**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

**Problem Set 4**

**Due**: Friday, March 8, 2018, in class
This problem set is worth **120 points**

1. **Opacity due to Thomson scattering [15 pts]** (from Choudhuri, page 59)

   Consider an atmosphere of completely ionized hydrogen having the same mass density as Earth's atmosphere at sea level ($\rho = 1.23$ kg m$^{-3}$). Calculate the path length over which a beam of light would be attenuated to half of its original intensity, due to Thomson scattering by free electrons.

2. **Protons or photons? [10 pts]**

   At the center of the Sun, the density is approximately 150 g cm$^{-3}$ and the temperature is about $15 \times 10^6$ K. Which is larger: the number density of protons, or the number density of photons? Give an order of magnitude estimate of each.

3. **Isothermal atmosphere [20 pts]**

   Suppose the photosphere of a particular star (or planet) can be modeled as a parallel-plane atmosphere with a constant temperature $T$ down to very large optical depth. Further assume the gravitational acceleration $g$ is constant throughout the photosphere, because most of the mass is in the deeper layers beneath the photosphere. The opacity is $\kappa_0$, a given constant. The gas pressure $P = \rho kT/\mu m_p$ (with $\mu$ the mean molecular weight) is much larger than the radiation pressure.

   (a) [10 pts] Find $P(\tau)$, the pressure as a function of vertical optical depth, in terms of given quantities and fundamental constants. You may assume the surface gravity is constant throughout the photosphere, and that $P = P_0$ at $\tau = 2/3$. (*Hint:* Combine the equation of hydrostatic equilibrium with the relationship between $\tau$ and $z$. Recall that the opacity is constant.)

   (b) [10 pts] Let $z$ be the vertical height in the photosphere, increasing away from the center of the star, with $z = 0$ at the fiducial level where $\tau = 2/3$. Show that $\rho(z) \propto \rho_0 \exp(-z/H)$. Find $H$ in terms of given quantities and fundamental constants.

4. **The Eddington limit [20 pts]**

   A star with sufficiently high radiation pressure will spontaneously eject material from its surface. This sets a practical limit on the maximum luminosity of a star of a given mass.

   (a) [14 pts] Start with the radiative diffusion equation and the equation for hydrostatic equilibrium. Assume the opacity to be frequency-independent, and show that the luminosity at which the radiation pressure gradient equals the hydrostatic pressure gradient is given by

   $$L_{\mathrm{Edd}} = \frac{4\pi GMc}{\kappa}, \tag{1}$$

   where $M$ is the stellar mass. This is the "Eddington luminosity."

   (b) [6 pts] For ionized hydrogen, a minimum value for $\kappa$ arises from Thomson scattering, which has cross-section $\sigma_T = 6.65 \times 10^{-25}$ cm$^2$. Show that for this case

   $$L_{\mathrm{Edd}} \approx 3 \times 10^4 \, L_\odot \left(\frac{M}{M_\odot}\right), \tag{2}$$

   where $L_\odot = 3.839 \times 10^{33}$ erg s$^{-1}$ and $M_\odot = 1.989 \times 10^{33}$ g.

5. **A fictional star [20 pts]**

Consider a star of luminosity $L$ with density distribution $\rho = \rho_0 \times (R/r)$, where $R$ is the star's outer radius. Please don't ask how it manages to have such a simple density profile; this star exists only in the homework universe.

All of the star's energy is generated from a very small region near $r = 0$, and is transported entirely by radiation (not convection). The opacity is dominated by electron (Thomson) scattering, with opacity $\kappa_T$.

(a) [4 pts] What is the star's effective temperature $T_{\text{eff}}$, in terms of the given quantities and fundamental constants?

(b) [16 pts] Solve for the temperature as a function of $r$, in terms of $\rho_0, T_{\text{eff}}, R, \kappa_T$, and fundamental constants. For the outer boundary condition, assume $r = R$ represents the $\tau = 2/3$ level of the photosphere, and use the gray-atmosphere result $T(\tau = 2/3) = T_{\text{eff}}$.

6. **Pressure due to both particles and radiation [30 pts]**

The pressure $P$ that appears in the equations of stellar structure may include contributions from both gas pressure and radiation pressure. For Sun-like stars, gas pressure is dominant. However, the centers of more massive stars are hot enough for radiation to provide a significant fraction of the total pressure. In this problem you will investigate this regime using a simplified model in which the gas pressure is a specified fraction of the total pressure. We will limit our attention to fully ionized material of solar composition (mean molecular weight $\mu \approx 0.6$), where the gas and radiation field are at the same temperature, and the ideal gas law is applicable.

(a) [10 pts] First, let us get a feeling for the densities and temperatures involved. Write an expression for $\beta \equiv P_{\text{gas}}/(P_{\text{gas}} + P_{\text{rad}})$, in terms of the gas density $\rho$ and temperature $T$. Using this result, obtain the relationship between density and temperature in a gas for which $\beta = 0.1$, i.e., gas provides only 10% of the total pressure. Evaluate the temperature required to satisfy this condition for $\rho = 10$ g cm$^{-3}$.

(b) [10 pts] Instead of a gas-radiation fluid of fixed density and temperature, instead consider one of fixed density and total pressure. Derive a quartic equation for $\beta$ in terms of $P$ and $\rho$. This is known as the *Eddington quartic*. Show that $\beta \to 1$ for sufficiently small $P$, and $\beta \to 0$ for sufficiently large $P$. What is the characteristic pressure separating these two regimes, in terms of $\rho$ and the fundamental constants $h$, $c$ and $m_p$? You may find it useful to recall that the Stefan-Boltzmann constant is defined as $\sigma = 2\pi^5 k^4/(15c^2h^3)$. (*Hint*: Begin by writing $P_{\text{gas}} = \beta P$, and using the ideal gas law rewrite the equation you derived for $\beta$ using $P$ instead of $T$.)

(c) [10 pts] Let $E_{\text{int}} = E_{\text{gas}} + E_{\text{rad}}$ be the total internal energy in a star, including both gas and radiation. Assume that $\beta$ is uniform throughout the star (not very realistic). Using the same steps that are used to prove the virial theorem (see Sec. 3.2.2 of Chaudhuri), show that

$$E_{\text{tot}} = E_{\text{int}} + E_{\text{grav}} = \frac{\beta}{2} E_{\text{grav}},$$

where $E_{\text{grav}}$ is the gravitational binding energy. What can you conclude about how strongly bound a massive, radiation-dominated star is compared to a low-mass star with negligible radiation pressure?

*Hint*: To set this problem up, first use the fact that the gravitational binding energy is

$$E_{\text{grav}} = \int_0^M \Phi_{\text{grav}}(M) dM = -4\pi G \int_0^R r\rho(r)M(r)\,dr$$

where $\Phi(M)$ is the gravitational potential at the surface of a sphere of mass $M$, and $M(r)$ is the mass contained in a sphere of radius $r$. Note we do not know $M(r)$, so use the equation of hydrostatic equilibrium plus integration by parts to rewrite this as an integral over pressure $P$. The pressure $P$ is the sum of the gas pressure $P_{\text{gas}}$ and the radiation pressure $P_{\text{rad}}$.

Next, use the fact that the density associated with the internal energy of the gas is $u_{\text{gas}} = \frac{3}{2} n_{\text{gas}} k_B T$ and that the internal energy associated with the radiation is $u_{\text{rad}} = aT^4$, with $a = 4\sigma/c$ (and $\sigma$ is the Stefan-Boltzmann constant). Relate these two energy densities to their associated pressures. With some labor, the result you need to prove will follow.

(d) **Course feedback. [5 pts]** Are you getting what you hoped for out of 8.901? What is working well for you? What improvements would you like to see?