot radius (in units of $R_\odot/100$) as a function of $\log_{10} \rho_c$.

(e) [7 pts] Plot radius (in $R_\odot/100$) as a function of mass (in $M_\odot$).

(f) [7 pts] For a model with $M = M_\odot$, plot the density as a function of radial distance. Repeat for $M = 1.3\ M_\odot$.