MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

Problem Set 6

Due: Friday, April 5, 2019, in class
This problem set is worth **70 pts**

1. **Hydrogen burning is the main thing [10 pts]**

   Consider two hypothetical stars of the same mass and (constant) luminosity. The stars are originally pure hydrogen. In Star A, fusion proceeds until the entire star is converted into ^{56}Fe. In Star B, fusion does not proceed all the way to ^{56}Fe. Instead, fusion is halted after all the hydrogen is converted into ^{4}He. How much longer is the lifetime of Star A, compared to Star B?

   You will find it useful to know the masses of the relevant nuclei:

$$\begin{aligned} m_{\rm H} &= 1.0078250 \text{ amu} \\ m_{^4{\rm He}} &= 4.0026032 \text{ amu} \\ m_{^{56}{\rm Fe}} &= 55.9349421 \text{ amu} \end{aligned}$$

   where the amu is approximately m_p (it is defined as one-twelfth of the mass of an unbound carbon-12 nucleus in its ground state).

2. **Electron degeneracy pressure for an arbitrary degree of relativistic motion [20 pts]** (based on Choudhuri 5.2)

   The degeneracy pressure of an electron gas is given by Equation (5.5) of Choudhuri,

$$P = \frac{8\pi}{3h^3}\int_0^{p_{\rm F}} \frac{p^4c^2}{\sqrt{p^2c^2+m_e^2c^4}}\,dp.$$

   (a) Work out this integral by substituting $p = m_e c \sinh\theta$ and show that the general expression for the electron degeneracy pressure is equal to

$$P = \frac{\pi m_e^4 c^5}{3h^3}\, f(x),$$

   where

$$f(x) \equiv x(2x^2-3)\sqrt{x^2+1} + 3\sinh^{-1} x$$

   and $x \equiv p_{\rm F}/m_e c$.

   (b) Evaluate $f(x)$ numerically for various values of x and use these numerical values to make a plot of $\log P$ versus $\log(\rho/\mu_e)$. Indicate regions of the plot corresponding to the two limiting equations $P \propto \rho^{5/3}$ for nonrelativistic motion, and $P \propto \rho^{4/3}$ for ultrarelativistic motion.

3. **Polytropic relation between core density and temperature [20 pts]**

   Previously you derived some useful analytic relations for a polytropic stellar model. Here you will use them to derive the relation between core density (ρ_c) and core temperature (T_c) that a star of a given mass M is expected to obey. As discussed in class, this relationship constrains the core's trajectory to a particular locus in the space of $\log\rho_c$ and $\log T_c$.

   (a) Show that the central pressure can be written in terms of M and ρ_c as

$$P_c = (4\pi)^{1/3} G M^{2/3} \rho_c^{4/3} F(n), \tag{1}$$

where F is a function of the polytropic index n that you should specify; it will involve ξ_1, the coordinate of the surface, as well as the function ϕ_n and/or its derivative(s). Show further that for n ranging from 1 to 3.5 (encompassing most of the realistic range of pressure/density profiles), the function F is nearly equal to 0.2 (within $\approx$30%).

(b) Show that if the ideal gas equation of state is applicable, then one expects

$$\log \rho_c = 3 \log T_c - 2 \log M + \text{ constant.}$$

(c) Show that if nonrelativistic degeneracy pressure is dominant, then one expects

$$\log \rho_c = 2 \log M + \text{ constant.}$$

4. **Extremes of the main sequence [20 pts]**

Near the "lower" end of the main sequence is a star with $M = 0.072\ M_\odot$, $\log_{10} T_{\rm eff} = 3.23$ and $\log_{10}(L/L_\odot) = -4.3$. Near the "upper" end is a star with $M = 85\ M_\odot$, $\log T_{\rm eff} = 4.705$ and $\log_{10}(L/L_\odot) = +6.006$.

(a) Estimate the hydrogen-burning lifetime of each star. For the 85 $M_\odot$ star, assume that only the innermost 10% of the hydrogen is available for burning. For the $0.072\ M_\odot$ star, assume the interior is entirely convective and all of its hydrogen becomes available for burning.

(b) Estimate the radius of each star, and the radius ratio.

(c) Compare the Eddington luminosity of each star to its actual luminosity. For the low-mass star, use $\kappa = 0.01$ cm^2 g^{-1}. For the high-mass star, assume the opacity is dominated by electron scattering. For which star is radiation pressure significant?