14 An Introduction to Nuclear Fusion

14.1 *Useful References*

- Choudhuri, Secs. 4.1–4.2
- Kippenhahn, Weiger, and Weiss, 2nd ed., Chap. 18
- Hansen, Kawaler, and Trimble, Sec. 6.2

14.2 *Introduction*

Commercial nuclear fusion may be perpetually 50 years away, but stellar fusion has powered the universe for billions of years and (for the lowest-mass stars) will continue to do so for trillions of years to come.

Our two goals here are (1) to understand ϵ, the volumetric energy production rate (see Eq. 242), and how it depends on ρ and T; and (2) to identify and describe the key nuclear reaction pathways that are important in stars.

14.3 *Nuclear Binding Energies*

Stars derive their energy from the fusion of individual atomic nuclei, as we described briefly in Sec. 10.6. Fusion involves true elemental transmutation of the sort that the ancients could only dream of. For better or for worse, our own discussions of this natural alchemy will involve relatively more considerations of the detailed physics involved and relatively less boiling of one's one urine.

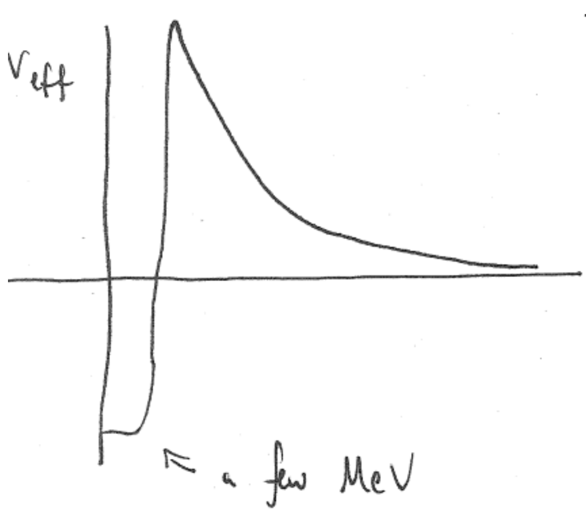

Figure 27: Rough sketch of the nuclear potential. Coulomb repulsion dominates at large separations, and is overwhelmed by Strong nuclear attraction at the smallest separations.

For one nucleus to reach another and fuse, it must overcome the strong Coulomb repulsion generated by the two positively-charged nuclei. Fig. 27 gives a rough sketch of the situation: Coulomb repulsion dominates at large separations, but it is overwhelmed by Strong nuclear attraction at the smallest separations. The fundamental nuclear size is set by the typical radius of protons and neutrons, $r_p \approx r_n \approx 0.8$ fm, as

$$r_{\rm nuc} \approx 2 r_p A^{1/3} \tag{324}$$

where A is the number of nucleons (neutrons plus protons, also approximately the atomic weight). We will also deal shortly with Z, the nuclear charge (i.e., number of protons).

The key thing that matters is the nuclear binding energy. Strong force has typical binding energies of a few MeV per nucleon. For astrophysical purposes we don't need to descend all the way into the realm of detailed nuclear physics. For our purposes an empirically-calibrated, semiclassical model (the "Bethe-Weizäcker formula) is sufficiently accurate. This posits that a nucleus' binding energy E_B is

$$E_B \approx a_V A - a_S A^{2/3} - \frac{a_C Z^2}{A^{1/3}} - a_A \frac{(A-2Z)^2}{A} \tag{325}$$

Each of the terms in Eq. 325 has a particular signifance. These are:

a_V	≈ 14 MeV	Volumetric term, describes bulk assembly of the nucleus.
a_S	≈ 13.1 MeV	Surface term, since surface nucleons have few neighbors.
a_C	≈ 0.58 MeV	Coulomb term, describes mutual repulsion of protons.
a_A	≈ 19.4 MeV	Asymmetry term, preferring $N_n = N_p$ (Fermi exclusion).

This model does a decent job: Eq. 325 correctly demonstrates that the nuclei with the greatest binding energy *per nucleon* have $Z \sim 25$. In fact the most tightly-bound, and thus most stable, nucleus is that of iron (Fe) with $Z = 26$, $A = 56$. Thus elements near Fe represent an equilibrium state toward which all nuclear processes will try to direct heavier or lighter atoms. For example, we will see that lighter atoms (from H on up) typically fuse into elements as high as Fe but no higher (except in unusual circumstances).

14.4 *Let's Get Fusing*

The Big Bang produced a universe whose baryonic matter was made of roughly 75% H and 25% He, with only trace amounts of heavier elements. Stellar fusion created most of the heavier elements, with supernovae doing the rest. For fusion to proceed, something must occur to either fuse H or He. Since He will have a $4\times$ greater Coulomb barrier, we'll focus on H; nonetheless we immediately encounter two huge problems.

Problem one is the huge Coulomb barrier shown in Fig. 27. At the separation of individual nucleons, the electronic (or protonic) repulsion is $e^2/\text{fm} \sim 1$ MeV, of roughly comparable scale to the strong nuclear attraction at shorter scales. But how to breach this Coulomb wall? Even at the center of the Sun where

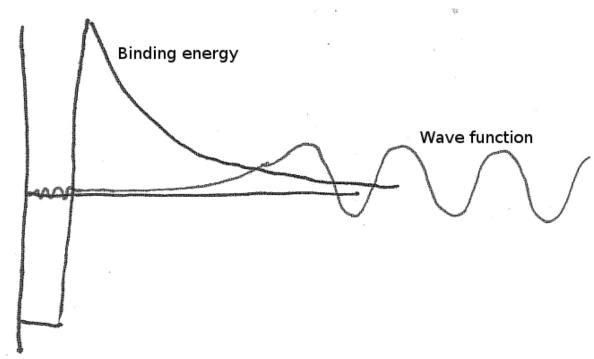

Figure 28: Rough sketch of quantum tunnelling, with the wave function (just barely) penetrating the forbidding Coulomb barrier.

$T_c = 1.5 \times 10^7$ K (Sec. 10) the typical thermal energies per particle are of order $k_B T_C \sim 1$ keV — a thousand times too low. (Problem Set 5 will show that even all the way out on the tail of the Maxwell-Boltzmann distribution, there are zero nuclei in a star with thermal energy sufficient to cross the barrier.) The second problem is that the fusion product of two protons would be ^{2}He, an isotope so unstable it is not entirely clear whether it has ever been observed.

Problem one: quantum tunnelling

The first problem was solved by recognizing that at the nuclear scale one doesn't climb a mountain — rather, one tunnels through it. Quantum mechanics states that each particle has a wave function $\Psi(x)$ given by the Schrödinger Equation, and the probability of finding the particle at x is $\propto |\Psi(x)|^2$. When the particle's energy is less than required to classically overcome an energy barrier, the wavefunction decays exponentially but remains nonzero. To order of magnitude, the protons only need to get close enough to each other that their thermal de Broglie wavelengths overlap; when this happens, tunneling becomes plausible (as sketched in Fig. 28).

The Coulomb barrier for two protons separated by a de Broglie wavelength λ_D (which may be substantially larger than the nucleon size of ∼1 fm) is

$$(326) \quad E_C = \frac{e^2}{\lambda_D} = \frac{e^2 p}{h}.$$

If the protons only need enough thermal energy to reach a separation of λ_D, then proton's required thermal momentum will be

$$(327) \quad p \approx \sqrt{2 m_p E_C}.$$

Solving for E_C, we see that

$$E_C^2 = \frac{e^4 p^2}{h^2} \approx \frac{2e^4 m_p E_C}{h^2} \quad (328)$$

and so

$$E_C \approx \frac{2e^4 m_p}{h^2}. \quad (329)$$

If the nuclei have thermal energies of order that given by Eq. 329, then quantum tunneling may happen. It turns out that E_C is of order a few keV, comparable to the thermal energy in the Sun's core.

The discussion above was merely phenomenological, but a more rigorous approach is the so-called "WKB approximation." WKB is described in more detail in advanced reference texts. Under certain fairly reasonable assumptions, the quantum wavefunction Ψ can be expressed as

$$\Psi(x) \propto \exp\left[\frac{i}{\hbar}\int \sqrt{2m\,(E - V(x))}dx\right] \quad (330)$$

with probability $\propto |\Psi|^2$, as noted above. When one calculates the full probability by integrating from the classical turning point tothe bound state, one finds that

$$P \approx e^{-bE^{-1/2}} \quad (331)$$

where

$$b = \frac{1}{\hbar} Z_1 Z_2 e^2 \sqrt{2\mu} \quad (332)$$

where μ is the reduced mass of the system. So as the typical particle energy E increases the probability of tunnelling becomes exponentially more likely. On the other hand, more strongly charged particles have larger Coulomb barriers and more massive particles have smaller de Broglie wavelengths; both of these effects will tend to make tunnelling more difficult to achieve.

The probability of tunnelling given by Eq. 331 directly relates to the cross section for nuclear interactions $\sigma(E)$. This is given approximately by

$$\sigma(E) \approx \lambda_D^2 e^{-bE^{-1/2}} \quad (333)$$

where

$$\lambda_D = \frac{h}{p} \propto \frac{1}{\sqrt{E}}. \quad (334)$$

This is pretty close. More accurate is a very similar form:

$$\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}} \quad (335)$$

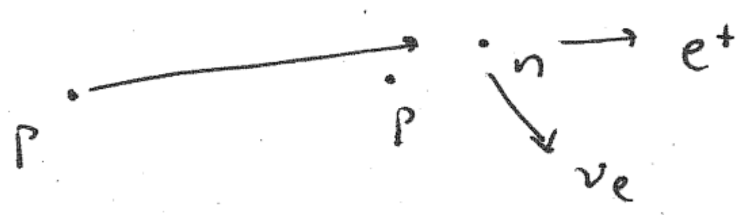

Figure 29: Rough sketch of inverse beta decay: $p + p$ yields $p + n + e^+ + \nu_e$.

where $S(E)$ is essentially a fudge factor (albeit one that varies only slowly with E).

Problem two: avoiding the ^{2}He trap

The key to our second problem (that the product of H + H, ^{2}He, is incredibly unstable) lies in the humble neutron. Given sufficient neutrons we could form the stable isotope ^{2}H (deuterium) instead of ^{2}He and open up new reaction pathways.

The challenge is that the neutron halflife is only 15 min, after which they undergo beta decay and produce $p + e^- + \bar{\nu}_e$. The opportunity lies in a related reaction, inverse beta decay. In this process (sketched in Fig. 29) two of the many, common protons interact via the weak process. The full reacion is

$p + p \rightarrow p + n + e^+ + \nu_e$

and perhaps surprisingly, this can provide all the neutrons we need to produce sufficient ^{2}H to make the universe an interesting place to be. The cross-section is tiny (it's a weak process):

$$\sigma_{\mathrm{p-p}} \approx 10^{-22} \text{ barnes} = 10^{-46} \text{ cm}^2 \quad (336)$$

(recall that the electron scattering, or Thomson, cross section of Sec. 9.5 was $\sigma_T = 0.67$ barns!). Put another way, the reaction rate in the Sun will be just once per $\sim$few Gyr, per proton. But it's enough, and once we have ^{2}H we can start producing heavier (and more stable) He isotopes: fusion becomes energetically feasible. Thus the solution to the ^{2}He fusion barrier is similar in a way to the H$^-$ story of Sec. 9.5: to get everything right required both hydrogen and some imagination.

14.5 *Reaction pathways*

With these most basic rudiments of nuclear considerations laid out, we can now start to consider some of the reaction pathways that might produce the energy we need to support the stars we see. This means that we're going to return to our volumetric energy production rate, ϵ, of Eq. 242 and determine what the quantity really means.

Imagine we have a reaction

$$\mathrm{A} + \mathrm{B} \rightarrow \mathrm{C}$$

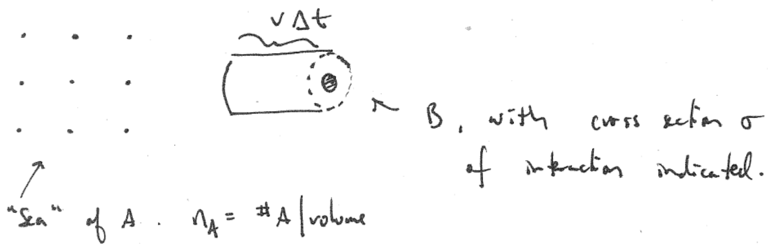

Figure 30: Particle B, moving at velocity v and impinging on a sea of particles A.

which releases Q units of energy and has some cross-section σ. What is ϵ? Fig. 30 shows the situation shortly before the reaction has occurred, with a particle B moving at speed v relative to a sea of particles A with number density n_A. In a short period of time Δt, the total volume contributing to the reaction is

$$V_{\text{eff}} = \sigma v \Delta t \tag{337}$$

and so the total reaction rate (per time, per particle B) will be

$$\frac{n_A v \Delta t \sigma}{\Delta t} = n_A v \sigma. \tag{338}$$

Since we have n_B B particles per volume, the volumetric reaction rate (reactions per time, per volume) will then be

$$r_{AB} = n_A n_B v \sigma. \tag{339}$$

Since each interaction liberates an amount of energy Q, the volumetric power density (energy per time per volume) will then be

$$\epsilon = Q n_A n_B v \sigma. \tag{340}$$

To be more accurate, we need to account for the fact that there is not a single relative velocity v but rather two separate velocity distributions (for particles A and B). Each distribution is given by Eq. 95,

$$\Phi_v = 4\pi\, n \left(\frac{m}{2\pi\, k_B T_{kin}} \right)^{3/2} v^2 \exp\left(-\frac{m v^2}{2\, k_B T_{kin}} \right) dv$$

It is an intruiging (and not too onerous) exercise to show that for two species in thermal equilibrium (and at the same T) that both obey Eq. 95, then their relative velocities also follow the same distribution but with m now replaced by μ, the reduced mass.

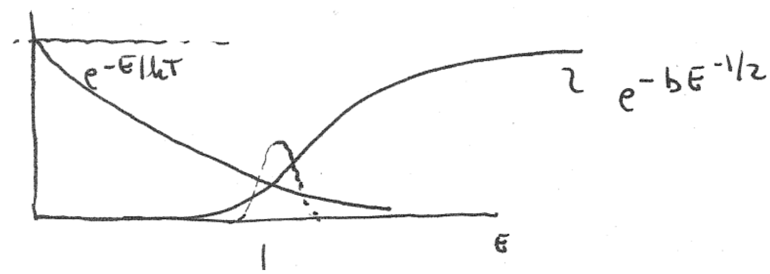

Figure 31: The two exponential terms in Eq. 346 cancel out everywhere but in a narrow region of overlap: the Gamow-Teller peak of nuclear energy production.

Alternatively, Eq. 95 can also be written in terms of energy instead of v:

(341) $$\Phi_E = \sqrt{\frac{2}{\pi}}(k_B T)^{-3/2} E^{1/2} e^{-E/k_B T}$$

and it must be true that

(342) $$\Phi_v dv = \Phi_E dE.$$

Then, instead of the simple Eq. 340 we have instead

(343) $$\epsilon = Q n_A n_B \int_0^\infty \sigma(E) v(E) \Phi_E dE$$

where $\sigma(E)$ comes from Eq. 335 and

(344) $$v(E) = \sqrt{2E/\mu}.$$

Neglecting everything but the T and E dependences, we then have

(345)
$$\epsilon \propto Q n_A n_B T^{-3/2} \int_0^\infty \frac{S(E)}{E} e^{-bE^{-1/2}} E^{1/2} E^{1/2} e^{-E/k_B T} dE$$

(346)
$$\propto Q n_A n_B T^{-3/2} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/k_B T} dE$$

The two exponentials inside the integral combine in an interesting way. As shown in Fig. 31, they cancel each other out everywhere but in a fairly narrow energy regime. This region of overlap, where ϵ reaches its greatest value, is known as the **Gamow-Teller peak**. This feature is all that's left after we've combined the Maxwell-Boltzmann distribution with the energy needed for tunnelling to proceed.

To actually solve the integral in Eq. 346 and calculate ϵ directly is more tricky. A straightforward and reasonable simplification is to approximate the Gamow-Teller peak as a normal (Gaussian) profile centered on the energy E_0 where ϵ reaches its maximum. The result is then

$$\epsilon \propto Q n_A n_B T^{-2/3} S(E_0) e^{-B T_6^{-1/3}} \tag{347}$$

where

$$B = 42.6 (Z_1 Z_2)^{2/3} \left(\frac{A_1 A_2}{A_1 + A_2} \right)^{1/3} \tag{348}$$

and $T_6 = T/(10^6 \mathrm{K})$.

Eq. 347 will go to zero in the limit of both large and small T, indicating that there is some optimal temperature range for any particular nuclear reaction. (Technically, this is an optimal range in T-n space.) The formula above can also be recast in logarithmic form, by noting that

$$\ln \epsilon = -\frac{2}{3} \ln T_6 - B T_6^{-1/3} + C \tag{349}$$

$$= -\frac{2}{3} \ln T_6 - B \exp\left(-\frac{1}{3} \ln T_6 \right) + C \tag{350}$$

It is common to then speak of a power index ν that describes the steepness of the dependence of ϵ on T, such that

$$\epsilon \propto T^{\nu}. \tag{351}$$

The index ν can be calculated

$$\nu \equiv \frac{d(\ln \epsilon)}{d \ln T_6} = -\frac{2}{3} + \frac{B}{3} T_6^{-1/3}. \tag{352}$$

For one of the main reaction chains in the Sun, $\nu \approx 3.8$ — so nuclear power generation depends fairly strongly on the central T. $\nu \approx 5$ for the lightest nuclei, and for other, much larger values of B involving more massive nuclei ν can take on much larger values, up to $n \approx 20$. This occurs in the most massive stars, which therefore show an extraordinary dependence of ϵ on T.

Traditionally the reactions described above are termed **thermonuclear** because the reaction rates and power generation depend most strongly on T. In dense environments with large compositions of heavy nuclei, electron shielding leads to an additional dependence on density as well, such that

$$\epsilon \propto \left(\frac{\rho}{\rho_0} \right)^{\lambda} \left(\frac{T}{T_0} \right)^{\nu}. \tag{353}$$

One then obtains

$$\nu = \frac{B}{2} T_6^{-1/3} - \frac{2}{3} - \frac{E_D}{k_B T}; \quad \lambda = 1 + \frac{1}{3} \frac{E_D}{k_B T} \tag{354}$$

and E_D is the electrostatic energy when two nuclei are separated by the radii of their electron clouds, r_D,

$$E_D = \frac{Z_1 Z_2 e^2}{r_D}. \tag{355}$$

At high densities and moderate temperatures, ν decreases and λ steepens considerably. This is the regime of **pyconuclear** reactions whose rates depend primarily on density not temperature. These conditions are typically seen in the latest stages of stellar evolution.