MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

Problem Set 5
Due: Friday, March 22, 2018
This problem set is worth **125 points**

1. **Stability against convection [10 pts]**

   (a) In lecture, we derived the condition

   $$\left|\frac{dT}{dr}\right| < \frac{T}{P}\left(1 - \frac{1}{\gamma_a}\right)\left|\frac{dP}{dr}\right|$$

   for stability against convection. Using the appropriate equation(s) of stellar structure and noting the sign of the radial gradients, show that this can be recast as a condition on the luminosity profile:

   $$L(r) < \left(1 - \frac{1}{\gamma_a}\right)\frac{64\pi\sigma_{\rm SB}T^4 GM(r)}{3\kappa_R P}$$

   (b) Show that to avoid convection in a stellar region where the equation of state is that of an ideal monatomic gas, the luminosity at a given radius must be limited by

   $$L(r) < 1.22 \times 10^{-18}\frac{\mu T^3}{\kappa_R \rho}M(r)$$

   where μ is the mean molecular weight, $T(r)$, κ_R is the Rosseland mean opacity, and $M(r)$ is the mass enclosed at radius r. All quantities are measured in the appropriate cgs units.

2. **Polytropes: Analytic calculations [25 pts]**

   Polytropes are simple models of self-gravitating bodies, based on the assumption $P = K\rho^{1+1/n}$. This assumption leads to a single differential equation for the density profile that can be nondimensionalized to give the *Lane-Emden equation*,

   $$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\phi_n}{d\xi}\right] = -\phi_n^n$$

   where ϕ_n is related to the star's density by $\rho = \rho_c\phi_n^n$, and ξ is related to radius by $r = \lambda_n\xi$ (λ_n is given below). Some objects can be approximated as polytropes with an appropriate choice for n. For example $n = 1$ describes a brown dwarf or giant planet pretty well; $n = 3/2$ describes a white dwarf; $n = 3$ does OK describing the Sun. In this problem, you will work out some useful mathematical properties of this model.

   (a) Show that the total mass of a polytropic star is

   $$M = 4\pi\rho_c\lambda_n^3\xi_1^2\left|\frac{d\phi_n}{d\xi}\right|_{\xi=\xi_1}.$$

   The factor λ_n is defined as

   $$\lambda_n \equiv \left[(n+1)\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right]^{1/2}$$

   (you may assume this form), and ξ_1 specifies the outer radius of the star: $\phi_n(\xi_1) = 0$.

   (b) Show that the ratio of the mean density to the central density is

   $$\frac{\langle\rho\rangle}{\rho_c} = \frac{3}{\xi_1}\left|\frac{d\phi_n}{d\xi}\right|_{\xi=\xi_1}.$$

(c) Show that the central pressure is

$$P_c = \frac{GM^2}{R^4}\left[4\pi(n+1)\left|\frac{d\phi_n}{d\xi}\right|^2_{\xi=\xi_1}\right]^{-1}.$$

Notice that this justifies the scaling of central pressure with mass and radius we found using a crude order of magnitude estimate in an earlier lecture.

(d) For a polytrope that also obeys the ideal gas equation of state, show $\phi_n = T/T_c$, where T_c is the central temperature.

3. **Polytropes: Numerical calculations [35 pts]**

In this problem you will calculate the density structure of various polytropes, including a model of the Sun. Numerically integrate the Lane-Emden equation to find $\phi_n(\xi)$ for polytropic indices of $n = 1.0$, 1.5, 2.0, 2.5, 3.0, and 3.5. One possible approach is to break up the second-order differential equation into two first-order equations,

$$\frac{d\phi_n}{d\xi} = u, \quad \frac{du}{d\xi} = -\phi_n^n - \frac{2u}{\xi}.$$

Then use a 4th-order Runge-Kutta integration scheme to find $\phi_n(\xi)$. The boundary conditions at the center are $u(0) = 0$ and $\phi_n(0) = 1$. The surface of the star is defined by $\phi_n(\xi_1) = 0$.

(a) Show that near the center of the star,

$$\phi_n(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \cdots$$

To do this, first show that the polynomial expansions of $\phi_n(\xi)$ contain only even terms in ξ. Then substitute such a polynomial into the Lane-Emden equation and find the first three coefficients.

For the rest of this problem, use this expansion to start your numerical calculation at small nonzero ξ.

(b) Plot the dimensionless temperature $\phi_n(\xi)$ and the dimensionless density $\phi_n^n(\xi)$ for all 6 values of n. It would be best to overlay all the temperature plots on a single set of axes, and all the density plots on another.

(c) Compute for each model the dimensionless potential energy $\Omega \equiv E_{\rm grav}/(-GM^2/R)$ and the dimensionless moment of inertia $k \equiv I/MR^2$. Tabulate ξ_1, $-(d\phi_n/d\xi)_{\xi_1}$, Ω, and k for each of the 6 polytropic models.

Next, you will use an $n = 3$ polytrope as a model of the Sun. One purpose of this exercise is to practice re-dimensionalizing your dimensionless solution. Another is to perform some order-of-magnitude checks on the applicability of this model.

Set the dimensional scales using the "known" values for the central density, temperature, and hydrogen mass fraction: $\rho_c = 158$ g cm^{-3}, $T_c = 15.7 \times 10^6$ K, $X = 0.6$.

(c) How do the total mass and radius of the model star compare to the actual Sun's mass and radius?

(d) Plot the following quantities as a function of $r/R_\odot$ (with $R_\odot$ being the radius of the model star): (i) $\log_{10} T$ with T in Kelvin; (ii) $\log_{10}\rho$ with ρ in g cm^{-3}.

(e) Compute the implied nuclear luminosity of the polytropic model. Take the nuclear energy generation rate per unit volume to be

$$\epsilon_V = (2.46 \times 10^6)\,\rho^2 X^2 T_6^{-2/3} \exp(-33.81\,T_6^{-1/3})\ \text{erg s}^{-1}\ \text{cm}^{-3},$$

where ρ is in g cm^{-3}, T_6 is the temperature in units of 10^6 K, and $X = 0.6$ is the hydrogen mass fraction. First, write the calculation as the product of a dimensioned constant and a dimensionless integral involving ϕ_n and ξ. (For the T_c inside the integral you can use 15.7×10^6 K.) Show the value of your constant, and the form of the dimensionless integral. Then, evaluate the nuclear luminosity in erg s^{-1}. Compare to the actual luminosity of 3.839×10^{33} erg s^{-1}.

4. **Overcoming the Coulomb barrier [15 pts]**

   In this problem you will show that classical mechanics predicts that hydrogen fusion cannot happen in the Sun.

   (a) Suppose two protons approach each other with equal speeds. What is the minimum speed needed to overcome the Coulomb barrier and collide, neglecting quantum effects? Take the radius of a proton to be $\approx$1 fermi $= 10^{-13}$ cm.

   (b) Assuming the proton speeds obey a Maxwell-Boltzmann distribution

   $$p(v) = \sqrt{\frac{2}{\pi}\left(\frac{m_p}{kT}\right)^3} v^2 \exp(-m_p v^2/2kT)$$

   with $T = 15.7 \times 10^6$ K (the central temperature of the Sun), what is the most probable speed? How does it compare to your answer to Prob. 4a?

   (c) You might wonder whether a small minority of protons in the tail of the M-B distribution could fuse. Give an order of magnitude estimate for the number of protons in the Sun, and for the number of those protons that are energetic enough to fuse. You may find it useful to know that for large u_0,

   $$\frac{4}{\sqrt{\pi}} \int_{u_0}^{\infty} u^2 e^{-u^2} du \approx \frac{2}{\sqrt{\pi}} u_0 e^{-u_0^2}.$$

5. **Tunneling through the Coulomb barrier [10 pts]**

   Now we compute the quantum-mechanical probability for two nuclei to tunnel through the Coulomb barrier.

   Let the two nuclei have charges Z_1, Z_2 and atomic masses A_1, A_2. Assume the interaction potential between the two nuclei is $Z_1 Z_2 e^2/r$, i.e., ignore the nuclear force until the nuclei are essentially touching (at a separation of a few fermi).

   Calculate the tunneling probability using the WKB approximation,

   $$\text{Trans. Prob.} \simeq \exp\left[-2\int_{r_{\min}}^{r_{\max}} \sqrt{\frac{2\mu(V-E)}{\hbar^2}}\, dr\right],$$

   where μ is the reduced mass (not the mean molecular weight) and $r_{\max}$ is the classical turning point, and you may approximate $r_{\min} \approx 0$. You should find that the probability varies as $\exp(-bE^{-1/2})$ where b is a constant.

6. **Nuclear binding energies [10 pts]**

   The Q value of a nuclear reaction is the amount of energy released (or absorbed) in the reaction; $Q > 0$ means energy is released. Compute the Q value in MeV for each of the following nuclear reactions.

   (a) ${}^{12}\mathrm{C} + {}^{12}\mathrm{C} \to {}^{24}\mathrm{Mg}$

   (b) ${}^{12}\mathrm{C} + {}^{12}\mathrm{C} \to {}^{16}\mathrm{O} + 2{}^{4}\mathrm{He}$

   (c) ${}^{19}\mathrm{F} + {}^{1}\mathrm{H} \to {}^{16}\mathrm{O} + {}^{4}\mathrm{He}$

   (d) ${}^{1}\mathrm{H} + {}^{1}\mathrm{H} \to {}^{2}\mathrm{H} + e^+ + \nu$

   (e) ${}^{15}\mathrm{N} + {}^{1}\mathrm{H} \to {}^{12}\mathrm{C} + {}^{4}\mathrm{He}$

   A useful source for this problem is the NIST table of atomic weights and compositions:

   `http://physics.nist.gov/cgi-bin/Compositions/stand_alone.pl`

   You'll need to convert from atomic mass units to MeV.

7. **Temperature dependence of thermonuclear reaction rates [20 pts]**

Next you will derive the leading-order dependence of a thermonuclear reaction rate on temperature. For the reaction A + B → C which liberates an energy Q, the rate of energy production per unit volume can be written

$$\epsilon_V \ [\mathrm{erg\ s^{-1}\ cm^{-3}}] = Q n_{\mathrm{A}} n_{\mathrm{B}} \langle \sigma v \rangle,$$

where $\langle \sigma v \rangle$ is the product of the cross-section and relative velocity, averaged over the Maxwell-Boltzmann distribution of relative energies. The cross-section may be written

$$\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}},$$

where the exponential factor arises from the tunneling probability, the E in the denominator arises from the inverse square of the de Broglie wavelength, and $S(E)$ represents the purely nuclear energy dependence.

(a) Show that ϵ_V is proportional to

$$\epsilon_V \propto Q n_A n_B T^{-3/2} \int_0^\infty S(E) \ \exp\left[-(bE^{-1/2} + E/kT)\right] dE,$$

with the same constant b that appeared in the previous problem.

Since the integral cannot be done analytically, we will need to make an approximation. The quantity $(bE^{-1/2} + E/kT)$ in the exponent is a falling function of E plus a rising function of E. The minimum in this quantity corresponds to a maximum in the value of the exponential. For most situations in stellar interiors, the only significant contributions to the integral occur when $(bE^{-1/2} + E/kT)$ is near its minimum E_0. Therefore, we will expand the exponent in a Taylor series about E_0, and we will assume that $S(E)$ is nearly constant over the narrow range surrounding E_0.

(b) Show that the Taylor series for the exponent is of the form:

$$-(bE^{-1/2} + E/kT) = -\frac{3b^{2/3}}{(4kT)^{1/3}} - f(T)(E - E_0)^2 + \ldots$$

where $f(T)$ is a function of the temperature.

(c) Complete the integration of over the Gaussian to get an expression for the temperature dependence of $\rho\epsilon$. You may drop any numerical prefactors, but be careful not to drop any factors that depend on temperature. You should find that the result is proportional to $e^{-B/T_6^{1/3}}$, where T_6 is the temperature expressed in millions of degrees K, and B is a constant. Show in particular that B is

$$B = 42.6 \ (Z_1 Z_2)^{2/3} \left(\frac{A_1 A_2}{A_1 + A_2}\right)^{1/3},$$

where Z_1 and Z_2 are the atomic numbers (nuclear charges) or the reactants, and A_1 and A_2 are their atomic masses (sum of neutron and proton numbers).