Problem Set 4

Due: Friday, March 8, 2018, in class
This problem set is worth 120 points

1. Opacity due to Thomson scattering [15 pts] (from Choudhuri, page 59)
   Consider an atmosphere of completely ionized hydrogen having the same mass density as Earth’s atmosphere at sea level ($\rho = 1.23 \text{ kg m}^{-3}$). Calculate the path length over which a beam of light would be attenuated to half of its original intensity, due to Thomson scattering by free electrons.

2. Protons or photons? [10 pts]
   At the center of the Sun, the density is approximately $150 \text{ g cm}^{-3}$ and the temperature is about $15 \times 10^6 \text{ K}$. Which is larger: the number density of protons, or the number density of photons? Give an order of magnitude estimate of each.

3. Isothermal atmosphere [20 pts]
   Suppose the photosphere of a particular star (or planet) can be modeled as a parallel-plane atmosphere with a constant temperature $T$ down to very large optical depth. Further assume the gravitational acceleration $g$ is constant throughout the photosphere, because most of the mass is in the deeper layers beneath the photosphere. The opacity is $\kappa_0$, a given constant. The gas pressure $P = \rho kT/\mu m_p$ (with $\mu$ the mean molecular weight) is much larger than the radiation pressure.

   (a) [10 pts] Find $P(\tau)$, the pressure as a function of vertical optical depth, in terms of given quantities and fundamental constants. You may assume the surface gravity is constant throughout the photosphere, and that $P = P_0$ at $\tau = 2/3$. (Hint: Combine the equation of hydrostatic equilibrium with the relationship between $\tau$ and $z$. Recall that the opacity is constant.)

   (b) [10 pts] Let $z$ be the vertical height in the photosphere, increasing away from the center of the star, with $z = 0$ at the fiducial level where $\tau = 2/3$. Show that $\rho(z) \propto \rho_0 \exp(-z/H)$. Find $H$ in terms of given quantities and fundamental constants.

4. The Eddington limit [20 pts]
   A star with sufficiently high radiation pressure will spontaneously eject material from its surface. This sets a practical limit on the maximum luminosity of a star of a given mass.

   (a) [14 pts] Start with the radiative diffusion equation and the equation for hydrostatic equilibrium. Assume the opacity to be frequency-independent, and show that the luminosity at which the radiation pressure gradient equals the hydrostatic pressure gradient is given by

   \[ L_{\text{Edd}} = \frac{4\pi GMc}{\kappa}, \]  

   where $M$ is the stellar mass. This is the “Eddington luminosity.”

   (b) [6 pts] For ionized hydrogen, a minimum value for $\kappa$ arises from Thomson scattering, which has cross-section $\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$. Show that for this case

   \[ L_{\text{Edd}} \approx 3 \times 10^4 L_\odot \left( \frac{M}{M_\odot} \right), \]

   where $L_\odot = 3.839 \times 10^{13} \text{ erg s}^{-1}$ and $M_\odot = 1.989 \times 10^{33} \text{ g}$. 


5. **A fictional star [20 pts]**

Consider a star of luminosity \( L \) with density distribution \( \rho = \rho_0 \times (R/r) \), where \( R \) is the star’s outer radius. Please don’t ask how it manages to have such a simple density profile; this star exists only in the homework universe.

All of the star’s energy is generated from a very small region near \( r = 0 \), and is transported entirely by radiation (not convection). The opacity is dominated by electron (Thomson) scattering, with opacity \( \kappa_T \).

(a) [4 pts] What is the star’s effective temperature \( T_{\text{eff}} \), in terms of the given quantities and fundamental constants?

(b) [16 pts] Solve for the temperature as a function of \( r \), in terms of \( \rho_0, T_{\text{eff}}, R, \kappa_T \), and fundamental constants. For the outer boundary condition, assume \( r = R \) represents the \( \tau = 2/3 \) level of the photosphere, and use the gray-atmosphere result \( T(\tau = 2/3) = T_{\text{eff}} \).

6. **Pressure due to both particles and radiation [30 pts]**

The pressure \( P \) that appears in the equations of stellar structure may include contributions from both gas pressure and radiation pressure. For Sun-like stars, gas pressure is dominant. However, the centers of more massive stars are hot enough for radiation to provide a significant fraction of the total pressure. In this problem you will investigate this regime using a simplified model in which the gas pressure is a specified fraction of the total gas pressure.

We will limit our attention to fully ionized material of solar composition (mean molecular weight \( \mu \approx 0.6 \)), where the gas and radiation field are at the same temperature, and the ideal gas law is applicable.

(a) [10 pts] First, let us get a feeling for the densities and temperatures involved. Write an expression for \( \beta \equiv P_{\text{gas}}/(P_{\text{gas}} + P_{\text{rad}}) \), in terms of the gas density \( \rho \) and temperature \( T \). Using this result, obtain the relationship between density and temperature in a gas for which \( \beta = 0.1 \), i.e., gas provides only 10% of the total pressure. Evaluate the temperature required to satisfy this condition for \( \rho = 10 \) g cm\(^{-3} \).

(b) [10 pts] Instead of a gas-radiation fluid of fixed density and temperature, instead consider one of fixed density and total pressure. Derive a quartic equation for \( \beta \) in terms of \( P \) and \( \rho \). This is known as the *Eddington quartic*. Show that \( \beta \to 1 \) for sufficiently small \( P \), and \( \beta \to 0 \) for sufficiently large \( P \). What is the characteristic pressure separating these two regimes, in terms of \( \rho \) and the fundamental constants \( h, c \) and \( m_p \)? You may find it useful to recall that the Stefan-Boltzmann constant is defined as \( \sigma = 2\pi^5 k^4/(15c^2 h^3) \). *(Hint: Begin by writing \( P_{\text{gas}} = \beta P \), and using the ideal gas law rewrite the equation you derived for \( \beta \) using \( P \) instead of \( T \)).*

(c) [10 pts] Let \( E_{\text{int}} = E_{\text{gas}} + E_{\text{rad}} \) be the total internal energy in a star, including both gas and radiation. Assume that \( \beta \) is uniform throughout the star (not very realistic). Using the same steps that are used to prove the virial theorem (see Sec. 3.2.2 of Chaudhuri), show that

\[
E_{\text{tot}} = E_{\text{int}} + E_{\text{grav}} = \frac{\beta}{2} E_{\text{grav}},
\]

where \( E_{\text{grav}} \) is the gravitational binding energy. What can you conclude about how strongly bound a massive, radiation-dominated star is compared to a low-mass star with negligible radiation pressure?

*Hint:* To set this problem up, first use the fact that the gravitational binding energy is

\[
E_{\text{grav}} = \int_0^M \Phi_{\text{grav}}(M)dM = -4\pi G \int_0^R r \rho(r)M(r) dr
\]

where \( \Phi(M) \) is the gravitational potential at the surface of a sphere of mass \( M \), and \( M(r) \) is the mass contained in a sphere of radius \( r \). Note we do not know \( M(r) \), so use the equation of hydrostatic equilibrium plus integration by parts to rewrite this as an integral over pressure \( P \). The pressure \( P \) is the sum of the gas pressure \( P_{\text{gas}} \) and the radiation pressure \( P_{\text{rad}} \).

Next, use the fact that the density associated with the internal energy of the gas is \( u_{\text{gas}} = \frac{3}{2} n_{\text{gas}} k_B T \) and that the internal energy associated with the radiation is \( u_{\text{rad}} = a T^4 \), with \( a = 4\sigma/c \) (and \( \sigma \) is the Stefan-Boltzmann constant). Relate these two energy densities to their associated pressures. With some labor, the result you need to prove will follow.
(d) **Course feedback. [5 pts]** Are you getting what you hoped for out of 8.901? What is working well for you? What improvements would you like to see?