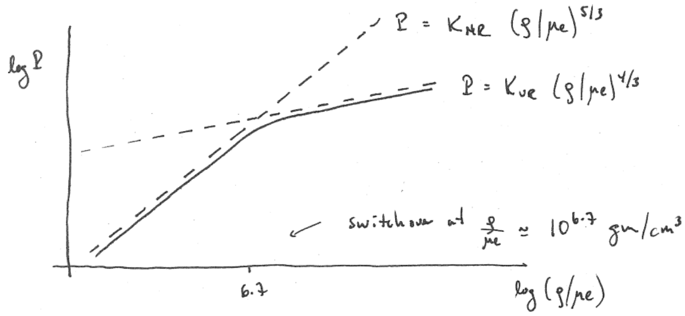


Figure 34: Pressure $P$ vs. density $\rho$ in the non-relativistic (NR) and ultra-relativistic (UR) limits. A switchover occurs at high densities above $\rho/\mu_e \approx 10^{6.7}$ g cm$^{-3}$.

of $p$, so we have one less $p$ in our integral for pressure:

(403)
$$P = \frac{1}{3}\int_0^{P_f} cp\frac{2}{h^3}4\pi p^2 dp$$

(404)
$$= \left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{8m_p^{4/3}}\left(\frac{\rho}{\mu_e}\right)^{4/3}$$

(405)
$$= K_{\mathrm{UR}}\left(\frac{\rho}{\mu_e}\right)^{4/3}.$$

So the ultra-relativistic degenerate gas is also a polytrope, but now with a slightly shallower index $\gamma = 4/3$.

## 16.4 *Implications of Degeneracy Pressure*

So our discussion of polytropes in Sec. 13 was fruitless; it now turns out that they give an exact description of the behavior of a degenerate gas. The polytrope indices in the two cases, above, 5/3 vs. 4/3, seem close enough together that there might not be much difference. But comparison to Eqs. 296 and 297 show that the slightly lower index of 4/3 makes all the difference: a fully relativistic and degenerate gas will tend toward instability and collapse.

In the equations of state Eqs. 402 and 405 above, the degeneracy pressure will dominate over the gas pressure so long as

(406)

$$P_{deg} \gg P_{gas}$$

(407)

$$K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3} \gg \frac{\rho kT}{\mu_e m_p}.$$

And assuming a fully ionized medium (so $\mu_e = 1/2$), we then require

(408) $$\frac{\rho}{\mu_e} \gg 750 \text{ g cm}^{-3}\left(\frac{T}{10^7\ K}\right)^{3/2}$$

which is quite similar to our earlier estimate of $n \ll \lambda_D^{-3}$ (Eq. 381, and Sec. 16.2). As the density of a degenerate gas is increased, Fig. 34 demonstrates that the equation of state will switch over from non-relativistic (Eq. 402) to ultra-relativistic (Eq. 405) above densities $\rho/\mu_e \approx 10^{6.7}$ g cm$^{-3}$ or (equivalently) when

(409) $$p_F \approx m_e c = \left(\frac{3nh^3}{8\pi}\right)^{1/3}.$$

## 16.5 *Comparing Equations of State*

As we start moving into stellar evolution, we will encounter wildly different regimes of pressure, density, and temperature. Which equation of state dominates in each regime? We've seen several examples so far:

| Type | EOS | | Ideal gas | Temp. dependence |
|---|---|---|---|---|
| NR degeneracy pressure | $K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3}$ | = | $\frac{\rho}{\mu_e}\frac{kT}{m_p}$ | $T \propto \rho^{2/3}$ |
| Rel degeneracy pressure | $K_{UR}\left(\frac{\rho}{\mu_e}\right)^{4/3}$ | = | $\frac{\rho}{\mu_e}\frac{kT}{m_p}$ | $T \propto \rho^{1/3}$ |
| Radiation pressure | $\frac{4\sigma}{3c}T^4$ | = | $\frac{\rho}{\mu_e}\frac{kT}{m_p}$ | $T \propto \rho^{1/3}$ |

Note that the temperature for radiation pressure and ultra-relativistic degeneracy pressure have the same dependence on temperature; however, the coefficient is larger for the radiation pressure case.

One additional case we haven't yet discussed is: when does treatment as a gas break down? This turns out to happen when Coulomb interactions become increasingly important. Or equivalently, when

(410)

$$E_C \approx E_{Th}$$

(411)

$$\frac{e^2}{a} \approx n^{1/3}e^2 = kT$$

which implies that again, $T \propto \rho^{1/3}$ — but with a smaller coefficient that for

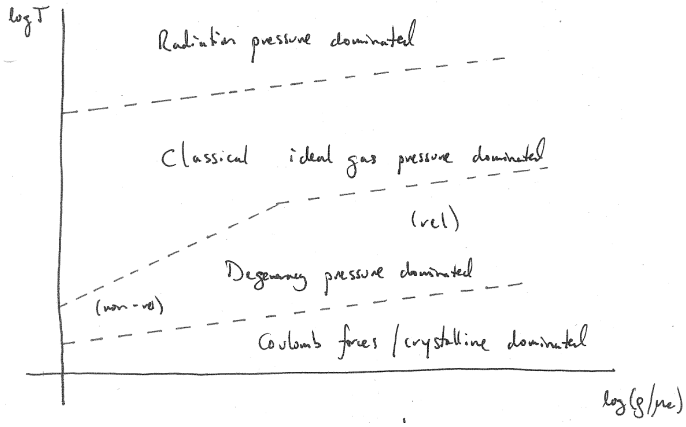


Figure 35: Different regimes in stellar interiors.

the ultra-relativistic degenerate gas. Fig. 35 summarizes all these different regimes.

Note that degeneracy pressure (like any good polytropic equation of state) is independent of temperature. So it halts stellar contraction even with no power generation. If nuclear power is somehow generated in a degenerate medium, there are interesting consequences:

- **Non-degenerate:** When extra energy is produced, the star expands and cools thanks to the virial theorem. Thus energy production will decreaes: negative feedback.
- **Degenerate star:** Extra energy production leads to no expansion of the star. The only place the energy can go is into heating the gas, so its temperature goes up – and thus energy production will increase as well. Positive feedback!

The positive feedback in the degenerate case can accelerate so rapidly that an entire star can become unbound. In other cases, the star will merely be heated up so much that the degenerate state is destroyed; then negative feedback via the virial theorem can once again come into play.

# 17 Stellar Evolution

## 17.1 *Useful References*

- Prialnik, 2nd ed., Ch. 7

## 17.2 *Introduction*

It's finally time to combine much of what we've introduced in the past weeks to address the full narrative of stellar evolution. This sub-field of astrophysics traces the changes to stellar composition and structure on nuclear burning timescales (which can range from Gyr to seconds). We'll start with a fairly schematic overview – first of the core, where the action is, then zoom out to the view from the surface, where the physics in the core manifest themselves as observables via the equations of stellar structure (Sec. 11.5).

Critical in our discussion will be the density-temperature plane. For a given composition (which anyway doesn't vary too widely for many stars), a star's evolutionary state can be entirely determined solely by the conditions in the core, $T_c$ and $\rho_c$. Fig. 36 introduces this plane, in which (as we will see) each particular stellar mass traces out a characteristic, parametric curve.[2]

## 17.3 *The Core*

There are several key ingredients for our "core view." These include:

- **Equation of State.** This could be an ideal gas ($P \propto \rho T$), a non-relativistic degenerate gas (Eq. 402, $P \propto \rho^{5/3}$), an ultra-relativistic degenerate gas

[2]We will see that it costs us very little to populate Fig. 36; that is, there's no need to stress over the $T$-$\rho$ price.

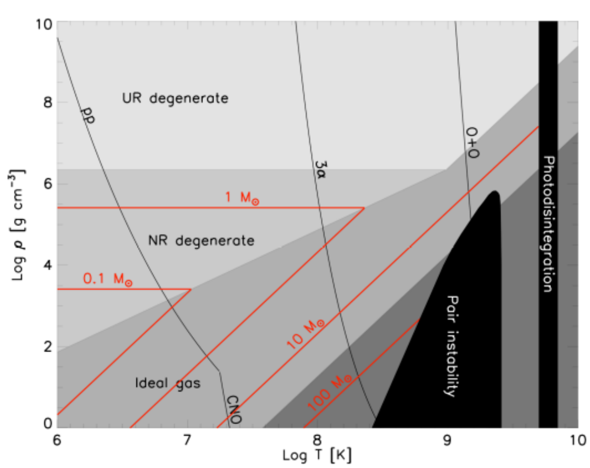


Figure 36: The plane of stellar density and temperature. See text for details.

(Eq. 405, $P \propto \rho^{4/3}$), or radiation pressure (Eq. 258; $P \propto T^4$). Which equation of state dominates depends on where we are in the $(\rho_c, T_c)$ plane – see Fig. 36.

- **Nuclear Reactions.** The active nuclear reaction pathways, and related reaction rates, depend steeply on $T$ and sometimes on $\rho$ as well(as discussed in Sec. 14.5), with a general form of $\epsilon = \epsilon_0 \rho^m T^n$. A given reaction pathway will "ignite" and rapidly start fusing when a given set of $T$ and $\rho$ are reached.
- **Energy Efficiency.** As we saw in Sec. 14.3 (see also Eq. 326), successively later stages of nuclear burning are incrementally less efficient. Each time the nuclear burning ratchets up to the next pathway, less and less binding energy can be liberated per nucleon. Thus $\tau_{nuc}$ of Eq. 207 gets shorter and shorter as the end draws near.
- **Stability.** In Sec. 12 we encountered several examples of stellar instabilities. For example, hydrostatic equilibrium breaks down for $\gamma_{ad} < 4/3$ (Eqs. 296 and 297). More generally, we will have to be on the lookout for cases when radiation pressure, degeneracy pressure, and/or ionization become particularly important; these will often correspond to significant dynamical upheavals in the star.
- **Nuclear Runaway.** Another kind of instability occurs if fusion occurs in a degenerate medium (we saw it at the end of Sec. 16). In this case we get a **fusion flash**: all available nuclear fuel is consumed on a thermal conduction timescale (just a few seconds) once burning begins.

## 17.4 *Equations of State*

Our discussion of stellar polytrope models (Sec. 13) is useful here for giving us a sense of what parts of Fig. 36 certain stars will occupy. What kind of polytrope? Because of the aforementioned stability constraints, for a star to be decently approximated by a polytrope requires

$$\frac{4}{3} < \gamma < \frac{5}{3}. \tag{412}$$

A useful relation comes from assuming a polytrope that is approximately in hydrostatic equilibrium. In this case, we obtain:

$$P_c = G M^{2/3} \rho^{2/3} (4\pi)^{1/3} F(n) \tag{413}$$

where $F(n)$ varies only slowly: from 0.233 to 0.145 for $n$ in the range (1, 3.5).

### Core is ideal gas

If we assume the core is an ideal gas, then this means

$$\frac{\rho_c k T_c}{\mu_e m_p} \approx \frac{1}{2} G M^{2/3} \rho_c^{4/3}. \tag{414}$$

Thus regardless of the evolutionary state, for a given star we have the relation

(415) $T_c \propto M^{2/3} \rho_c^{1/3}$

or equivalantly,

(416) $\log \rho_c = 3 \log T_c - 2 \log M + C.$

Thus a star of given mass will lie along a diagonal line in the (logarithmic) plane of Fig. 36, with higher-mass stars lying increasingly toward the right (higher $T$). It seems that while lower-mass stars will approach and perhaps enter the degenerate regime, more massive stars never do.

**Core is degenerate**

For those stars whose cores do reach the (non-relativistic) degenerate zone, then degeneracy pressure must be responsible for Eq. 413's pressure calculated from stellar structure considerations. Thus

(417) $K_{NR} \rho_c^{5/3} \approx \frac{1}{2} G M^{2/3} \rho_c^{4/3}.$

This result implies instead that

(418) $\rho_c \propto M^2.$

or equivalently,

(419) $\log \rho_c = 2 \log M + C.$

(Note that we would need to assume a somewhat different polytropic index to gain insight into the ultra-relativistic degenerate zone; but the non-relativistic assumption still gets the point across).

As we saw before, the structure of a degenerate object is independent of its temperature – so a degenerate core of a given mass has a fixed, maximum central density. Note that Eq. 418 also implies that a degenerate object's density increases as the square of its mass, which means that if we add mass to a degenerate white dwarf or neutron star its radius actually decreases. Furthermore, degenerate stars apparently lie along purely horizontal tracks in Fig. 36, with more massive stars at higher densities.

## 17.5 *Nuclear Reactions*

We've now seen how Fig. 36 can be populated with tracks representing the central conditions for a range of stars. We can now also populate the $T$-$\rho$ diagram with a set of orthogonal curves describing nuclear energy production in the cores of our stars.

What is a useful characteristic energy production rate $\epsilon$ to use for these tracks? We know that in the absence of fusion, stars can be (briefly) heated via gravitational (Kelvin-Helmholtz) contraction. We'll therefore require that nuclear burning produces enough energy to overcome that contraction. In other words,

$$(420)\quad \tau_{KH} \sim \frac{GM^2/R}{L} \sim \frac{u}{\epsilon_V}$$

where

$$(421)\quad u = \frac{3}{2}nkT$$

and the volumetric and mass-based energy rates are related as

$$(422)\quad \epsilon_V = \epsilon_m \rho.$$

This implies that the relevant energy production rate of for any given fusion chain, and over a wide range of stars, is approximately

$$(423)\quad \epsilon_m \sim 10 \text{ erg g}^{-1} \text{ s}^{-1}.$$

The features of these curves in Fig. 36 can be described as follows:

- **p-p chain**: At high $\rho$, ignites at fairly low $T$.
- **CNO cycle**: Dominates at sufficiently high $T$ and low $\rho$. Thus given the right raw materials, CNO can actually ignite first as a cloud condenses to form a star.
- **3$\alpha$**: Ignites at $\sim 10^8$ K, with a very steep $T$ dependence.
- **O+O**: Ignites at $\sim 10^9$ K, with an even steeper (practically vertical) slope.

## 17.6 *Stability*